10.)

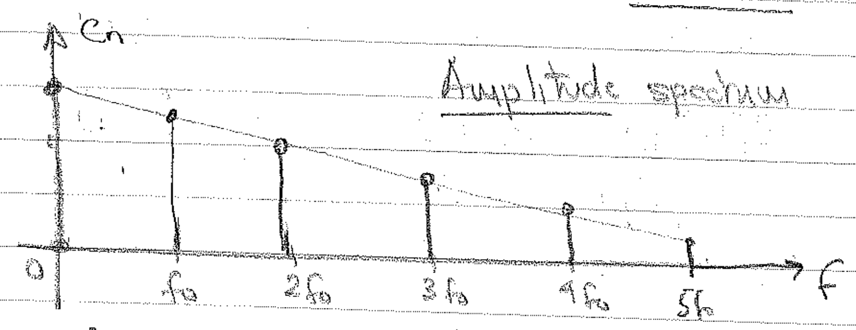
ECE 5201 Linear systems — Lecture 13

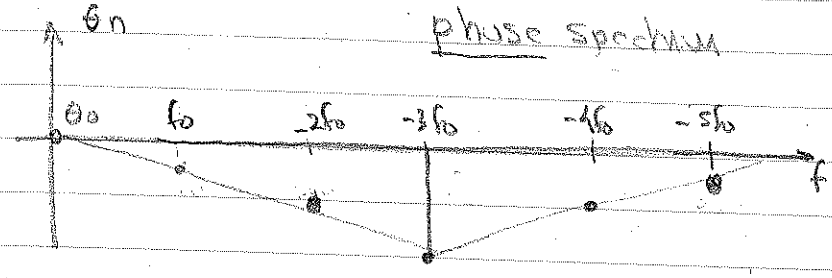
Furier spectrum

1) FORM II (compact trigonometric form)

$$X(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n) \qquad (*)$$

(*) - shows the building blocks of signal $X(t)$. The signal is build out of sinusoidals that have frequencies $n\omega_0 = n\,2\pi f_0$ and magnitudes C_n and θ_n. A convinient representation of (*) is <u>one sided</u> fourier spectrum

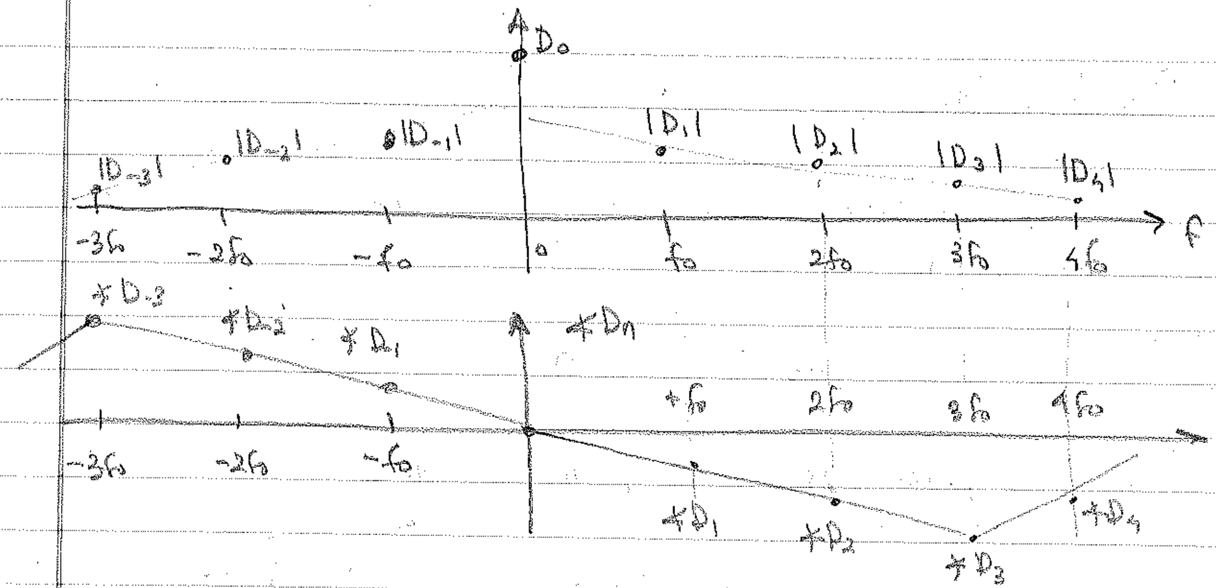




2) FORM III (exponential form)

