ECE 5201 Linear Systems

Lecture 10

Unit impulse response

Consider n^{th}-order difference equation

$$(E^N + a_1 E^{N-1} + \dots + a_{N-1} E + a_N)\, y[n] = (b_0 E^N + b_1 E^{N-1} + \dots + b_N)\, x[n] \quad (*)$$

The response of the system to input $x[n]$

$$y[n] = y_{zi}[n] + y_{zs}[n]$$

$y_{zi}[n]$ - obtained through analysis of natural modes (characteristic poly...)
$y_{zs}[n]$ - obtained as a convolution between input & impulse response

Impulse response may be obtained 2 ways.

1) Iterative evaluation of difference equation $(*)$
2) closed form solution of difference equation $(*)$

1) Iterative evaluation:

$$Q(E) \cdot h[n] = P(E)\, \delta[n] \quad (\star)$$

Subject to:

$$h[-1] = h[-2] = \dots = h[-N] = 0$$

Equation $(\star)$ can be easily evaluated for every $n \geq 0$ and lead provide impulse response of the system

Example 3.11. Find the impulse response of the system given by

$$y[n] - 0.6y[n-1] - 0.16\, y[n-2] = 5x[n]$$

To determine impulse response we substitute $x[n] = \delta[n]$

$h[n] - 0.6h[n-1] - 0.16h[n-2] = 5\delta[n]$, or

$h[n] = 0.6h[n-1] + 0.16h[n-2] + 5\delta[n]$

n	$\delta[n]$	$h[n-1]$	$h[n-2]$	$h[n]$	
0	1	0	0	5	
1	0	5	0	3	$0.6\times5+0.16\times0$
2	0	3	5	2.6	$0.6\times3+0.16\times5$
3	0	2.6	3	2.04	$0.6\times2.6+0.16\times3$
4	0	2.04	2.6	1.64	$0.6\times2.04+0.16\times2.6$
5	0	1.64	2.04	1.3104	$0.6\times1.64+0.16\times2.04$
6	0	...			

Pro: Impulse response is easily evaluated. With a simple script one can easily determine value of impulse response for any n.

Con: The method does not provide closed form of the impulse response. No additional insight that we obtain from the closed form.

2) Closed form of the impulse response

$$h[n] = \underbrace{A_0\delta[n]}_{\text{direct coupling of input-output}} + \underbrace{y_c[n]\cdot u[n]}_{\text{response in the form of natural modes}} \quad \text{- format of impulse response} \quad (+)$$

Substitution of (+) into (*) one obtains

$$Q(E)\cdot\left(A_0\delta[n] + y_c[n]\cdot u[n]\right) = P(E)\cdot\delta[n]$$

$$Q(E)\,A_0\delta[n] + \underbrace{Q(E)\cdot y_c[n]\cdot u[n]}_{0} = P(E)\cdot\delta[n]$$

Therefore

$$A_0 Q(E)\cdot\delta[n] = P(E)\cdot\delta[n]$$, or

$$A_0\left(\delta(n+N) + a_1\delta(n+N-1) + \dots + a_N\delta[n]\right) = b_0\delta[n+N] + \dots + b_N\delta[n]$$

This equation is true for every n. Setting $n=0$, one obtains

$$A_0\left(\delta(N) + a_1\delta(N-1) \dots + a_N\delta[0]\right) = b_0\,\delta[N] + \dots + b_N\delta(0)$$, or

$$A_0 a_N \delta[0] = b_N \Rightarrow A_0 = b_N/a_N$$

Therefore

$$h[n] = b_N/a_N\,\delta[n] + y_c[n]\cdot u[n]$$

subject to initial conditions $h[0], h[1], \dots, h[N-1]$ that may be determined through recursive evaluation of the difference equation

