ECE 5201 Linear Systems   Lecture 12

Periodic signals

$x(t)$ - periodic if $x(t+nT_0) = x(t)$, $n = 0, \pm 1, \ldots$

T_0 - fundamental period or just period of $x(t)$

Fourier series:

Every periodic signal can be decomposed into a series in a form given by.

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t) + b_n \sin(2\pi n f_0 t) \quad (*)$$

where $f_0 = 1/T_0$ - fundamental frequency and

T_0 - period of signal $x(t)$

Coefficients in the expansion given by $(*)$ can be determined using

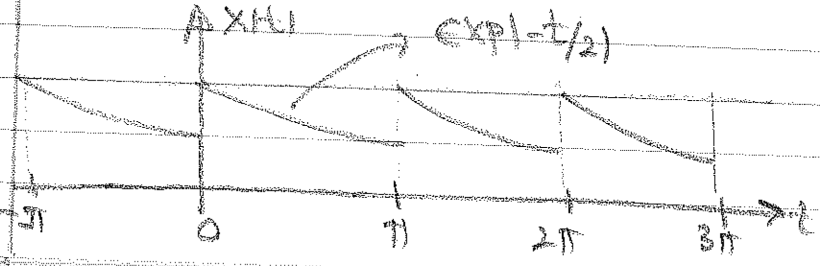
$$a_0 = \frac{1}{T}\int_T x(t)\,dt$$

$$a_n = \frac{2}{T}\int_T x(t)\cos(2\pi n f_0 t)\,dt$$

$$b_n = \frac{2}{T}\int_T x(t)\sin(2\pi n f_0 t)\,dt$$

Form given in $(*)$ is referred to as the trigonometric form of the Fourier series

Example 6.1. Consider the signal given in the figure. Determine its FS expansion



$$X(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t) + b_n \sin(2\pi n f_0 t)$$

$$f_0 = 1/T_0 = 1/\pi$$

Step1 $a_0 = \frac{1}{T}\int_T X(t)\,dt = \frac{1}{\pi}\int_0^{\pi} \exp(-t/2)\,dt/2 \times 2$

$$= \frac{2}{\pi} \exp(-t/2)\Big|_{\pi}^{0} = \frac{2}{\pi}\left(1 - \exp(-\frac{\pi}{2})\right) = 0.5043$$

Step2. $a_n = \frac{2}{T}\int_T X(t)\cos(2\pi n f_0 t)\,dt =$

$$= \frac{2}{\pi}\int_0^{\pi} \exp(-t/2)\cos(2\pi n \tfrac{1}{\pi} t)\,dt =$$

$$= \frac{2}{\pi}\int_0^{\pi} \exp(-t/2)\cos(2nt)\,dt =$$

$$= \frac{1}{\pi}\int_0^{\pi} \exp(-t/2)\,\frac{\exp(j2nt) + \exp(-j2nt)}{2}\,dt =$$

$$= \frac{1}{\pi}\int_0^{\pi} \exp(-(\tfrac{1}{2} - j2n)t)\,dt + \frac{1}{\pi}\int_0^{\pi} \exp(-(\tfrac{1}{2} + j2n)t)\,dt =$$

$$= \frac{1}{\pi}\left\{ \frac{\exp(-(1/2 - j2n)t)}{\frac{1}{2} - j2n}\Big|_{\pi}^{0} + \frac{\exp(-(1/2 + j2n)t)}{1/2 + j2n}\Big|_{\pi}^{0} \right\}$$

$$= \frac{1}{\pi}\left\{ \frac{1 - \exp(-\pi/2)\exp(+j2n\pi)}{\frac{1}{2} - j2n} + \frac{1 - \exp(-\pi/2)\exp(j2n\pi)}{\frac{1}{2} + j2n} \right\}$$

$$= \frac{1}{\pi}\left[1 - \exp(-\pi/2)\right] \frac{\frac{1}{2} + j2n + \frac{1}{2} - j2n}{(1/2)^2 + (2n)^2} = \frac{1}{\pi}\left[1 - \exp(-\pi/2)\right] \frac{\pi}{\frac{1}{4} + 4n^2}$$

$$= 0.5043 \times \frac{2}{1 + 16n^2}$$

Step 8. $b_n = \frac{2}{T}\int_T x(t)\sin(2\pi n f_0 t)\,dt =$

$$= \frac{2}{\pi}\int_\pi \exp\left(-\frac{t}{2}\right)\sin\left(2\pi n \frac{1}{\pi} t\right)dt$$

$$= \cdots = 0.5043\ \frac{8n}{1+16n^2}$$

Therefore

$$x(t) = 0.5043 + \sum_{n=1}^{\infty}\left(0.5043\ \frac{2}{1+16n^2}\right)\cos(2nt) + \left(0.5043\ \frac{8n}{1+16n^2}\right)\sin(2nt)$$

$$x(t) = 0.5043\left[1+\sum_{n=1}^{\infty}\frac{2}{1+16n^2}\cos(2nt) + \frac{8n}{1+16n^2}\sin(2nt)\right]$$

