Theory of Set Differential Equations in Metric Spaces

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Preface

The study of analysis in metric spaces has gained importance in recent times. It is realized that many results of differential calculus and set valued analysis, including the inverse function theorem do not really rely upon the linear structure and therefore can be adapted to the nonlinear case of metric spaces and exploited. Moreover, the concept of the differential equation governing evolution in metric spaces has been suitably formulated.

Multivalued differential equations (now known as set differential equations (SDEs)) generated by multivalued differential inclusions have been introduced in a semi-linear metric space, consisting of all nonempty, compact, convex subsets of an initial finite or infinite dimensional space. The basic existence and uniqueness results of such SDEs have been investigated and their solutions have compact, convex values. Also, these generated SDEs have been employed as a tool to prove the existence of solutions, in a unified way, of multivalued differential inclusions. The multifunctions involved in this set up are compact, but not necessarily convex, subsets of the base space utilized.

Because of the fact that fuzzy set theory and its applications have been extensively investigated, due to the increase of industrial interest in fuzzy control, the initiation of the theory of fuzzy differential equations (FDEs) in an appropriate metric space has recently been accomplished. In view of the inherent disadvantage resulting from the fuzzification of the derivative employed in the original formulation of FDEs, an alternative formulation based upon a family of multivalued differential inclusions derived from the fuzzy maps involved in the FDEs, is recently suggested to reflect the rich behaviour of the corresponding ordinary differential equation before fuzzification.

The investigation of the theory of SDEs as an independent discipline, has certain advantages. For example, when the set is a single valued mapping, it is clear that the Hukuhara derivative and the integral utilized in formulating the SDEs reduce to the ordinary vector derivative and the integral, and therefore the results obtained in the framework of SDEs become the corresponding results of ordinary differential systems if the base space is $\mathbb{R}^n$. On the other hand, if the base space is a Banach space, we get from the corresponding SDE's the differential equations in a Banach space. Moreover, one has only a semilinear metric space to work with in the SDE set up, compared to the complete normed linear space that one employs in the usual study of an ordinary differential system. As indicated earlier,the SDEs that are generated by multivalued differential inclusions when the needed convexity is missing, form a natural vehicle for proving the existence results for multivalued differential inclusions. Also, one can utilize SDEs profitably to investigate FDEs. Consequently, the study of the theory of SDEs has recently been growing very rapidly and is still in the initial stages. Nonetheless, there exists sufficient literature to warrant assembling the existing fundamental results in a unified way to understand and appreciate the intricacies and advantages involved, so as to pave the way for further advancement of this important branch of differential equations as an independent subject area. It is with this spirit we see the importance of the present monograph. As a result, we provide a systematic account of recent development, describe the current state of the useful theory, show the essential unity achieved and initiate several new extensions to other types of SDEs.

In Chapter 1, we assemble the preliminary material providing the necessary tools including the calculus for set valued maps relevant to the later development. Chapter 2 is devoted to the investigation of the fundamental theory of SDEs such as various comparison principles, existence and uniqueness, continuous dependence, existence of extremal solutions suitably introducing a partial order in the metric space, monotone iterative technique using lower and upper solutions and global existence under the continuity assumption for SDEs. We also discuss, utilizing the method of nonsmooth analysis, existence and flow invariance results without any continuity assumption, in terms of Euler solutions. Finally, we consider the case of upper semicontinuity in the framework of Caratheodory and prove an existence result in a general set up.

In Chapter 3, we extend Lyapunov stability theory to SDEs, employing Lyapunov-like functions, proving first suitable comparison results in terms of such functions. The stability and boundedness criteria are obtained by choosing appropriate initial values in terms of the Hukuhara difference to eliminate the undesirable part of the solutions of SDEs, so that the rich behaviour of the corresponding ODEs, from which SDEs are generated, is preserved. The methods of vector Lyapunov-like functions and the perturbing Lyapunov-like functions are discussed in detail. Also, employing lower semicontinuous Lyapunov-like functions and utilizing nonsmooth analysis, stability results are described under weaker assumptions.

Chapter 4 deals with the interconnection between SDEs and fuzzy differential equations(FDEs). For this purpose, necessary tools are provided for formulating FDEs, and basic results are proved, including the stability theory of Lyapunov. Then the interconnection between FDEs and SDEs is explored via a sequence of multivalued differential inclusions, suitably generating SDEs as described earlier. The impulsive effects are then incorporated in FDEs and then it is shown how impulses can help to improve the qualitative behaviour of solutions of FDEs. Hybrid fuzzy differential equations are introduced and their stability properties are discussed. Another concept of differential equations in metric spaces is considered which can be applied to the study of FDEs.

Chapter 5 is devoted to initiate several topics in the setup of SDEs such as impulsive SDEs, SDEs with time delay, set difference equations, and SDEs involving causal maps, which cover several types of SDEs including integro-differential equations. Some important basic results are provided for each type of SDEs. We then introduce Lyapunov-like functions whose values are in some metric space, prove suitable comparison results and study stability theory in this general set up. This study includes the methods of single, vector, matrix and cone-valued Lyapunov-like functions by an appropriate choice of the metric space. Since the basic space utilized to define the metric space $(K_c(\mathbb{R}^n), D)$ is restricted, for convenience of understanding, to $\mathbb{R}^n$, we indicate how one can extend most of the results described when we choose a Banach space E instead of $\mathbb{R}^n$, so that we have the corresponding metric space $(K_c(E), D)$ to work with. Finally, notes and comments are provided for each chapter.

Some of the important features of the monograph as follows:

1. It is the first book that attempts to describe the theory of set differential equations as an independent discipline.;

2. It incorporates, the recent general theory of set differential equations, discusses the interconnections between set differential equations and fuzzy differential equations and uses both smooth and nonsmooth analysis for investigation.

3. It exhibits several new areas of study by providing the initial apparatus for further advancement.

4. It is a timely introduction to a subject that follows the present trend of studying analysis and differential equations in metric spaces.

This monograph will be very useful to those experts and their doctoral students who work in Nonlinear Analysis, in general. It will also be a good reference book to Engineering and Computer Scientists since it also covers fuzzy dynamics as a subset.

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Chapter 1

Preliminaries

1.1 Introduction

Recently the study of set differential equations (SDEs) was initiated in a metric space and some basic results of interest were obtained. The investigation of set differential equations as an independent subject has some advantages. For example, when the set is a single valued mapping, it is easy to see that the Hukuhara derivative, and the integral utilized in formulating the SDEs reduce to the ordinary vector derivative and the integral and therefore, the results obtained in this new framework become the corresponding results in ordinary differential systems. Also, we have only a semilinear complete metric space to work with in the present setup, compared to the normed linear space that one employs in the usual study of ordinary differential systems.

Furthermore, SDEs that are generated by multivalued differential inclusions, when the multivalued functions involved do not possess convex values, can be used as a tool for studying multivalued differential inclusions. Moreover, one can utilize SDEs indirectly to investigate profitably fuzzy differential equations, since the original formulation of fuzzy differential equations suffers from grave disadvantage and does not reflect the rich behavior of corresponding differential equations without fuzziness. This is due to the fact that the diameter of any solution of a fuzzy differential equation increases as time increases because of the necessity of the fuzzification of the derivative involved.

In order to formulate the set differential equations in a metric space, we need some background material, since the metric space involved consists of all nonempty compact, convex sets in finite or infinite dimensional space. In Section 1.2, we define the necessary ingredients of such sets restricting ourselves to the Euclidean n-space $\mathbb{R}^n$. Since the difference of any two sets in $K_c(\mathbb{R}^n)$ (set of all nonempty, compact, convex sets in $\mathbb{R}^n$) is not defined in general, conditions for the existence of the difference are provided in this section. Section 1.3 introduces the Hausdorff metric $D[\cdot,\cdot]$ for $K_c(\mathbb{R}^n)$ and lists its properties. Support functions are defined in Section 1.4, where they are utilized to create a mapping that makes it possible to embed the metric space $(K_c(\mathbb{R}^n), D)$ into a complete cone in a specified Banach space.

In Section 1.5, the continuity and measurability properties of mappings into the metric space are dealt with. Section 1.6 investigates the concept of differentiation of such mappings and its behavior. In Section 1.7, we consider the theory of integration of these mappings and the needed properties. Section 1.8 summarizes the corresponding situation when the elements of the metric space considered are from a Banach space. Notes and comments are listed in Section 1.9.