Example 3.12 Consider the system given by difference equation

$$y[n] - 0.6\,y[n-1] - 0.16\,y[n-2] = 5x[n]$$, or

$$(E^2 - 0.6E - 0.16)\cdot y[n] = 5\cdot x[n]$$

Step 1 $Q(\lambda) = \lambda^2 - 0.6\lambda - 0.16$

$$\lambda_{1/2} = \frac{0.6 \pm \sqrt{0.36 + 0.64}}{2}, \quad \lambda_1 = 0.8, \quad \lambda_2 = -0.2$$

Step 2 $h[n] = b_N/a_N\,\delta[n] + y_c[n]\,u[n]$

Since $b_N = 0$

step 2. $h[n] = [c_1(-0.2)^n + c_2 \cdot 0.8^n] \times u[n]$

$$\left.\begin{array}{ll} n=0 & c_1 + c_2 = 5 \\ n=1 & -0.2c_1 + 0.8c_2 = 3 \end{array}\right\} \rightarrow$$ see example 3.11 (page 69)

Solving the system yields $c_1 = 1$, $c_2 = 4$, and therefore

$h[n] = [(-0.2)^n + 4 \times (0.8)^n] \cdot u[n]$

Exercise E3.14. Find impulse response of following LTID systems

a) $y[n+1] - y[n] = x[n]$

$y[n] - y[n-1] = x[n-1]$, $b_N = 1$, $a_N = -1$

step 1. $Q(E) = E - 1$

$Q(\lambda) = \lambda - 1 = 0 \Rightarrow \lambda = 1$ $y_c[n] = C \cdot 1^n u[n] = C\,u[n]$

step 2. $h[n] = \frac{b_N}{a_N}\delta[n] + y_c[n] \cdot u[n]$

$h[n] = -1\,\delta[n] + C\,u[n]$

step 3. Initial conditions

$h[0] - h[-1] = \delta[-1]$

$h[0] - 0 = 0 \Rightarrow h[0] = 0$

$h[0] = -1\,\delta[0] + C\,u[n] = -1 + C = 0 \Rightarrow C = 1$

Impulse response $h[n] = -1\,\delta[n] + u[n] = u[n-1]$

(72)

d) $y[n] = 2x[n] - 2x[n-1]$, or
$y[n+1] = 2x[n+1] - 2x[n]$

$Q(E) = E + 0$
$P(E) = 2E - 2$

In this case $a_N = 0$, and the impulse response cannot be determined using formula

$$h[n] = \frac{b_N}{a_N}\delta[n] + y_c[n]\cdot u[n]$$

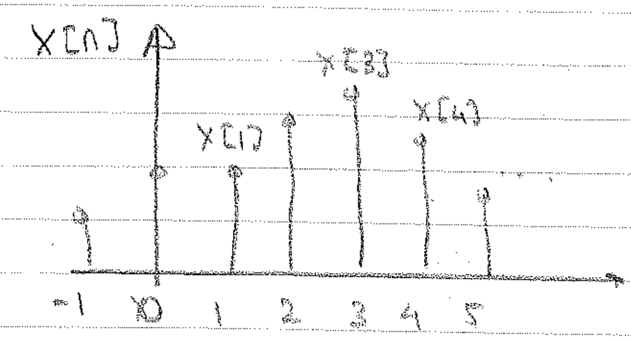
Using direct evaluation:

$$h[n] = 2\delta[n] - 2\delta[n-1]$$

$h[0] = 2\delta[0] - 2\delta[-1] = 2$
$h[1] = 2\delta[1] - 2\delta[0] = -2$
$h[2] = 2\delta[2] - 2\delta[1] = 0 \ \ldots$

$$h[n] = 2\delta[0] - 2\delta[1]$$

System response to external inputs



$$x[n] = \sum_{m=-\infty}^{+\infty} x[m]\delta[n-m] =$$

$= \cdots x[-1]\delta[n+1] + x[0]\delta[n] + \cdots + x[n]\delta[0] + \cdots$

Selected by delta

$$\delta[n] \rightarrow \boxed{\begin{matrix}\text{LDT} \\ \text{system}\end{matrix}} \rightarrow h[n]$$ Using superposition

$$x[n] = \sum_{m=-\infty}^{+\infty} x[m]\delta[n-m] \longrightarrow y[n] = \sum_{m=-\infty}^{+\infty} x[m]\cdot h[n-m]$$