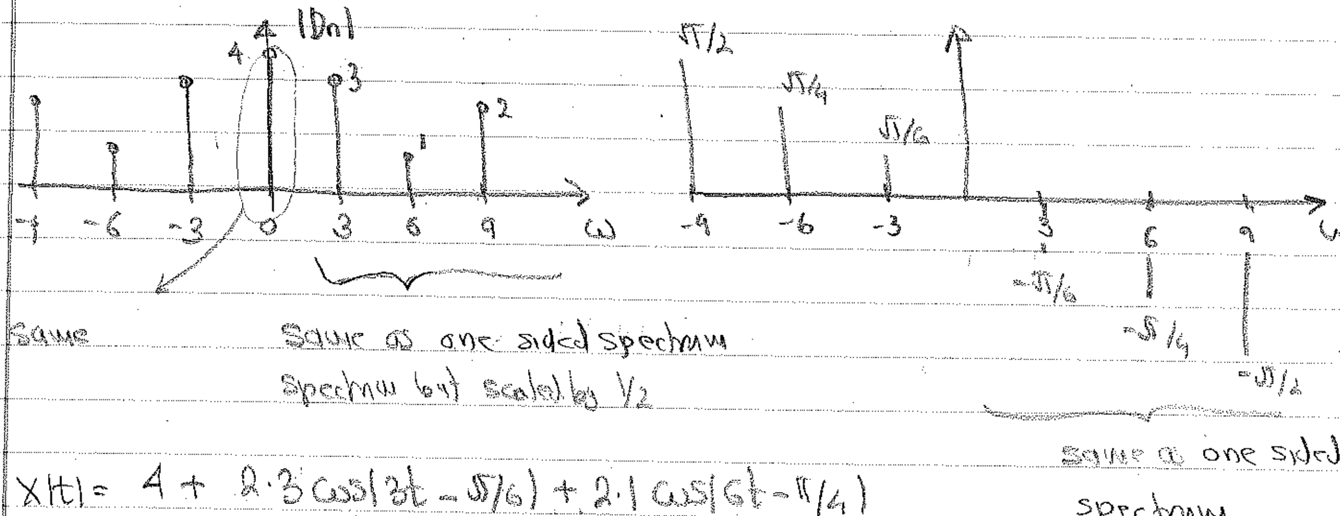
$$X(t) = \sum_{n=-\infty}^{+\infty} D_n\, e^{j2\pi n f_0 t} = \sum_{n=-\infty}^{+\infty} |D_n|\, e^{j(2\pi n f_0 t + \angle D_n)} \qquad (**)$$

(**) - shows that the signal can be decomposed of complex exponentials. Each complex exponential is characterized with its magnitude ($|D_n|$) its frequency parameter $2\pi n f_0$ and its phase $\angle D_n$. This representation leads to double sided Fourier spectrum



<u>Note:</u>
- * Amplitude spectrum of two sided spectrum has even symmetry
- * Phase spectrum of two sided spectrum plot has odd symmetry
- * for $f > 0$, amplitude spectrum of two sided plot is a scaled version of single sided plot with scaling of $1/2$
- * for $f > 0$, phase spectra in both plots are the same.