Alternative forms of Fourier series

Compact trigonometric form:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos(2\pi n f_0 t) + b_n\sin(2\pi n f_0 t)$$

Using $A\cos(\omega t) + B\sin\omega t = (A^2+B^2)^{1/2}\cos(\omega t - \text{atan}\,B/A)$

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n^2+b_n^2)^{1/2}\cos(2\pi n f_0 t - \text{atan}\,b_n/a_n)$$

Define $C_0 = a_0$

$C_n = \sqrt{a_n^2+b_n^2}$

$\theta_n = \text{atan}(-b_n/a_n)$

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n\cos(2\pi n f_0 t + \theta_n)$$

Exponential form.

$$X(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t) + b_n \sin(2\pi n f_0 t) =$$

$$= a_0 + \sum_{n=1}^{\infty} \left(a_n \frac{e^{j2\pi n f_0 t} + e^{-j2\pi n f_0 t}}{2} + b_n \frac{e^{j2\pi n f_0 t} - e^{-j2\pi n f_0 t}}{j2} \right)$$

$$= a_0 + \sum_{n=1}^{\infty} \left(\frac{a_n}{2} + \frac{b_n}{j2} \right) e^{j2\pi n f_0 t} +$$

$$+ \sum_{n=1}^{\infty} \left(\frac{a_n}{2} - \frac{b_n}{j2} \right) e^{-j2\pi n f_0 t}$$

$$= a_0 + \sum_{n=1}^{\infty} \frac{1}{2}(a_n - j b_n) e^{j2\pi n f_0 t} + \sum_{n=1}^{\infty} \frac{1}{2}(a_n + j b_n) e^{-j2\pi n f_0 t}$$

$$= \sum_{n=-\infty}^{+\infty} D_n e^{+j2\pi n f_0 t}$$

Where $D_0 = a_0$

$D_n = \frac{1}{2}(a_n - j b_n)$, $n > 0$

$D_n = \frac{1}{2}(a_n + j b_n)$, $n < 0$

$n = 0$ $\quad D_0 = \frac{1}{T} \int_T X(t)\, dt$

$$D_n = \frac{1}{2} \left[\frac{2}{T} \int_T X(t) \cos(2\pi f_0 n t)\, dt + \right.$$

$$\left. - j \frac{2}{T} \int_T X(t) \sin(2\pi n f_0 t)\, dt \right] =$$

$$= \frac{1}{T} \int_T X(t) \left[\cos(2\pi n f_0 t) - j \sin(2\pi n f_0 t) \right] dt$$

8)

$$D_n = \frac{1}{T}\int_T x(t)\, e^{-j2\pi n f_0 t}\, dt$$

Summary

1) Trigonometric form

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f_0 t) + b_n \sin(2\pi n f_0 t)$$

$f_0 = 1/T_0$ → fundamental frequency

$$a_0 = \frac{1}{T}\int_T x(t)\, dt$$

$$a_n = \frac{2}{T}\int_T x(t)\cos(2\pi n f_0 t)\, dt \qquad b_n = \frac{2}{T}\int_T x(t)\sin(2\pi n f_0 t)\, dt$$

2) Compact trigonometric form

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(2\pi n f_0 t + \theta_n),$$

- this form does not have direct calculation formulas

3) Exponential form

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j2\pi n f_0 t}$$

$$D_n = 1/T \int_T x(t)\, e^{-j2\pi n f_0 t}\, dt, \quad \forall n$$

Conversion formulas.

1→2) $C_n = \sqrt{a_n^2 + b_n^2}$ $\qquad a_n = C_n \cos(\theta_n)$

$\theta_n = \operatorname{atan}(-b_n/a_n)$ $\qquad b_n = C_n \sin(\theta_n)$

1→3

$D_n = (a_n - jb_n)/2$ $\qquad a_n = D_n + D_n^*$

$\qquad\qquad\qquad\qquad\qquad b_n = j(D_n - D_n^*)$

2→3 $\quad C_n = 2|D_n|$ $\qquad D_n = \frac{1}{2}\left[C_n\cos(\theta_n) - j\,C_n\sin(\theta_n)\right]$

$\theta_n = \measuredangle D_n$ $\qquad\qquad = \frac{1}{2} C_n e^{-j\theta_n}$

Physical insight (effects of symmetry)

1) $a_0 = \frac{1}{T}\int_T x(t)\,dt \Rightarrow$ average of periodic signal over duration of one period.