1.2 Compact Convex Subsets of $\mathbb{R}^n$

We shall consider the following three spaces of nonempty subsets of $\mathbb{R}^n$, namely,

(i) $K_c(\mathbb{R}^n)$ consisting of all nonempty compact convex subsets of $\mathbb{R}^n$;

(ii) $K(\mathbb{R}^n)$ consisting of all nonempty compact subsets of $\mathbb{R}^n$;

(iii) $C(\mathbb{R}^n)$ consisting of all nonempty closed subsets of $\mathbb{R}^n$.

Recall that a nonempty subset A of $\mathbb{R}^n$ is convex if for all $a_1,\ a_2 \in A$ and all $\lambda \in [0,1]$, the point

$$a = \lambda a_1 + (1-\lambda)a_2 \tag{1.2.1}$$

belongs to A. For any nonempty subset A of $\mathbb{R}^n$, we denote by co$A$ its *convex hull,* that is the totality of points a of the form (1.2.1) or, equivalently, the smallest convex subset containing A. Clearly

$$A \subseteq co\ A = co(co\ A) \tag{1.2.2}$$

with $A = coA$ if A is convex. Moreover, co$A$ is closed (compact) if A is closed (compact).

Let A and B be two nonempty subsets of $\mathbb{R}^n$ and let $\lambda \in I\!R$. We define the following Minkowski addition and scalar multiplication by

$$A + B = \{a + b : a \in A,\ b \in B\} \tag{1.2.3}$$

and

$$\lambda A = \{\lambda a :\ a \in A\}. \tag{1.2.4}$$

Then we have the following proposition.

Proposition 1.2.1 *The spaces $C(\mathbb{R}^n)$, $K(\mathbb{R}^n)$ and $K_c(\mathbb{R}^n)$ are closed under the operations of addition and scalar multiplication. In fact, the following properties hold:*

(i) $A+\theta = \theta+A = A$, $\theta \in \mathbb{R}^n$, *is the zero element of* $\mathbb{R}^n$, *treated as a singleton.*

(ii) $(A + B) + C = A + (B + C)$

(iii) $A + B = B + A$

(iv) $A + C = B + C$ *implies* $A = B$

(v) $1 \cdot A = A$

(vi) $\lambda(A + B) = \lambda A + \lambda B$

(vii) $(\lambda + \mu)A = \lambda A + \mu A$

where $A, B, C \in K_c(\mathbb{R}^n)$, λ, $\mu \in \mathbb{R}_+$.

Proof We only give the proof of (iv), the rest being simple to prove.

Let A, B, $C \in K_c(\mathbb{R}^n)$. We show that $A \neq B$ implies $A + C \neq B + C$. Suppose, for example, that there exists a point $a \in A$ which does not belong to B. Through a pass hyperplanes which are disjoint from B. Let one of these hyperplanes be P. Let P' be the support hyperplane of C, which is parallel to P and such that, if we move P' parallel to itself onto P, C moves on a compact convex set which is located on the same side of P as B. If c is a point of $C \cap P'$, then $a + c \notin B + C$. Hence the proof.

In general, $A + (-A) \neq \{\theta\}$. This fact is illustrated by the following example.

Example 1.2.1 *Let* $A = [0, 1]$ *so that* $(-1)A = [-1, 0]$, *and therefore*

$$A + (-1)A = [0, 1] + [-1, 0] = [-1, 1].$$

Thus, adding (-1) *times a set does not constitute a natural operation of subtraction.*

This leads us to the following definition.

Definition 1.2.1 *For a fixed* A *and* B *in* $K_c(\mathbb{R}^n)$ *if there exists an element* $C \in K_c(\mathbb{R}^n)$ *such that* $A = B + C$ *then we say that the Hukuhara Difference of* A *and* B *exists and is denoted by* $A - B$.

When the Hukuhara difference exists it is unique. This follows from (iv) of Proposition 1.2.1.

The following example explains the above definition.

Example 1.2.2 *From Example 1.2.1, we get*

$$[-1, 1] - [-1, 0] = [0, 1] \ \textit{and} \ [-1, 1] - [0, 1] = [-1, 0].$$

Note that the Hukuhara difference $A - B$ is different from the set

$$A + (-B) = \{a + (-b) : \ a \in A, \ b \in B\}.$$

The next proposition provides the necessary and sufficient condition for the existence of the Hukuhara difference $A - B$.

Proposition 1.2.2 *Let $A, B \in K_c(\mathbb{R}^n)$. For the difference $A - B$ to exist, it is necessary and sufficient to have the following condition. If $a \in \partial A$, there exists at least a point c such that*

$$a \in B + c \subset A. \tag{1.2.5}$$

Proof Necessity: Suppose the difference $A - B$ exists. Let $C = A - B$. Then, $A = B + C$. If $a \in \partial A$, $a \in B + C$, that is, $a = b + c$ where $b \in B$ and $c \in C$. Also, if $z \in B$, then $z + c \in A$ and therefore (1.2.5) is satisfied.

Sufficiency: Suppose (1.2.5) holds. Consider the set $C = \{x : B + x \subseteq A\}$. Clearly C is compact and we have $B + C \subseteq A$. Now, if d and $d' \in C$, then we have $B + d \subseteq A$ and $B + d' \subseteq A$, from which we obtain

$$(1 - \lambda)(B + d) + \lambda(B + d') \subset A, \text{ for } 0 \leq \lambda \leq 1. \tag{1.2.6}$$

We can write the L.H.S of (1.2.6) as $B + z$ with $z = (1-\lambda)d + \lambda d'$. Hence $z \in C$ and C is convex.

Let $u \in A$. A straight line through u meets ∂A at two points a and a'. By hypothesis there exist elements d and d' in C such that $a \in B + d$, and $a' \in B + d'$. We can write $u = (1 - \lambda)a + \lambda a'$ with $0 < \lambda < 1$. Then $u \in B + x$, where $x = (1 - \lambda)d + \lambda d' \in C$. Hence $A \subseteq B + C$. Thus $A = B + C$ and the proof is complete.

We note that a necessary condition for the Hukuhara difference $A - B$ to exist is that some translate of B is a subset of A. However, in general, the Hukuhara difference need not exist as is seen from the following example.

Example 1.2.3 $\{0\} - [0, 1]$ *does not exist, since no translate of [0,1] can ever belong to the singleton set* $\{0\}$.

1.3 The Hausdorff Metric

Let x be a point in $\mathbb{R}^n$ and A a nonempty subset of $\mathbb{R}^n$. The distance $d(x, A)$ from x to A is defined by

$$d(x, A) = \inf\{\|x - a\| : a \in A\}. \tag{1.3.1}$$

Thus $d(x, A) = d(x, \bar{A}) \geq 0$ and $d(x, A) = 0$ if and only if $x \in \bar{A}$, the closure of $A \subseteq \mathbb{R}^n$.

We shall call the subset

$$S_\epsilon(A) = \{x \in \mathbb{R}^n : d(x, A) < \epsilon\} \tag{1.3.2}$$

an $\epsilon-$neighborhood of A. Its closure is the subset

$$\bar{S}_\epsilon(A) = \{x \in \mathbb{R}^n : d(x, A) \leq \epsilon\}. \tag{1.3.3}$$

In particular, we shall denote by

$$\bar{S}_1^n = \bar{S}_1(\theta), \tag{1.3.4}$$

which is obviously a compact subset of $\mathbb{R}^n$. Note also that

$$\bar{S}_\epsilon(A) = A + \epsilon \bar{S}_1^n, \tag{1.3.5}$$

for any $\epsilon > 0$ and any nonempty subset A of $\mathbb{R}^n$. We shall for convenience sometimes write $S(A, \epsilon)$ and $\bar{S}_\epsilon(A)$.