Define convolution in DT domain

$$x[n] * h[n] = \sum_{m=-\infty}^{+\infty} x[m]\cdot h[n-m] \qquad (*)$$

<u>Some properties of convolutional sum.</u>

1) commutative property $x_1[n] * x_2[n] = x_2[n] * x_1[n]$

2) distributive property $x_1[n] * (x_2[n] + x_3[n]) = x_1[n] * x_2[n] + x_1[n] * x_3[n]$

3) associative property $x_1[n] * (x_2[n] * x_3[n]) = (x_1[n] * x_2[n]) * x_3[n]$

4) the shifting property:

If $x_1[n] * x_2[n] = c[n]$,

then $x_1[n-m] * x_2[n-p] = c[n-m-p]$

5) Convolution with an impulse

$$x[n] * \delta[n] = x[n].$$

Equation $(*)$ is valid in general case. If the system is causal and the input is causal, then the output is calculated as.

$$y[n] = \sum_{m=0}^{n} x[m]\, h[n-m] = \sum_{m=0}^{n} x[n-m]\cdot h[m].$$

Example 3.13. Consider

$$x[n] = 0.8^n u[n]$$
$$y[n] = 0.3^n u[n]$$

Determine $x[n] * y[n]$

$$c[n] = x[n] * y[n] = \sum_{m=-\infty}^{+\infty} x[m] \cdot y[n-m] =$$

$$= \sum_{m=-\infty}^{+\infty} 0.8^m u[m] \cdot 0.3^{n-m} \cdot u[n-m] =$$

$$= \sum_{m=0}^{n} 0.8^m \, 0.3^{n-m} = 0.3^n \sum_{m=0}^{n} (0.8/0.3)^m$$

$$c[n] = 0.3^n \left[1 + \left(\frac{0.8}{0.3}\right) + (0.8/0.3)^2 + \cdots + (0.8/0.3)^n \right]$$

$$c[n] \cdot \left(\frac{0.8}{0.3}\right) = 0.3^n \left[\left(\frac{0.8}{0.3}\right) + (0.8/0.3)^2 + \cdots + (0.8/0.3)^n + (0.8/0.3)^{n+1} \right]$$

$$c[n] \left(1 - 0.8/0.3\right) = 0.3^n \left(1 - (0.8/0.3)^{n+1}\right)$$

$$c[n] = 0.3^n \frac{1 - (0.8/0.3)^{n+1}}{1 - 0.8/0.3} = \frac{0.8^{n+1} - 0.3^{n+1}}{0.5}$$

General results:

$$S_n = 1 + a + a^2 + \cdots + a^n = \frac{1 - a^n}{1 - a}$$

$$S_\infty = 1 + a + a^2 + \cdots + a^n + \cdots = \frac{1}{1-a}, \quad |a| < 1$$

Note: Text book provides a table with some commonly encountered convolutional pairs. Excellent exercise is to validate the entries in the table

Response to complex inputs

In general case $x[n] = x_r[n] + j x_i[n]$

$h[n] = h_r[n] + j h_i[n]$

$$y[n] = x[n] * h[n] = (x_r[n] + j x_i[n]) * (h_r[n] + j h_i[n]) =$$

$$= (x_r[n] * h_r[n] - x_i[n] * h_i[n]) + j(x_i[n] * h_r[n] + x_r[n] * h_i[n]).$$

Problems:

3.7-1	3.8-1
3.7-2	3.8-2
3.7-4	3.8-3
	3.8-4
	3.8-5
	3.8-7