Let \( n \) denote any fixed nonnegative integer. Consider the DE \( x^2 y'' + xy' + (x^2 - n^2) y = 0 \), \( n = 0, 1, 2, \ldots \).

Writing in the standard form on \((0, \infty)\),

\[ y'' + \frac{1}{x} y' + \left(\frac{x^2 - n^2}{x^2}\right) y = 0. \]

\( p(x) = \frac{1}{x}, \quad q(x) = \frac{x^2 - n^2}{x^2} \) do not have convergent power series expansions in a neighborhood of \( x = 0\).

However, \( x p(x) = 1, \quad x^2 q(x) = x^2 - n^2 \) have such a \( x = 0 \), is said to be a **regular singular point** for the DE. In this case, we seek a solution of the form

\[ y(x) = x^c \sum_{j=0}^{\infty} a_j x^j = \sum_{j=0}^{\infty} a_j x^{c+j}, \quad \text{for the DE}. \]

Differentiating, we get

\[ y'(x) = \sum_{j=0}^{\infty} (c+j) a_j x^{c+j-1}, \quad y''(x) = \sum_{j=0}^{\infty} (c+j)(c+j-1) a_j x^{c+j-2} \]

Substituting in the DE,

\[ \sum_{j=0}^{\infty} (c+j)(c+j-1) a_j x^{c+j} + \sum_{j=0}^{\infty} (c+j) a_j x^{c+j} + \sum_{j=0}^{\infty} a_j x^{c+j+2} - \sum_{j=0}^{\infty} n^2 a_j x^{c+j} = 0 \]

\[ \Rightarrow \sum_{j=0}^{\infty} \left[(c+j)^2 - n^2\right] a_j x^{c+j} + \sum_{j=2}^{\infty} a_{j-2} x^{c+j} = 0. \]

Multiplying throughout by \( x^c \), we get

\[ (c^2 - n^2) a_0 + \left[(c+1)^2 - n^2\right] a_1 x + \sum_{j=2}^{\infty} \left[\left[(c+j)^2 - n^2\right] a_j + a_{j-2} x^{c+j+2}\right] = 0 \]

\[ \Rightarrow \text{coefficients of each power of } x \text{ vanishes.} \]

Assuming \( a_0 \neq 0 \), we get
\[ C^2 - n^2 = 0 \]

\[ \Rightarrow c = \pm n \]

Choose \( c = n \). Then, \( q_n = 0 \).

And

\[ \left( (n+1)^2 - n^2 \right) a_j + a_{j-2} = 0, \quad j = 2, 3, \ldots \]

\[ \Rightarrow a_j = -\frac{1}{j(2n+j)} a_{j-2}, \quad j = 2, 3, \ldots \]

Recurrence relation

Remark: If \( c = -n \), then, the above solution is not well defined for all \( j \).

Since \( a_1 = 0 \), we must have \( a_3 = a_5 = \ldots = a_{2k+1} = 0 \)

So, for \( j = 2k \), \( k = 1, 2, \ldots \), we have

\[ a_{2k} = -\frac{1}{k(2n+k)} a_{2k-2} \]

Equate the product of the LHS to the product on the RHS and cancelling common terms, we get

\[ a_{2k} = \frac{(-1)^k}{k!(n+1) \ldots (n+k)} 2^k a_0, \quad k = 1, 2, \ldots \]

Thus:

\[ g(x) = a_0 x^n + \sum_{k=1}^{\infty} a_{2k} x^{n+2k} \]

is a solution of the DE.

This series converges absolutely for all \( x \) if

\[ \lim_{k \to \infty} \left| \frac{a_{2(k+1)} x^{n+2(k+1)}}{a_{2k} x^{n+2k}} \right| = \lim_{k \to \infty} \frac{1}{(k+1)(n+k+1)} \left( \frac{|x|^2}{2} \right)^2 = 0. \]
The series represent a continuous function and is differentiable at most \( n \), any number of times.

\[
y(x) = a_0 \, x^n + a_0 \sum_{k=1}^{\infty} \frac{(-1)^k}{k! \, (n-1)! \cdots (n+k)!} \frac{1}{2^{2k}} \, x^{n+2k}
\]

Choose \( a_0 = \frac{1}{n! \, 2^n} \)

\[
J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \, (n+k)!} \left( \frac{x}{2} \right)^{n+2k}
\]

Bessel function of first kind of order \( n \).

\[
J_n(-x) = (-1)^n \, J_n(x)
\]

**Remark:**

**Special case** \( n = 0 \).

\[
x^2 y'' + x y' + x^2 y = 0
\]

\[
\Rightarrow \quad x y'' + y' + x y = 0
\]

\[
J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \, (k+1)!} \left( \frac{x}{2} \right)^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k \, x^{2k}}{(k!)^2 \, 2^{2k}}
\]

\[
(k!)^2 \, 2^{2k} = 2^2 \, 4^2 \, 6^2 \cdots (2k)^2
\]

\[
J_0(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k \, x^{2k}}{2^2 \, 4^2 \, 6^2 \cdots (2k)^2}
\]

Similarly,

\[
J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \, x^{2k+1}}{k! \, (k+1)! \, 2^{2k+1}}
\]

\[
= \frac{x}{2} - \frac{x^3}{2^3 \, 1! \, 2!} + \frac{x^5}{2^5 \, 2! \, 3!} - \cdots
\]