A solution of the Bessel equation of order zero, that is \( Y_0(\infty) \) from \( Y_0(x) \) is given by:

\[
Y_0(x) = \frac{2}{\pi} \left[ (\ln x^2 + 3) J_0(x) + \frac{x^2}{2^2} - \frac{x^4}{2^4 2^2} (1 + \frac{1}{2}) \right.
\]

\[
+ \frac{x^6}{2^6 4^2 2^2} (1 + \frac{1}{2} + \frac{1}{3}) - \ldots \left. \right] \]

\( Y_0(x) \) is known as Weber's Bessel function of second kind of order zero.

\[ Y = 0.5772 \ldots \text{ Eul:er's constant} \]

Note that, \( Y_0(x) \) is unbounded as \( x \to 0^+ \).

A general solution of the Bessel equation of order zero is given by \( Y(x) = A J_0(x) + B Y_0(x) \). Thus \( A \) and \( B \) are arbitrary constants. Any solution for which \( B \neq 0 \) is unbounded as \( x \to 0^+ \).

Recurrence relations

\[
\frac{d}{dx} \left( x^n J_n(x) \right) = -n x^n J_{n+1}(x) \]

We know \( J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k! (n+k)!} \), \( n = 0, 1, 2, \ldots \)

So,

\[
\frac{d}{dx} \left( x^n J_n(x) \right) = \frac{d}{dx} \left[ \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k}}{k! (n+k)!} \frac{x^{2k}}{2^{n+2k}} \right]
\]

\[
= \sum_{k=1}^{\infty} \frac{(-1)^k}{k! (n+k)!} \frac{k x^{2k-1}}{2^{n+2k-1}}
\]

\[
= \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)! (n+k)!} \frac{x^{2k-1}}{2^{n+2k-1}}
\]

\[
= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k! (n+k+1)!} \frac{x^{2k+1}}{2^{n+2k+1}}
\]
\[= -x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (n+1+k)} \frac{x^{n+1+2k}}{2^{n+1+2k}}\]

\[= -x^n J_{n+1}(x).\]

\[
\frac{d}{dx} (x^n J_n(x)) = -x^n J_{n+1}(x), \quad n = 0, 1, 2, \ldots
\]

For \( n = 0 \), we have \( J_0'(x) = -J_1(x) \).

(2) \[
\frac{d}{dx} (x^n J_n(x)) = x^n J_{n-1}(x), \quad n = 1, 2, \ldots
\]

Integrating (2), we get \[
\int_0^x s^n J'(s) ds = x^n J_n(x), \quad n = 1, 2, \ldots
\]

For \( n = 1 \), \[
\int_0^x s J_0(s) ds = x J_1(x)
\]

Reduction Formula

(4) \[
\int_0^x s^n J_0(s) ds = x^n J_1(x) + (n-1) x^{n-1} J_0(x) - (n-1)^2 \int_0^x s^{n-2} J_0(s) ds.
\]

(5) \[x \bar{J}_n(x) = n \bar{J}_n(x) - x \bar{J}_{n+1}(x), \quad n = 0, 1, 2, \ldots\]

(6) \[x \bar{J}_n(x) = -n \bar{J}_n(x) + x \bar{J}_{n-1}(x), \quad n = 1, 2, \ldots\]

(7) \[x \bar{J}_{n+1}(x) = 2n \bar{J}_n(x) - x \bar{J}_{n-1}(x), \quad n = 1, 2, \ldots\]
The Zeros of \( J_0(x) \)

Consider \[ x^{-2} y'' + y'(x) + x y(x) = 0. \] Bessel Equation of order zero.

Let \[ y = x^{1/2} u(x); \quad y'(x) = x^{-1/2} u'(x) - \frac{1}{2} x^{-3/2} u(x) \]

\[ y''(x) = x^{-1/2} u''(x) - \frac{1}{4} x^{-3/2} u'(x) + \frac{3}{4} x^{-5/2} u(x) \]

\[ x y'' + y'(x) + x y(x) \]

\[ = x^{1/2} u'' - x^{-1/2} u'(x) + \frac{3}{4} x^{-3/2} u'(x) \]

\[ - \frac{1}{4} x^{-3/2} u(x) + x^{1/2} u(x) = 0. \]

\[ \Rightarrow x^{1/2} u'' + \frac{1}{4} x^{-3/2} u'(x) + x^{1/2} u(x) = 0. \]

\[ \Rightarrow x^2 u'' + (x^2 + \frac{1}{4}) u(x) = 0. \]

Clearly, \( u(x) = \sqrt{x} J_0(x) \) is a solution of the above equation.

**Lemma:** The positive zeros of the function \( J_0(x) \), or the positive roots of the equation \( J_0(x) = 0 \), form an increasing sequence of numbers \( x_j \) \((j = 1, 2, \ldots)\) such that \( x_j \to \infty \) as \( j \to \infty \).

**Proof:** Discussed in the lecture.

\[ J_0(x_j) = 0. \]

\[ x_1 = 2.405, \quad x_2 = 5.520, \quad x_3 = 8.654, \quad x_4 = 11.79 \]

**Theorem:** Let \( n \) be any fixed nonnegative integer, \( n = 0, 1, 2, \ldots \)

The positive zeros of \( J_n(x) \) or positive roots of \( J_n(x) = 0 \)

form an increasing sequence of numbers \( x_j \) \((j = 1, 2, \ldots)\) such that \( x_j \to \infty \), as \( j \to \infty \).