note: usually one could guess the value of a_0 — DC component of the signal

2) $x(t) = \underbrace{\underbrace{a_0}_{DC} + \sum_{n=1}^{\infty} a_n\cos(2\pi n f_0 t)}_{\text{even part}} + \underbrace{\sum_{n=1}^{\infty} b_n\sin(2\pi n f_0 t)}_{\text{odd part}}$

$x_e(t) = \frac{1}{2}(x(t) + x(-t))$

$= \frac{1}{2}\Big[a_0 + \sum_{n=1}^{\infty} a_n\cos(2\pi n f_0 t) + \sum_{n=1}^{\infty} b_n\sin(2\pi n f_0 t)$

$\quad a_0 + \sum_{n=1}^{\infty} a_n\cos(2\pi n f_0\cdot(-t)) + \sum_{n=1}^{\infty} b_n\sin(2\pi n f_0(-t))\Big]$

$= a_0 + \sum_{n=1}^{\infty} a_n\cos(2\pi n f_0 t)$

$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$

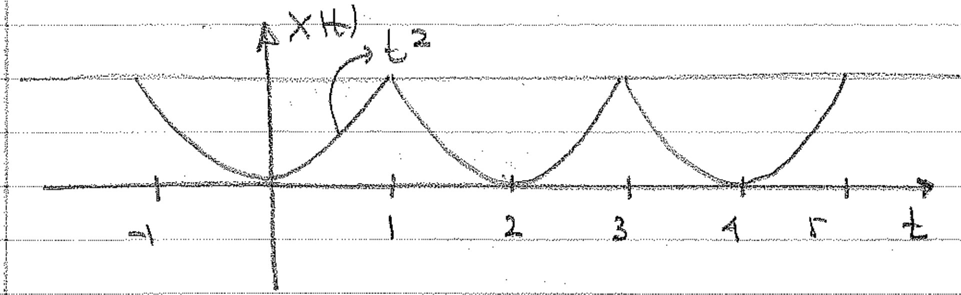
$= \frac{1}{2}\Big[a_0 + \sum_{n=1}^{\infty} a_n\cos(2\pi n f_0 t) + \sum_{n=1}^{\infty} b_n\sin(2\pi n f_0 t)$

$\quad a_0 + \sum_{n=1}^{\infty} a_n\cos(2\pi n f_0(-t)) + \sum_{n=1}^{\infty} b_n\sin(2\pi n f_0(-t))\Big]$

$= \sum_{n=1}^{\infty} b_n\sin(2\pi n f_0 t)$

Note: 1) If the function is even - sine part of FS is zero
2) If the function is odd - cosine part of FS is zero.

Example: Determine all 3 forms of FS for the function in figure



a) $T_0 = 2 \quad f_0 = 1/2$

STEP1: $a_0 = \frac{1}{T}\int_T x(t)\,dt = \frac{2}{2}\int_0^1 t^2\,dt = \frac{t^3}{3}\Big|_0^1 = 1/3$

STEP2: Function has even symmetry $\Rightarrow b_n = 0 \ \forall n$

STEP3: $a_n = \frac{2}{T}\int_T x(t)\cos(2\pi n f_0 t)\,dt = \frac{2\times 2}{2}\int_0^1 t^2\cos(2\pi n f_0 t)\,dt$

$$= 2\int_0^1 t^2\cos(2\pi n f_0 t)\,dt$$

Using table integral $\int x^2\cos(ax)\,dx = \frac{2x\cos(ax)}{a^2} + \frac{a^2x^2-2}{a^3}\sin ax + c$

$$a_n = 2\cdot\left\{\frac{2t\cos(2\pi n \tfrac{1}{2} t)}{(2\pi n \tfrac{1}{2})^2}\Big|_0^1 + \frac{(2\pi n \tfrac{1}{2})^2 t^2 - 2}{(2\pi n \tfrac{1}{2})^3}\sin(2\pi n \tfrac{1}{2} t)\Big|_0^1\right\}$$

$$= 4\frac{(-1)^n}{(\pi n)^2} + 0$$

$$x(t) = \frac{1}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{(\pi n)^2}\cos(\pi n t)$$

b) $C_0 = a_0$

$$C_n = \sqrt{a_n^2 + b_n^2} = |a_n| \text{, since } b_n = 0$$

$$C_n = \frac{4}{\pi^2 n^2}, \quad \theta_n = \begin{cases} 0 & - n \text{ even} \\ \pi & - n - \text{odd} \end{cases}$$

$$x(t) = 1/3 + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \cos(\pi n \cdot t + n\pi)$$

c) $$D_n = \frac{1}{2}(a_n - jb_n) = \frac{1}{2} a_n = \frac{2}{\pi^2 n^2}$$

$$x(t) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} \frac{2}{\pi^2 n^2} e^{+j2\pi n f_0 t} + 1/3$$

Problems:

6-1-2
6-1-3
6-1-4
6-1-5
6-1-6