Now, let A and B be nonempty subsets of $\mathbb{R}^n$. We define the Hausdorff separation of B from A by

$$d_H(B, A) = \sup\{d(b, A) : b \in B\} \tag{1.3.6}$$

or, equivalently

$$d_H(B, A) = \inf\{\epsilon > 0 : B \subseteq A + \epsilon \bar{S}_1^n\}.$$

We have $d_H(B, A) \geq 0$ with $d_H(B, A) = 0$ if and only if $B \subseteq \bar{A}$. Also, the triangle inequality

$$d_H(B, A) \leq d_H(B, C) + d_H(C, A),$$

holds for all nonempty subsets A, B and C of $\mathbb{R}^n$. In general, however

$$d_H(A, B) \neq d_H(B, A).$$

We define the *Hausdorff* distance between nonempty subsets A and B of $\mathbb{R}^n$ by

$$D(A, B) = \max\{d_H(A, B), d_H(B, A)\}, \tag{1.3.7}$$

which is symmetric in A and B. Consequently,

(a) $D(A, B) \geq 0$ with $D(A, B) = 0$ if and only if $\bar{A} = \bar{B}$;
(b) $D(A, B) = D(B, A)$;
(c) $D(A, B) \leq D(A, C) + D(C, B)$, (1.3.8)

for any nonempty subsets A, B and C of $\mathbb{R}^n$.

If we restrict our attention to nonempty closed subsets of $\mathbb{R}^n$, we find that the Hausdorff distance (1.3.7) is a metric known as the *Hausdorff metric.* Thus $(C(\mathbb{R}^n), D)$ is a metric space.

In fact, we have

Proposition 1.3.1 *$(C(\mathbb{R}^n), D)$ is a complete separable metric space in which $K(\mathbb{R}^n)$ and $K_c(\mathbb{R}^n)$ are closed subsets. Hence, $(K(\mathbb{R}^n), D)$ and $(K_c(\mathbb{R}^n), D)$ are also complete separable metric spaces.*

The following properties of the Hausdorff metric will be useful later.

We start by stating a proposition dealing with the invariance of the Hausdorff metric.

Proposition 1.3.2 *If A, $B \in K_c(\mathbb{R}^n)$ and $C \in K(\mathbb{R}^n)$ then,*

$$D(A + C, B + C) = D(A, B). \tag{1.3.9}$$

We need the following result which deals with the law of cancellation to proceed further.

Lemma 1.3.1 *Let* $A,\ B \in K_c(\mathbb{R}^n)$ *and* $C \in K(\mathbb{R}^n)$ *and* $A + C \subseteq B + C$, *then* $A \subseteq B$.

Proof Let a be any element of A. We need to show that $a \in B$. Given any $c_1 \in C$, we have $a + c_1 \in B + C$, that is, there exist $b_1 \in B$ and $c_2 \in C$ with $a+c_1 = b_1+c_2$. For the same reason, $b_2 \in B$ and $c_3 \in C$ with $a+c_2 = b_2+c_3$ must exist. Repeat the procedure indefinitely and sum the first n of the equations obtained. We get

$$na + \sum_{i=1}^{n} c_i = \sum_{i=1}^{n} b_i + \sum_{i=2}^{n+1} c_i$$

which implies

$$na + c_1 = \sum_{i=1}^{n} b_i + c_{n+1}.$$

Then,

$$a = \frac{1}{n}\sum_{i=1}^{n} b_i + \frac{c_{n+1}}{n} - \frac{c_1}{n}.$$

Set $x_n = \frac{1}{n}\sum_{i=1}^{n} b_i$. Thus

$$a = x_n + \frac{c_{n+1}}{n} - \frac{c_1}{n}.$$

We observe that $x_n \in B$ for all n, because B is convex and $\frac{c_{n+1}}{n} - \frac{c_1}{n} \to 0$ as C is compact. Thus x_n converges to a. But since B is compact, $a \in B$. Thus, if $A + C = B + C$ then $A = B$. This completes the proof of the lemma.

Proof of Proposition 1.3.2. Let $\lambda \geq 0$ and S denote the closed unit sphere of the space. Consider the following inequalities

(1) $A + \lambda S \supset B$,

(2) $B + \lambda S \supset A$,

(3) $A + C + \lambda S \supset B + C$,

(4)$B + C + \lambda S \supset A + C$.

Put $d_1 = D(A, B)$ and $d_2 = D(A + C, B + C)$. Then d_1 is the infimum of the positive numbers λ for which (1) and (2) hold. Similarly, d_2 is the infimum of the positive numbers λ for which (3) and (4) hold. Since (3) and (4) follow from (1) and (2) respectively, by adding C, we have $d_1 \geq d_2$. Conversely, since by Lemma 1.3.1, canceling C is allowed in (3) and (4), we obtain $d_1 \leq d_2$, which proves the proposition.

Proposition 1.3.3 *If* $A, B \in K(\mathbb{R}^n)$

$$D(co\ A, co\ B) \leq D(A, B). \tag{1.3.10}$$

If $A,\ A',\ B,\ B' \in K_c(\mathbb{R}^n)$ *then*

$$D(tA, tB) = tD(A, B) \text{ for all } t \geq 0, \tag{1.3.11}$$

$$D(A + A', B + B') \leq D(A, B) + D(A', B'), \tag{1.3.12}$$

Further,

$$D(A - A', B - B') \leq D(A, B) + D(A', B'), \tag{1.3.13}$$

provided the differences $A - A'$ *and* $B - B'$ *exist. Moreover with* $\beta = \max\{\lambda,\ \mu\}$, *we have*

$$D(\lambda A, \mu B) \leq \beta D(A, B) + |\lambda - \mu|[D(A, \theta) + D(B, \theta)] \tag{1.3.14}$$

and

$$D(\lambda A, \lambda B) = \lambda D(A - B, \theta),\ \text{if } A - B \text{ exists.} \tag{1.3.15}$$

Proof Since (1.3.10) is obvious, we begin with the proof of (1.3.11). For all $a \in A$ and $u \in A'$, compactness of B and B' ensures that there exist $b(a) \in B$ and $v(u) \in B'$ such that

$$\inf_{b \in B} |a - b| = |a - b(a)|; \quad \inf_{v \in B'} |u - v| = |u - v(u)|. \tag{1.3.16}$$

From the relation

$$|a + u - b(a) - v(u)| \leq |a - b(a)| + |u - v(u)|$$

and (1.3.16), it follows that

$$\sup_{a \in A,\, u \in A'} \inf_{b \in B,\, v \in B'} |a + u - b - v| \leq \sup_{a \in A} \inf_{b \in B} |a - b| + \sup_{u \in A'} \inf_{v \in B'} |u - v|.$$

From the above and the analogous inequality obtained by interchanging the roles of A with B and A' with B', we obtain (1.3.11).

We now prove (1.3.13).

Using Proposition 1.3.2, we find that

$$\begin{aligned} D(A - A', B - B') &= D(A - A' + A' + B',\ B - B' + B' + A') \\ &= D(A + B',\ B + A') \\ &\leq D(A, B) + D(A', B'), \end{aligned}$$

which follows from (1.3.11).

To prove (1.3.14), consider, for $\lambda - \mu \geq 0$,

$$D(\lambda A, \mu B) \leq \mu D(A, B) + (\lambda - \mu) D(A, \theta),$$

and if $\lambda - \mu \leq 0$, then

$$D(\lambda A, \mu B) \leq \lambda D(A, B) + (\mu - \lambda) D(B, \theta).$$

The relations above put together prove (1.3.14).

The proof of (1.3.15) follows from Proposition 1.3.2.

Next, we define the magnitude of a nonempty subset of A of $\mathbb{R}^n$ by

$$\|A\| = \sup\{\|a\| : a \in A\}, \tag{1.3.17}$$

or equivalently,

$$\|A\| = D(\theta, A). \tag{1.3.18}$$

Here, $\|A\|$ is finite, and the supremum in (1.3.17) is attained when $A \in K(\mathbb{R}^n)$.

From (1.3.10) it obviously follows that

$$\|tA\| = t\|A\|, \quad \text{for all} \quad t \geq 0. \tag{1.3.19}$$

Moreover, (1.3.8) and (1.3.18) yield

$$|\|A\| - \|B\|| \leq D(A, B), \tag{1.3.20}$$

for all $A, B \in K(\mathbb{R}^n)$.

We say that a subset $\mathcal{U} \in K(\mathbb{R}^n)$ or $K_c(\mathbb{R}^n)$ is uniformly bounded if there exists a finite constant $c(\mathcal{U})$ such that

$$\|A\| \leq c(\mathcal{U}), \quad \text{for all} \quad A \in \mathcal{U}. \tag{1.3.21}$$

We then have the following simple characterization of compactness.