<u>Exercise. E6.4.</u> The exponential (two sided) spectrum of the signal is given in the figure. Determine expression for the trigonometric Fourier series.



$$x(t) = 4 + 2\cdot 3\cos(3t - \pi/6) + 2\cdot 1\cos(6t - \pi/4) + 2\cdot 2\cos(9t - \pi/2)$$

$$x(t) = 4 + 6\cos(3t - \pi/6) + 2\cos(6t - \pi/4) + 4\cos(9t - \pi/2)$$

Parseval's theorem

Consider a periodic signal $x(t)$ with period $T_0 = 1/f_0$. Its Fourier representation is given as:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t) + b_n \sin(2\pi n f_0 t)$$

This is a power signal with power given as:

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_T |x(t)|^2 dt = \frac{1}{T_0} \int_{T_0} |x(t)|^2 dt$$

$$P_x = \frac{1}{T_0} \int_{T_0} \left| a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t) + b_n \sin(2\pi n f_0 t) \right|^2 dt$$

$$= \frac{1}{T_0} \int_{T_0} \Big[a_0^2 + \Big(\sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t) \Big)^2 + \Big(\sum_{n=1}^{\infty} b_n \sin(2\pi n f_0 t) \Big)^2 +$$

$$2a_0 \sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t) + 2a_0 \sum_{n=1}^{\infty} b_n \sin(2\pi n f_0 t) +$$

$$\sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t) \sum_{m=1}^{\infty} b_n \sin(2\pi m f_0 t) \Big] dt$$

$$= \frac{1}{T_0} \int_{T_0} a_0^2 \, dt + \quad \ldots\ldots I_1$$

$$\frac{1}{T_0} \int_{T_0} \Big(\sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t) \Big)^2 dt + \quad \ldots\ldots I_2$$

$$1/T_0 \int_{T_0} \Big(\sum_{n=1}^{\infty} a_n \sin(2\pi n f_0 t) \Big)^2 dt + \quad \ldots\ldots I_3$$

$$\frac{2a_0}{T_0} \int_{T_0} \sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t) \, dt + \quad \ldots\ldots I_4$$

$$\frac{2a_0}{T_0} \int_{T_0} \sum_{n=1}^{\infty} b_n \sin(2\pi n f_0 t) \, dt + \quad \ldots\ldots I_5$$

$$\frac{1}{T_0} \int_{T_0} \sum_{n=1}^{\infty} a_n \cos(n w_0 t) \sum_{m=1}^{\infty} b_n \sin(n w_0 t) \, dt \quad \ldots I_6$$

$I_1 = a_0^2$ — power of the DC component

$I_4 = 0$ — average of cosines over integer # of periods

$I_5 = 0$ — average of sines over integer # of periods

$$I_2 = \frac{1}{T_0}\int_{T_0}\Big(\sum_{n=1}^{\infty} a_n\cos(n\omega_0 t)\Big)^2 dt =$$

$$= \frac{1}{T_0}\int_{T_0}\Big[\sum_{n=1}^{\infty} a_n^2\cos^2 n\omega_0 t + \sum_{\substack{n=1 \\ n\neq m}}^{\infty}\sum_{m=1}^{\infty} a_n\cos(n\omega_0 t)\, a_m\cos(m\omega_0 t)\Big]dt$$

$$= \sum_{n=1}^{\infty}\Big(\frac{1}{T_0}\int_{T_0} a_n^2\cos^2 n\omega_0 t\, dt\Big) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{T_0}\int_{T_0} a_n a_m\cos(n\omega_0 t)\cos(m\omega_0 t)\,dt$$

$$= \sum_{n=1}^{\infty}\frac{a_n^2}{2} + 0$$

$$I_3 = \sum_{n=1}^{\infty}\frac{b_n^2}{2}$$

$$I_6 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{T_0}\int_{T_0} a_n\cos(n\omega_0 t)\, b_n\sin(n\omega_0 t)\,dt = 0$$

Therefore:

1) $x(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)$

$P_x = a_0^2 + \sum_{n=1}^{\infty}\frac{a_n^2 + b_n^2}{2}$ — power of the signal

2) $x(t) = C_0 + \sum_{n=1}^{\infty} C_n\cos(n\omega_0 t + \theta_n)$

$P_x = C_0^2 + \sum_{n=1}^{\infty} C_n^2/2$ — power of the signal

3) $x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$

$P_x = \sum_{n=-\infty}^{\infty} |D_n|^2$ — power of the signal

→ Statements of Parseval's theorem.

