1. Answer any FIVE questions. Precise and complete answers are a must for full credit. Show all your work. Calculators are NOT allowed.

1. State/ Define the following precisely. (5 x 1 = 5)

   (i) Cauchy Sequence: A sequence \( \{a_n\} \) is said to be Cauchy if for every \( \varepsilon > 0 \) there exists a \( N \) such that \( |a_n - a_m| < \varepsilon \) for all \( n, m \geq N \).

   (ii) \( \lim_{n \to \infty} a_n = -\infty \) if for every \( M > 0 \), there exists a \( N \) such that \( a_n < -M \), for all \( n \geq N \).

   (iii) \( \{a_n\} \) converges to \( A \) with rate of convergence \( \mathcal{O}(b_n) \) if and only if \( |a_n - A| \leq K |b_n| \) for sufficiently large values of \( n \) and some positive constant \( K \).

   (iv) \( \liminf_{n \to \infty} a_n \) where \( \{a_n\} \) is a bounded sequence: Let \( T \) be the set of all subsequential limits of \( \{a_n\} \). \( \inf T \) is called the limit inferior of \( \{a_n\} \).

   (v) Bolzano-Weierstrass Theorem for sequences: Every bounded sequence has a convergent subsequence.

2. Let \( \{a_n\} \) be a sequence such that every subsequence of \( \{a_n\} \) converges. Prove that \( \{a_n\} \) converges.

   Solution: Since \( \{a_n\} \) is a subsequence of itself, \( \{a_n\} \) converges! (5 points)

3. Let \( \{a_n\} \) be a Cauchy sequence. Suppose that \( S = \{a_n | n \in \mathbb{N}\} \) is finite. Show that \( \{a_n\} \) is an eventually constant sequence.

   Solution: Let \( S = \{a_1, a_2, \ldots, a_k\} \) and \( \varepsilon = \min |a_i - a_j| \) for \( i, j = 1, \ldots, k, i \neq j \). For this \( \varepsilon \) there exists a \( N \) so that \( |a_n - a_N| < \varepsilon \) for all \( n \geq N \). Since both \( a_n, a_N \in S \) this cannot happen unless \( a_n = a_N \) for all \( n \geq N \). Thus, \( a_n \) is an eventually constant sequence. (5 points)

4. Define the sequence \( \{a_n\} \) recursively by

   \[ a_{n+1} = \frac{1}{2} \left( a_n + \frac{\pi}{a_n} \right), \]

   with \( n \in \mathbb{N} \) and with \( a_1 = 1 \). Prove that \( \{a_n\} \) converges to \( \sqrt{\pi} \).

   Solution: \( a_{n+1} = \frac{1}{2} \left( a_n + \frac{\pi}{a_n} \right) = \frac{a_n^2 + \pi}{2a_n} \geq \sqrt{\pi}, \ n \geq 1 \). So, \( a_n \)'s are bounded below. And, \( a_{n+1} - a_n = \frac{\pi - a_n^2}{2a_n} \leq 0, \ n \geq 2 \). Hence, \( \{a_n, n \geq 2\} \) is monotone decreasing. Thus, \( \{a_n\} \) converges. Let \( \lim_{n \to \infty} a_n = L (\neq 0) \). Then, taking the limit as \( n \to \infty \) in the given relation we get \( L = \frac{L^2 + \pi}{2L} \). Solving for \( L \) we get \( L = \sqrt{\pi} \). (5 points)
5. Suppose that \( \{a_n\} \) converges to a nonzero number and a sequence \( \{b_n\} \) is such that \( \{a_n b_n\} \) converges. Prove that \( b_n \) must converge.

Solution: Let \( \lim_{n \to \infty} a_n = A \), \( \lim_{n \to \infty} a_n b_n = C \). Claim that \( b_n \) converges to \( C/A \). Consider,

\[
| b_n - \frac{C}{A} | = \left| \frac{a_n b_n - C}{a_n} \right|
= \left| \frac{(a_n b_n)A - a_n C}{a_n A} \right|
= \left| \frac{(a_n b_n)A - CA + CA - a_n C}{a_n A} \right|
\leq \left| \frac{(a_n b_n) - C}{a_n} \right| + \left| \frac{C}{a_n} \right| \left| a_n - A \right|
\]

There exists a \( n_1 \) such that for all \( n \geq n_1 \), \( |a_n| > \frac{|A|}{2} \) use this and make the terms \( |(a_n b_n) - C| \) and \( |a_n - A| \) suitably small to conclude that \( |b_n - \frac{C}{A}| < \varepsilon \). \( \quad \) (5 points)

6. Bonus Question: Find \( \lim \sup_{n \to \infty} r^n \) if (i) \( 0 < r < 1 \) (ii) \( r = 1 \) (iii) \( r > 1 \)

Solution:
(i) If \( 0 < r < 1 \) then \( a_n = r^n \to 0 \). So \( \lim \sup_{n \to \infty} r^n = 0 \).
(ii) If \( r = 1 \), then \( a_n = r^n \to 1 \). So \( \lim \sup_{n \to \infty} r^n = 1 \).
(iii) If \( r > 1 \) then \( \lim \sup_{n \to \infty} r^n \) doesn’t exist.