Proposition 1.3.4 *A nonempty subset $\mathcal{A}$ of the metric space $(K(\mathbb{R}^n), D)$ or $(K_c(\mathbb{R}^n), D)$, is compact if and only if it is closed and uniformly bounded.*

Set inclusion induces partial ordering on $K(\mathbb{R}^n)$. Write $A \leq B$ if and only if $A \subseteq B$, where $A, B \in K(\mathbb{R}^n)$. Then

$$\mathcal{L}(B) = \{A \in K(\mathbb{R}^n) : B \leq A\}, \quad \mathcal{U}(B) = \{A \in K(\mathbb{R}^n) : A \leq B\}, \tag{1.3.22}$$

are closed subsets of $K(\mathbb{R}^n)$ for any $B \in K(\mathbb{R}^n)$. In fact, from Proposition 1.3.4, $\mathcal{U}(B)$ is compact subset of $K(\mathbb{R}^n)$.

Proposition 1.3.5 *$\mathcal{U}(B)$ is a compact subset of $K(\mathbb{R}^n)$.*

This assertion remains true with $K_c(\mathbb{R}^n)$ replacing $K(\mathbb{R}^n)$ everywhere.

Sequences of nested subsets in $(K(\mathbb{R}^n), D)$ have the following useful intersection and convergence properties.

Proposition 1.3.6 *Let $\{A_j\} \subset K(\mathbb{R}^n)$ satisfy*

$$\cdots \subseteq A_j \subseteq \cdots A_2 \subseteq A_1.$$

Then $A = \cap_{j=1}^{\infty} A_j \in K(\mathbb{R}^n)$ and

$$D(A_n, A) \to 0 \text{ as } n \to \infty. \tag{1.3.23}$$

On the other hand, if $A_1 \subseteq A_2 \subseteq \cdots A_j \subseteq \cdots$ and $A = \cup_{j=1}^{\infty} A_j \in K(\mathbb{R}^n)$, then (1.3.23) holds.

1.4 Support Functions

Let A be a nonempty subset of $\mathbb{R}^n$. The support function of A is defined for all $p \in \mathbb{R}^n$ by

$$s(p, A) = \sup\{< p, a >: a \in A\}, \tag{1.4.1}$$

which may take the value $+\infty$ when A is unbounded. However, when A is a compact, convex subset of $\mathbb{R}^n$ the supremum is always attained and the support function $s(\cdot, A) : \mathbb{R}^n \to \mathbb{R}$ is well defined. Indeed,

$$|s(p, A)| \leq \|A\|\|p\|, \tag{1.4.2}$$

for all $p \in \mathbb{R}^n$, and

$$|s(p, A) - s(q, A)| \leq \|A\|\|p - q\|, \tag{1.4.3}$$

for all $p, q \in \mathbb{R}^n$.

Further, for all $p \in \mathbb{R}^n$,

$$s(p, A) \leq s(p, B), \quad \text{if } \; A \subseteq B, \tag{1.4.4}$$

and

$$s(p, co(A \cup B)) \leq \max\{s(p, A), s(p, B)\}. \tag{1.4.5}$$

The support function $s(p, A)$ is uniquely paired to the subset $A \in K_c(\mathbb{R}^n)$ in the sense that $s(p, A) = s(p, B)$ for all $p \in \mathbb{R}^n$ if and only if $A = B$ when A and B are restricted to $K_c(\mathbb{R}^n)$. It also preserves set addition and nonnegative scalar multiplication. That is, for all $p \in \mathbb{R}^n$,

$$s(p, A + B) = s(p, A) + s(p, B), \tag{1.4.6}$$

which, in particular reduces to

$$s(p, A + \{x\}) = s(p, A) + < p, x >, \tag{1.4.7}$$

for any $x \in \mathbb{R}^n$, and

$$s(p, tA) = ts(p, A), \quad t \geq 0. \tag{1.4.8}$$

For a fixed $A \in K_c(\mathbb{R}^n)$, $s(p, A)$ is positively homogeneous

$$s(tp, A) = ts(p, A), \quad t \geq 0, \tag{1.4.9}$$

for all $p \in \mathbb{R}^n$, and subadditive:

$$s(p_1 + p_2, A) \leq s(p_1, A) + s(p_2, A), \tag{1.4.10}$$

for all $p_1, p_2 \in \mathbb{R}^n$. Moreover, combining (1.4.13) and (1.4.14) we see that $s(\cdot, A)$ is a convex function, that is, it satisfies

$$s(\lambda p_1 + (1 - \lambda)p_2, A) \leq \lambda s(p_1, A) + (1 - \lambda)s(p_2, A), \tag{1.4.11}$$

for all $p_1, p_2 \in \mathbb{R}^n$ and $\lambda \in [0, 1]$.

The nonempty compact convex subsets of $\mathbb{R}^n$ are uniquely characterized by such functions.

Proposition 1.4.1 *For every continuous, positively homogeneous and subadditive function $s : \mathbb{R}^n \to \mathbb{R}$ there exists a unique nonempty compact convex subset*

$$A = \{x \in \mathbb{R}^n : <p, x> \leq s(p) \text{ for all } p \in \mathbb{R}^n\},$$

which has s as its support function.

The Hausdorff metric is related to the support function for $A, B \in K_c(\mathbb{R}^n)$, since we have

$$D(A, B) = \sup\{|s(p, A) - s(p, B)| : p \in S^{n-1}\}, \tag{1.4.12}$$

where $S^{n-1} = \{p \in \mathbb{R}^n : \|p\| = 1\}$ is the unit sphere in $\mathbb{R}^n$.

Let $C(S^{n-1})$ denote the Banach Space of continuous functions $f : S^{n-1} \to \mathbb{R}$ with the supremum norm

$$\|f\| = \sup\{|f(p)| : p \in S^{n-1}\}.$$

One can use the support function to embed the metric space $(K_c(\mathbb{R}^n), D)$ isometrically as a positive cone in $C(S^{n-1})$.

For this, define $j : K_c(\mathbb{R}^n) \to C(S^{n-1})$ by $j(A)(\cdot) = s(\cdot, A)$, for each $A \in K_c(\mathbb{R}^n)$.

From the properties of the support function, j is a univalent mapping satisfying

$$j(A + B) = j(A) + j(B), \tag{1.4.13}$$

and

$$j(tA) = tj(A), \quad t \geq 0, \tag{1.4.14}$$

with

$$\|j(A) - j(B)\| = D(A, B), \tag{1.4.15}$$

for all $A, B \in K_c(\mathbb{R}^n)$.

The desired positive cone is the image $j(K_c(\mathbb{R}^n))$ in $C(S^{n-1})$. Obviously j is continuous, as is its inverse

$$j^{-1} : j(K_c(\mathbb{R}^n)) \to K_c(\mathbb{R}^n).$$

1.5 Continuity and Measurability

We consider mappings F from a domain T in $\mathbb{R}^k$ into the metric space $(K_c(\mathbb{R}^n), D)$. Thus $F : T \to K_c(\mathbb{R}^n)$ or equivalently,

$$F(t) \in K_c(\mathbb{R}^n), \text{ for all } t \in T. \tag{1.5.1}$$

We shall call such a mapping F a (compact convex) *set valued mapping* from T to $\mathbb{R}^n$.

The usual definition of continuity of mappings between metric spaces applies here. We shall say that a set valued mapping F satisfying (1.5.1) is *continuous* at t_0 in T if for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that

$$D[F(t), F(t_0)] < \epsilon, \tag{1.5.2}$$

for all $t \in T$ with $\|t - t_0\| < \delta$.

Alternatively, we can write (1.5.2) in terms of the convergence of sequences, that is

$$\lim_{t_n \to t_0} D[F(t_n), F(t_0)] = 0, \tag{1.5.3}$$

for all sequences $\{t_n\}$ in T with $t_n \to t_0 \in T$.

Using the Hausdorff separation d_H and neighborhoods, we see that (1.5.2) is the combination of

$$d_H(F(t), F(t_0)) < \epsilon, \tag{1.5.4}$$

that is

$$F(t) \subset S_\epsilon(F(t_0)) \equiv F(t_0) + \epsilon S_1^n, \tag{1.5.5}$$

and

$$d_H(F(t_0), F(t)) < \epsilon, \tag{1.5.6}$$

that is

$$F(t_0) \subset S_\epsilon(F(t)) \equiv F(t) + \epsilon S_1^n, \tag{1.5.7}$$

for all $t \in T$ with $\|t - t_0\| < \delta$. As before, $S_1^n = \{x \in \mathbb{R}^n : \|x\| < 1\}$ is the open unit ball in $\mathbb{R}^n$. If the mapping F satisfies (1.5.4), (1.5.5) we say that it is *upper semicontinuous* at t_0, or that it is *lower semicontinuous* at t_0, if it satisfies (1.5.6), (1.5.7). Thus, F is continuous at t_0 if and only if it is both lower semicontinuous and upper semicontinuous at t_0. A set valued mapping can be lower semicontinuous without being upper semicontinuous, and vice versa.