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Bandwidth of the signal

$B_{p\%}$ – bandwidth that contains $p\%$ of the signal power.
– value of $p\%$ depends on application

Consider $x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$

$P_x = \frac{1}{T_0}\int_{T_0} |x(t)|^2\, dt$ – power of the signal

$P_x = C_0^2 + \sum_{n=1}^{\infty} C_n^2/2 = C_0^2 + C_1^2/2 + C_2^2/2 + \cdots + C_n^2/2 + \cdots$ (*)

C_0^2 – power in DC component
$C_1^2/2$ – power in 1st harmonic
$C_2^2/2$ – power in 2nd harmonic

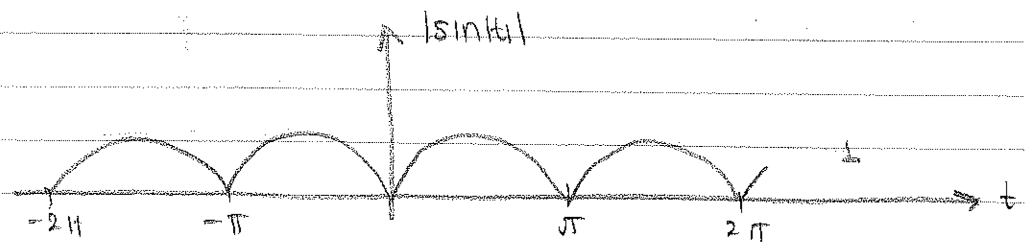
$P_0 = \frac{C_0^2}{P_x}$ – power in DC component

$P_1 = \frac{C_0^2 + C_1^2/2}{P_x}$ – power in DC component and 1st harmonic

$P_2 = \frac{C_0^2 + C_1^2/2 + C_2^2/2}{P_x}$ – power in DC component on first two harmonics

⋮

Example. Consider the signal given in figure. (full wave rectified sine). Determine the 95% bandwidth of the signal



Step 1: Determine fourier series of the signal

$T_0 = \pi$

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a) $a_0 = \frac{1}{\pi}\int_0^{\pi} \sin(t)\, dt = \frac{1}{\pi} \cos(t) \Big|_{\pi}^{0} = \frac{2}{\pi}$

b) signal is even $\Rightarrow$ $b_n = 0 \;\; \forall n$

c) $a_n = \frac{2}{\pi}\int_0^{\pi} \sin(t) \cdot \cos(2\pi n \frac{1}{\pi} t)\, dt =$

$$\frac{2}{\pi}\int_0^{\pi} \sin(t) \cos(2nt)\, dt = \frac{1}{\pi}\int_0^{\pi} \left(\sin[(1-2n)t] + \sin[(1+2n)t]\right) dt$$

$$= \frac{1}{\pi}\left(\frac{\cos[(1-2n)t]}{1-2n}\Big|_{\pi}^{0} + \frac{\cos[(1+2n)t]}{1+2n}\Big|_{\pi}^{0} \right) =$$

$$= 1/\pi \left(\frac{1-(-1)^{2n-1}}{1-2n} + \frac{1-(-1)^{2n+1}}{1+2n} \right)$$

$$= \frac{1}{\pi} \frac{(1+2n)\left[1-(-1)^{2n-1}\right] + (1-2n)\left[1-(-1)^{2n+1}\right]}{1-4n^2}$$

$$= \frac{1}{\pi} \frac{(1+2n)\cdot 2 + (1-2n)\cdot 2}{1-4n^2} = \frac{4/\pi}{1-4n^2}$$

$$x(t) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4/\pi}{1-4n^2} \cos(2nt)$$

Step 2

a) Power of the signal

$$P_x = \frac{1}{\pi}\int_0^{\pi} \sin^2(t)\, dt = \frac{1}{2\pi}\int_0^{\pi} (1-\cos(2t))\, dt = \frac{1}{2}$$

b)

$P_0 = (2/\pi)^2 = 0.4053$ $\quad P_0/P_x = 0.8106 \rightarrow 81.06\%$

$P_1 = (2/\pi)^2 + \left(\frac{4/\pi}{1-4}\right)^2 /2 = 0.4953$ $\quad P_1/P_x = 0.9907 \rightarrow 99.07\%$

Keeping DC + 1st harmonic captures 99.07% of power.