Example 1.5.1 *The set valued mapping F from $\mathbb{R}$ into $\mathbb{R}$ defined by*

$$F(t) = \begin{cases} \{0\}, & \text{for } t = 0, \\ [0, 1], & \text{for } t \in I\!R \setminus \{0\}, \end{cases}$$

is lower semicontinuous, but not upper semicontinuous, at $t_0 = 0$. On the other hand, $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(t) = \begin{cases} [0, 1], & \text{for } t = 0, \\ \{0\}, & \text{for } t \in \mathbb{R} \setminus \{0\}, \end{cases}$$

is upper semicontinuous, but not lower semicontinuous, at $t_0 = 0$.

Example 1.5.2 *A single valued mapping $f : \mathbb{R} \to \mathbb{R}_+$ is upper (lower) semicontinuous if the set valued mapping F defined by $F(t) = [0, f(t)]$ is upper (lower) semicontinuous.*

If the mapping is continuous, upper semicontinuous or lower semicontinuous at every $t_0 \in T$, we shall replace the qualifier 'at t_0' with 'on T', or omit it altogether.

We say that a set valued mapping F from T into $\mathbb{R}^n$ is *Lipschitz continuous* with *Lipschitz constant* L if

$$D\left[F(t'), F(t)\right] \ \le \ L\|t' - t\|, \tag{1.5.8}$$

for all $t', t \in T$. A Lipschitz continuous mapping is obviously continuous.

The distance $d(x, F(t))$ of $F(t)$ from a point $x \in \mathbb{R}^n$ satisfies

$$|d(x, F(t)) - d(y, F(t'))| \le \|x - y\| + D\left[F(t), F(t')\right], \tag{1.5.9}$$

for all $x, y \in \mathbb{R}^n$ and $t', t \in T$. Thus $d(\cdot, F(\cdot)) : \mathbb{R}^n \times T \to \mathbb{R}_+$ is continuous whenever F is continuous, and Lipschitz continuous whenever F is Lipschitz continuous. Since the magnitude $\|F(t)\| = D[F(t), \theta]$, a similar assertion holds for $\|F(\cdot)\| : T \to \mathbb{R}_+$ with the same Lipschitz constant when F is Lipschitz continuous.

We saw in Section 1.4 that the support function $s(\cdot, A)$ of an element $A \in K_c(\mathbb{R}^n)$ can be used to form an isometric embedding $j : K_c(\mathbb{R}^n) \to C(S^{n-1})$ with $j(A)(\cdot) = s(\cdot, A)$.

Thus, if $F : T \to K_c(\mathbb{R}^n)$, the support function $s(\cdot, F(t))$, for each $t \in T$ defines a mapping $j(F(\cdot)) : T \to C(S^{n-1})$. From (1.4.12) we have

$$\sup\{|s(p, F(t')) - s(p, F(t))| : p \in S^{n-1}\} = D\left[F(t'), F(t)\right]. \tag{1.5.10}$$

So $j(F(\cdot))$ is continuous or Lipschitz continuous (with the same Lipschitz constant) whenever F is continuous or Lipschitz continuous.

Combining (1.4.3) and (1.5.10) we obtain

$$|s(x, F(t')) - s(y, F(t))| \le \|F(t)\|\|x - y\| + D\left[F(t'), F(t)\right], \tag{1.5.11}$$

for all $t',\ t \in T$ and $x, y \in \bar{S}_1^n = \{x \in \mathbb{R}^n : \|x\| \le 1\}$.

Thus the support function $s(x, F(t))$, considered as a mapping $s(\cdot, F(\cdot)) : \bar{S}_1^n \times T \to \mathbb{R}$ is continuous or Lipschitz continuous whenever F is continuous or Lipschitz continuous.

Let $\mathcal{B}(\mathbb{R}^k)$ and $\mathcal{B}(K_c(\mathbb{R}^n))$ denote the $\sigma-$ algebras of Borel subsets of $\mathbb{R}^k$ and $(K_c(\mathbb{R}^n), D)$ respectively. Adopting the usual definition of Borel measurability of a mapping between metric spaces, we shall say that a mapping $F : T \to K_c(\mathbb{R}^n)$ is *measurable* if

$$\{t \in T : F(t) \subset \mathcal{B}\} \in \mathcal{B}(\mathbb{R}^k) \text{ for all } \mathcal{B} \in \mathcal{B}(K_c(\mathbb{R}^n)). \tag{1.5.12}$$

We shall write $F^{-1}(A) = \{t \in T : F(t) \cap A \ne \emptyset\}$ for any subset A of $\mathbb{R}^n$.

Then we have

Proposition 1.5.1 *The following assertions are equivalent:*

(i) $F : T \to K_c(\mathbb{R}^n)$ *is measurable;*

(ii) $F^{-1}(B) \in \mathcal{B}(\mathbb{R}^k)$ *for all* $B \in \mathcal{B}(\mathbb{R}^k)$;

(iii) $F^{-1}(O) \in \mathcal{B}(\mathbb{R}^k)$ *for all open subsets* O *of* $\mathbb{R}^n$;

(iv) $F^{-1}(C) \in \mathcal{B}(\mathbb{R}^k)$ *for all closed subsets* C *of* $\mathbb{R}^n$;

(v) $d(x, F(\cdot)) : T \to \mathbb{R}$ *is measurable for each* $x \in \mathbb{R}^n$;

(vi) $\|F(\cdot)\| : T \to \mathbb{R}$ *is measurable;*

(vii) $s(x, F(\cdot)) : T \to \mathbb{R}$ *is measurable for each* $x \in \mathbb{R}^n$.

In (v)-(vii) we mean measurability of the single valued mapping $T \mapsto \mathbb{R}$ with respect to the Borel $\sigma-$ algebras $\mathcal{B}(\mathbb{R}^k)$ and $\mathcal{B}(\mathbb{R})$. Such mappings are measurable if they are continuous. For set valued mappings we have

Proposition 1.5.2 $F : T \to K_c(\mathbb{R}^n)$ *is measurable if it is upper semicontinuous or lower semicontinuous, and hence if it is continuous.*

For set valued mappings, measurability is also preserved on taking limits.

Proposition 1.5.3 *Let* $F_i : T \to K_c(\mathbb{R}^n)$ *be measurable for* $i = 1, 2, 3, \cdots$ *and suppose that* $\lim_{t\to\infty} D[F_i(t), F(t)] = 0$ *for almost all* $t \in T$*. Then* $F : T \to K_c(\mathbb{R}^n)$ *is measurable.*

A *selector* of a set valued mapping F from T into $\mathbb{R}^n$ is a single valued mapping $f : T \to \mathbb{R}^n$ such that

$$f(t) \in F(t) \text{ for all } t \in T. \tag{1.5.13}$$

Proposition 1.5.4 *If* $F : T \to K_c(\mathbb{R}^n)$ *is measurable then it has a measurable selector* $f : T \to \mathbb{R}^n$.

The following result, known as the *Castaing Representation Theorem,* gives an additional characterization of measurability of a set valued mapping.

Theorem 1.5.1 $F : T \to K_c(\mathbb{R}^n)$ *is measurable if and only if there exists a sequence* $\{f_i\}$ *of measurable selectors of* F *such that*

$$F(t) = \overline{\bigcup\{f_i(t) : i = 1, 2, \cdots\}}, \tag{1.5.14}$$

for each $t \in T$.

If the set valued mapping is at least lower semicontinuous, then it has continuous selectors. In fact, any point in $F(t)$ is attainable by a continuous selector.

Proposition 1.5.5 *Let* $F : T \to K_c(\mathbb{R}^n)$ *be lower semicontinuous. Then for each* $x \in F(t)$ *and* $t \in T$ *there is a continuous selector* f *of* F *such that* $f(t) = x$.

On the other hand, a set valued mapping need not have a continuous selector if it is only upper semicontinuous.