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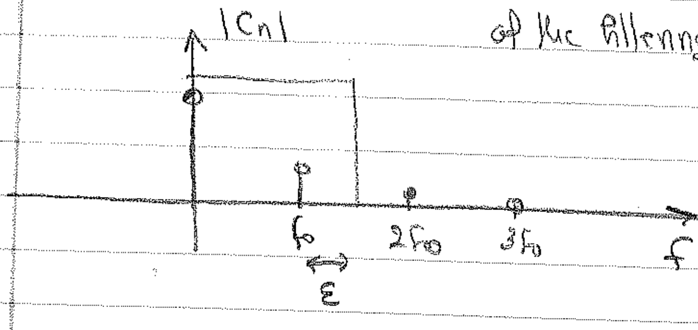
therefore 95% BW of the signal is

$BW_{95} = f_0 + \varepsilon$ f_0 - frequency of first harmonic ($f_0 = 1/\pi$ in this case)
ε - some value introduced to allow for transition of the filtering stages. $\varepsilon > 0$ and $\varepsilon < f_0$



<u>LTIC response to periodic inputs:</u>

$x(t) \longrightarrow \boxed{\begin{matrix} h(t) \\ H(f) \end{matrix}} \longrightarrow y(t)$ Let $H(f) = \int_{-\infty}^{+\infty} h(t) e^{-j2\pi f t} dt$

I) 1) $x(t) = A\cos(2\pi f_0 t)$, $\forall t$ - started at $t = -\infty$

$$y(t) = A \cdot |H(f)|_{f=f_0} \cos(2\pi f_0 t + \measuredangle H(f)|_{f=f_0})$$

2) $x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(2\pi n f_0 t + \theta_n)$

$$y(t) = C_0 |H(f)|_{f=0} + \sum_{n=1}^{\infty} C_n \underbrace{|H(f)||_{f=nf_0}}_{\text{magnitude response of the system at } f=f_0} \times \cos(2\pi n f_0 t + \theta_n + \underbrace{\measuredangle H(f)|_{f=nf_0}}_{\text{phase response of the system at } f=f_0})$$

II) 1) $x(t) = D e^{j2\pi f_0 t}$

$$y(t) = D \cdot |H(f)|_{f=f_0} e^{j(2\pi f_0 t + \measuredangle H(f)|_{f=f_0})}$$

21) $x(t) = \sum_{n=-\infty}^{+\infty} D_n e^{j2\pi n f_0 t}$

$$y(t) = \sum_{n=-\infty}^{+\infty} D_n \cdot |H(f)|_{f=f_0} e^{j(2\pi n f_0 t + \angle H(f)|_{f=f_0})}$$

Process.

$x(t)$ - periodic $\xrightarrow{\text{direct way}}$ $y(t) = x(t) * h(t)$

$\downarrow$ Fourier series

$x(t) = \sum_{n=-\infty}^{+\infty} D_n e^{jnf_0 t}$

$\xrightarrow{\text{frequency response}}$ $y(t) = \sum_{n=-\infty}^{+\infty} D_n H(nf_0) e^{j2\pi n f_0 t}$

where $H(f) = \int_{t=-\infty}^{+\infty} h(t) e^{-j2\pi f t}\, dt$ - frequency response of the system.

Problems.

6-3-2 6-3-9
6-3-3 6-3-10
6-3-4 6-4-2
6-3-5 6-4-3
6-3-7
6-3-12