Example 1.5.3 *The set valued mapping F from $\mathbb{R}$ to $\mathbb{R}$ defined by*

$$F(t) = \begin{cases} \{-1\}, & \text{if } t < 0, \\ [-1, 1], & \text{if } t = 0, \\ \{+1\}, & \text{if } t > 0, \end{cases}$$

is upper semicontinuous, but has no continuous selectors. Note that F is not lower semicontinuous at $t = 0$.

When the set valued mapping F is Lipschitz continuous, then it has Lipschitz continuous selectors satisfying the attainability property of Proposition 1.5.4.

Let $\leq$ be the partial ordering on $K_c(\mathbb{R}^n)$ induced by set inclusion, that is, $A \leq B$ if and only if $A \subseteq B$. We say that a mapping $F : T \to K_c(\mathbb{R}^n)$ has a $\leq -$ maximum at $t_0 \in T$ if

$$F(t) \leq F(t_0) \text{ for all } t \in T, \tag{1.5.15}$$

and a $\leq -$ minimum at $t_0 \in T$ if

$$F(t_0) \leq F(t) \text{ for all } t \in T. \tag{1.5.16}$$

1.6 Differentiation

We begin with the definition of the Hukuhara derivative.

Definition 1.6.1 *Let I be an interval of real numbers. Let a multifunction $U : I \to K_c(\mathbb{R}^n)$ be given. U is Hukuhara differentiable at a point $t_0 \in I$, if there exists $D_H U(t_0) \in K_c(\mathbb{R}^n)$ such that the limits*

$$\lim_{\Delta t \to 0^+} \frac{U(t_0 + \Delta t) - U(t_0)}{\Delta t} \tag{1.6.1}$$

and

$$\lim_{\Delta t \to 0^+} \frac{U(t_0) - U(t_0 - \Delta t)}{\Delta t} \tag{1.6.2}$$

both exist and are equal to $D_H U(t_0)$.

Clearly, implicit in the definition of $D_H U(t_0)$ is the existence of the differences $U(t_0 + \Delta t) - U(t_0)$ and $U(t_0) - U(t_0 - \Delta t)$, for all $\Delta t > 0$ sufficiently small. Using the difference quotient in (1.6.2) is not equivalent to using the difference quotient in

$$\lim_{\Delta t \to 0^-} \frac{U(t_0 + \Delta t) - U(t_0)}{\Delta t}, \tag{1.6.2$'$}$$

contrary to the situation for ordinary functions from I into a topological vector space. In general the existence of $A - B$, $A, B, \in K_c(\mathbb{R}^n)$ implies nothing about the existence of $B - A$.

The integral of a continuous multifunction $F : [a, b] \to K_c(\mathbb{R}^n)$ is defined in Hukuhara [1,2], and is shown that $D_H \int_a^t F(s)ds = F(t)$. In order that such a result holds, one must use (1.6.2) instead of (1.6.2$'$), since the difference quotient in (1.6.2$'$) may not exist, as shown in the following example.

Example 1.6.1 *Let* $A \in K_c(\mathbb{R}^n)$ *and define* $F(t) = A,\ t \in \mathbb{R}$; *then for any* $t > 0$, *we have* $\int_0^t F(s)ds = tA$. *Taking* $U(t) = tA,\ t > 0$ *we see that the difference quotient (1.6.2′) does not exist.*

The following proposition illustrates an important property of Hukuhara derivative.

Proposition 1.6.1 *If the multifunction* $U : I \to K_c(\mathbb{R}^n)$ *is Hukuhara differentiable on* I, *then the real valued function* $t \to diam(U(t)),\ t \in I$ *is nondecreasing on* I.

Proof If U is Hukuhara differentiable at a point $t_0 \in I$, then there is a $\delta(t_0) > 0$, such that $U(t_0 + \Delta t) - U(t_0)$ and $U(t_0) - U(t_0 - \Delta t)$ are defined for $0 < \Delta t < \delta(t_0)$. Since $A - B,\ A, B \in K_c(\mathbb{R}^n)$ is defined only if some translate of B is contained in A, thus $A - B$ exists only if $diam(A) \geq diam(B)$. Let $t_1, t_2 \in I$ be fixed with $t_1 < t_2$. Then for each $\tau \in [t_1, t_2]$ there is a $\delta(\tau) > 0$ such that $diam(U(s)) \leq diam(U(\tau))$, for $s \in [\tau - \delta(\tau), \tau]$, and $diam(U(s)) \geq diam(U(\tau))$, for $s \in [\tau, \tau + \delta(\tau)]$. The collection

$$\{I_\tau : \tau \in [t_1, t_2],\ I_\tau = (\tau - \delta(\tau), \tau + \delta(\tau))\},$$

forms an open covering of $[t_1, t_2]$. Choose a finite subcover $I_{\tau_1}, \cdots, I_{\tau_N}$ with $\tau_i < \tau_{i+1}$. We then arrive at $diam(U(t_1)) \leq diam(U(\tau_1))$ and $diam(U(\tau_N)) \leq diam(U(t_2))$. There is no loss in generality to assume $I_{\tau_i} \cap I_{\tau_{i+1}} \neq \emptyset,\ i = 1, \cdots, N-1$. Thus for each $i = 1, \cdots, N-1$, there exists an $s_i \in I_{\tau_i} \cap I_{\tau_{i+1}}$ with $\tau_i < s_i < \tau_{i+1}$, and hence

$$diam(U(\tau_i)) \leq diam(U(s_i)) \leq diam(U(\tau_{i+1})).$$

Therefore we have $diam(U(t_1)) \leq diam(U(t_2))$, which proves the proposition.

The example given below, utilizes the above proposition.

Example 1.6.2 *Let* $U(t) = (2 + \sin t)\bar{S}_1^n.(\bar{S}_1^n$ *is the closed unit ball in* $\mathbb{R}^n$). U *is not Hukuhara differentiable on* $(0, 2\pi)$, *since* $diam(U(t)) = 2(2 + \sin t)$ *is not non-decreasing on* $(0, 2\pi)$.

Remark 1.6.1 *Note that the existence of the limits in (1.6.1) and (1.6.2) is not used in the proof of proposition 1.6.1. In fact, instead of the hypothesis that* $U(t)$ *is Hukuhara differentiable on* I, *one could substitute the assumption that for each* $t \in I$, *the differences* $U(t + \Delta t) - U(t)$ *and* $U(t) - U(t - \Delta t)$ *both exist for all sufficiently small* $\Delta t > 0$.

Also, we observe that a multifunction $U : I \to K_c(\mathbb{R}^n)$ is Hukuhara differentiable on I, and $diam(U(t)) > 0$ for $t \in I$ need not imply U is monotone with respect to set inclusion. The following example illustrates this fact.

Example 1.6.3 *If* $U(t) = [t, 2t],\ 0 < t < 1$, *then* $D_H(t) = [1, 2],\ 0 < t < 1$ *and yet* $U(t_1) \not\subset U(t_2)$ *and* $U(t_2) \not\subset U(t_1)$ *for any* $t_1,\ t_2,\ 0 < t_1 < t_2 < 1$.

We now proceed to prove the following standard result on the real line for the Hukuhara derivative.

Proposition 1.6.2 *The set valued mapping U is constant if and only if we have*

$$D_H U = 0 \tag{1.6.3}$$

identically on I.

Proof If U is a constant, the result follows. Conversely, suppose (1.6.3) holds. For a fixed $t_0 \in (0,1)$, if $t > t_0$, using (1.3.15) we get

$$D[U(t), U(t_0)] = D[U(t) - U(t_0), \theta],$$

which gives

$$\lim_{t \to t_0^+} \frac{D[U(t), U(t_0)]}{t - t_0} = 0.$$

Similarly, if $t < t_0$, we obtain

$$\lim_{t \to t_0^-} \frac{D[U(t), U(t_0)]}{t - t_0} = 0$$

and hence,

$$\lim_{t \to t_0} \left| \frac{D[U(t), U(t_0)]}{t - t_0} \right| = 0 \tag{1.6.4}$$

Fixing $t_1 \in (0,1)$, from the inequality

$$|D[U(t_1), U(t)] - D[U(t_1), U(t_0)]| \leq D[U(t), U(t_0)],$$

upon dividing by $|t - t_0|$ and considering (1.6.4), we obtain that the real valued function $t \to D[U(t_1), U(t)]$ is a constant, and since it is zero at t_1, it must be identically zero.

1.7 Integration

Let $F : [0,1] \to K_c(\mathbb{R}^n)$ and let $S(F)$ denote the set of integrable selectors of F over $[0,1]$. Then the *Aumann integral* of F over $[0,1]$ is defined as

$$\int_0^1 F(t)dt = \overline{\left\{ \int_0^1 f(t)dt : f \in S(F) \right\}}. \tag{1.7.1}$$

If $S(F) \neq \emptyset$, then the Aumann integral exists and F is said to be *Aumann integrable.*

We shall say that F is *integrally bounded* on $[0,1]$ if there exists an integrable function $g : [0,1] \to \mathbb{R}$ such that

$$\|F(t)\| \leq g(t), \quad \text{for almost all} \quad t \in [0,1]. \tag{1.7.2}$$

If such an F has measurable selectors, then they are also integrable and $S(F)$ is nonempty.

Theorem 1.7.1 *If $F : [0,1] \to K_c(\mathbb{R}^n)$ is measurable and integrally bounded, then it is Aumann integrable over each $[a,s] \subset [0,1]$ with $\int_a^s F(t)dt \in K_c(\mathbb{R}^n)$ for all $s \in [a,1]$.*

For a set valued mapping F as in Theorem 1.7.1, the Castaing Representation Theorem 1.5.1 applies and provides a sequence $\{f_i\}$ of integrable selectors which are pointwise dense in F. Moreover,

$$\int_0^1 F(t)dt = \overline{\left\{ \int_0^1 f_i(t)dt : i = 1, 2, \cdots \right\}} \tag{1.7.3}$$

and so we need only consider these selectors to evaluate $\int_0^1 F(t)dt$.

The Aumann integrability of a mapping $F : [0,1] \to K_c(\mathbb{R}^n)$ is fundamentally related to the Bochner integrability of its support function.

Theorem 1.7.2 *Suppose that $F : [0,1] \to K_c(\mathbb{R}^n)$ is measurable. Then F is Aumann integrable if and only if $s(\cdot, F(\cdot)) : S^{n-1} \times [0,1] \to C(S^{n-1})$ is Bochner integrable, in which case*

$$s\left(\cdot, \int_0^1 F(t)dt\right) = \int_0^1 s(\cdot, F(t))dt, \tag{1.7.4}$$

where the integral on the right is Bochner integral.

From (1.7.4), we obtain the pointwise equality

$$s\left(p, \int_0^1 F(t)dt\right) = \int_0^1 s(p, F(t))dt \tag{1.7.5}$$

for all $p \in S^{n-1}$, where the integral on the right is now the Lebesgue integral.

Using Theorem 1.7.2, we find that the Aumann integral satisfies

$$\int_0^1 (F(t) + G(t))dt = \int_0^1 F(t)dt + \int_0^1 G(t)dt, \tag{1.7.6}$$

$$\int_a^c F(t)dt = \int_a^b F(t)dt + \int_b^c F(t)dt, \quad 0 \le a \le b \le c \le 1, \tag{1.7.7}$$

and

$$\int_0^1 \lambda F(t)dt = \lambda \int_0^1 F(t)dt, \quad \lambda \in \mathbb{R}, \tag{1.7.8}$$

for all Aumann integrable $F, G : [0,1] \to K_c(\mathbb{R}^n)$, with

$$\int_0^1 F(t)dt \subseteq \int_0^1 G(t)dt \quad \text{if} \quad F(t) \subseteq G(t) \quad \text{for all} \quad t \in [0,1]. \tag{1.7.9}$$

In addition, the Aumann integral uniquely determines its integrand.

Proposition 1.7.1 *If $F, G : [0,1] \to K_c(\mathbb{R}^n)$ are Aumann integrable with $\int_0^1 F(t)dt = \int_0^1 G(t)dt$, then $F(t) = G(t)$ for almost all $t \in [0,1]$.*

Similarly the following convergence properties can be established.

Theorem 1.7.3 *Let $F_i, F : [0,1] \to K_c(\mathbb{R}^n)$, $i = 1, 2, \cdots$, be measurable and uniformly integrally bounded. If $F_i(t) \to F(t)$ for all $t \in [0,1]$ as $i \to \infty$, then*

$$A_i = \int_0^1 F_i(t)dt \to A = \int_0^1 F(t)dt \quad as \quad i \to \infty. \tag{1.7.10}$$

Theorem 1.7.4 *Let $F_i : [0,1] \to K_c(\mathbb{R}^n)$, $i = 1, 2, \cdots$, be measurable and uniformly integrally bounded, and suppose that $A_i = \int_0^1 F_i(t)dt \to A \in K_c(\mathbb{R}^n)$ as $i \to \infty$. Then there exists a measurable mapping $F : [0,1] \to K_c(\mathbb{R}^n)$ such that $A = \int_0^1 F(t)dt$.*

Theorem 1.7.5 *If $F, G : [0,1] \to K_c(\mathbb{R}^n)$ are integrable, then so also is*

$$D[F(\cdot), G(\cdot)] : [0,1] \to \mathbb{R}$$

and

$$D\left[\int_0^1 F(t)dt, \int_0^1 G(t)dt\right] \leq \int_0^1 D[F(t), G(t)]dt. \tag{1.7.11}$$

Integration and differentiation of set valued mappings $F : [0,1] \to K_c(\mathbb{R}^n)$ are essentially inverse operations.

Proposition 1.7.2 *Let $F : [0,1] \to K_c(\mathbb{R}^n)$ be measurable and integrally bounded. Then*

$$\lim_{\Delta t \to 0^+} \frac{1}{\Delta t} \int_{t_0}^{t_0+\Delta t} F(t)dt = F(t_0), \tag{1.7.12}$$

for almost all $t_0 \in [0,1]$. In particular, (1.7.12) holds at all $t_0 \in [0,1)$ when F is continuous.

Theorem 1.7.6 *Let $F : [0,1] \to K_c(\mathbb{R}^n)$ be measurable and integrally bounded. Then $A : [0,1] \to K_c(\mathbb{R}^n)$ defined by*

$$A(t) = \int_0^t F(s)ds, \tag{1.7.13}$$

for all $t \in [0,1]$ is Hukuhara differentiable for almost all $t_0 \in (0,1)$, with the Hukuhara derivative $D_H A(t_0) = F(t_0)$.

A counterpart of the first Fundamental Theorem of Calculus

$$F(t_1) = F(t_0) + \int_{t_0}^{t_1} D_H F(t)dt, \quad 0 \leq t_0 \leq t_1 \leq 1, \tag{1.7.14}$$

holds for a Hukuhara differentiable $F : [0,1] \to K_c(\mathbb{R}^n)$ with continuous Hukuhara derivative $D_H F$ on $[0,1]$.

1.8 Subsets of Banach Spaces

Let E be a real Banach space with the norm $\|\cdot\|$ and the metric generated by it. Let $(2^E)_b$ be a collection of all nonempty bounded subsets of E with Hausdorff pseudometric,

$$D[A, B] = \max\left[\sup_{x\in B} d(x, A), \sup_{y\in A} d(y, B)\right], \tag{1.8.1}$$

where $d(x, A) = \inf[d(x, y) : y \in A]$, $A, B \in (2^E)_b$. Denote by $K(E)$ $(K_c(E))$ the collection of all nonempty compact (compact convex) subsets of E, which are considered as subspaces of $(2^E)_b$. We note that on $K(E)$, $(K_c(E))$ the topology of the space $(2^E)_b$ induces the Hausdorff metric. Also $K(E)$ $(K_c(E))$ is a complete metric space and $K(E)$ $(K_c(E))$ is separable if E is separable space.

It is known that if the space $K_c(E)$ is equipped with the natural algebraic operations of addition and nonnegative scalar multiplication, then $K_c(E)$ becomes a semilinear metric space which can be embedded as a complete cone in a corresponding Banach space. See Tolstonogov [1], and Brandao et. el [1].

Let $T = [0, a]$, $a > 0$. Then the mapping $F : T \to K(E)$ is said to be strongly measurable, if it is almost everywhere (a.e.) in T a point-wise limit of the sequence $F_n : T \to K(E)$, $n \geq 1$, of step mappings.

If $D(F(t), \Theta) \leq \lambda(t)$, a.e. on T, where $\lambda(t)$ is summable on T and Θ is the zero element of E, which is regarded as a one-point set, then F is said to be integrally bounded on T. For a set-valued mapping $F : T \to E$, we shall denote by $(\mathcal{A}) \int_{T_0} F(s)ds$ the integral in the sense of Aumann on the measurable set $T_0 \subset T$, that is,

$$(\mathcal{A}) \int_{T_0} F(s)ds = \left[\int_{T_0} f(s)ds : f \text{ is a Bochner integrable selector of F}\right].$$

For a strongly measurable mapping $F : T \to K_c(E)$, the integral $\int_{T_0} F(s)ds$ in the sense of Bochner is introduced in a natural way, since as pointed out earlier, $K_c(E)$ can be embedded as a complete cone into a corresponding Banach Space.

If a multifunction $F : T \to E$ with compact convex values is strongly measurable and integrally bounded then

$$\int_{T_0} F(s)ds = (\mathcal{A}) \int_{T_0} F(s)ds, \tag{1.8.2}$$

on the measurable set $T_0 \subset T$.

Let $A, B \in K_c(E)$. The set $C \in K_c(E)$ satisfying $A = B + C$ is known as the Hukuhara difference of the sets A and B and is denoted by the symbol $A - B$. We say that the mapping $F : T \to K_c(E)$ has a Hukuhara derivative $D_H F(t_0)$ at a point $t_0 \in T$ if there exists an element $D_H F(t_0) \in K_c(E)$ such that

$$\lim_{h\to 0^+} \frac{F(t_0 + h) - F(t_0)}{h}, \text{ and } \lim_{h\to 0^+} \frac{F(t_0) - F(t_0 - h)}{h}$$

exist in the topology of $K_c(E)$ and are equal $D_H F(t_0)$.

By embedding $K_c(E)$ as a complete cone in a corresponding Banach space and taking into account the result on differentiation of Bochner integral, we find that if

$$F(t) = X_0 + \int_0^t \Phi(s)\, ds, \quad X_0 \in K_c(E), \tag{1.8.3}$$

where $\Phi : T \to K_c(E)$ is integrable in the sense of Bochner, then $D_H F(t)$ exists a.e. on T and the equality

$$D_H F(t) = \Phi(t) \ \text{ a.e. on } \ T, \tag{1.8.4}$$

holds.

Let μ be Lebesgue measure, $\mathcal{L}$ be a $\sigma-$ algebra of Lebesgue measurable subsets of $J = [t_0, b],\ t_0 \geq 0,\ b \in (t_0, \infty)$, and E be a metrizable space.

A multiplication $F : J \to E$ with closed values is measurable if the set $F^{-1}(B) = \{t \in J; F(t) \cap B \neq \emptyset\}$ is measurable for each closed subset B of E. If $F : Y \to E$, where Y is a topological space, then F is measurable implies that F is measurable when Y is assigned the $\sigma-$algebra $\mathcal{B}_Y$ of Borel subsets of Y. Similarly, if $F : J \times Y \to E$, then the measurability of F is defined in terms of the product of $\sigma-$algebras $\mathcal{L} \otimes \mathcal{B}_Y$ generated by the sets $A \times B$, where $A \in \mathcal{L}$ and $B \in \mathcal{B}_Y$. If E is separable then for multifunction $F : J \to E$ with compact values the definitions of strong measurability and measurability are equivalent.

A multifunction from a topological space Y into space E is upper semicontinuous (usc) at a point $y_0 \in Y$ if, for any $\epsilon > 0$, there exists a neighborhood $U(y_0)$ of the point y_0 such that $F(y) \subset F(y_0) + \epsilon \cdot B$ for all $y \in U(y_0)$, where B is unit ball of E.

A multifunction $F : Y \to E$ is said to be usc if it is usc at any point $y_0 \in E$.

For a multifunction $F : Y \to E$ with compact values the definition of usc is equivalent to the following: the set $F^{-1}(U)$ is closed for each closed subset U of E.

Let us recall that the Hausdorff metric (1.8.1) satisfies the following properties:

$$D[U + W, V + W] = D[U, V], \tag{1.8.5}$$

$$D[\lambda U, \lambda V] = \lambda D[U, V], \tag{1.8.6}$$

$$D[U, V] \leq D[U, W] + D[W, V], \tag{1.8.7}$$

for all $U, V, W \in K_c(E)$ and $\lambda \in \mathbb{R}^+$. Also for $U \in K_c(E)$, we set

$$D[U, \Theta] = \|U\| = \sup[\|u\| : u \in U]. \tag{1.8.8}$$

1.9 Notes and Comments

The preliminary material including calculus of the set valued maps assembled in this chapter is taken from Arstein [1], Banks and Jacobs [1], De Blasi and

Iervolino [1], Diamond and Kloeden [1], Hukuhara [1,2], Radström [1] and Tolstonogov [1]. See also for further details Aubin and Frankowska [1], Aumann [1], Castaing and Valadier [1], Debreu [1], Hermes [1], Hausdorff [1], Lay [1] and Rockafeller [1].

Chapter 2

Basic Theory

2.1 Introduction

This chapter is devoted to the basic theory of set differential equations (SDEs). We begin Section 2.2 with the formulation of the initial value problem of SDEs in the metric space $(K_c(\mathbb{R}^n), D)$. Utilizing the properties of the Hausdorff metric $D[\cdot, \cdot]$ and employing the known theory of differential and integral inequalities, we establish a variety of comparison results, that are required for later discussion. Section 2.3 deals with the convergence of successive approximations of the initial value problem (IVP) under the general uniqueness assumption of Perron type, using the comparison function that is rather instructive. The continuous dependence of solutions relative to the initial conditions is also studied under the same conditions. In Section 2.4, we investigate an existence result of Peano's type and then consider the existence of extremal solutions of SDE. For this purpose, one needs to introduce a partial order in $(K_c(\mathbb{R}^n), D)$, prove the required comparison result for strict inequalities, and then, utilizing it, discuss the existence of extremal solutions. Having the notion of maximal solution for SDE, we then prove the comparison result analogous to the well known comparison theorem in the ordinary differential system.

The monotone iterative technique is considered for SDE in Section 2.5, employing the method of upper and lower solutions. The results considered are so general that they contain several special cases of interest. Section 2.6 contains a global existence result, and Section 2.7 considers the error estimate between the solutions and approximate solutions of SDEs.

In Section 2.8, we discuss the IVP of SDE without assuming any continuity of the function involved and obtain the existence of an Euler solution which reduces to the actual solution when the continuity assumption is made. Using the proximal normal aiming condition, we study in Section 2.9, the flow invariance of solutions relative to a closed set. Here weak and strong flow invariance results are considered in terms of nonsmooth analysis. Section 2.10 establishes an existence result for SDE when the function involved is upper semicontinuous.

The conditions are also provided to obtain the function involved in SDE from the multifunction with compact values only by suitably utilizing convexification. Notes and comments are provided in Section 2.11.

2.2 Comparison Principles

Let us consider the initial value problem (IVP) for the set differential equation

$$D_H U = F(t, U), \quad U(t_0) = U_0 \in K_c(\mathbb{R}^n),\ t_0 \geq 0, \tag{2.2.1}$$

where $F \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$ and $D_H U$ is the Hukuhara derivative of U.

The mapping $U \in C^1[J, K_c(\mathbb{R}^n)]$ where $J = [t_0, t_0 + a],\ a > 0$, is said to be a solution of (2.2.1) on J, if it satisfies (2.2.1) on J.

Since $U(t)$ is continuously differentiable, we have

$$U(t) = U_0 + \int_{t_0}^{t} D_H U(s) ds, \quad t \in J. \tag{2.2.2}$$

We therefore associate with the IVP(2.2.1) the following integral equation

$$U(t) = U_0 + \int_{t_0}^{t} F(s, U(s)) ds, \quad t \in J, \tag{2.2.3}$$

where the integral in (2.2.3) is the Hukuhara integral. Observe also that $U(t)$ is a solution of (2.2.1) if and only if it satisfies (2.2.3) on J.

Utilizing the properties of the Hausdorff metric $D[\cdot, \cdot]$ and the integral, and employing the known theory of differential and integral inequalities for ordinary differential equations, we shall first establish the following comparison principles, which we need for later discussion.

Theorem 2.2.1 *Assume that* $F \in C[J \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$ *and* $t \in J$, $U, V \in K_c(\mathbb{R}^n)$,

$$D[F(t, U), F(t, V)] \leq g(t, D[U, V]), \tag{2.2.4}$$

where $g \in C[J \times \mathbb{R}_+, \mathbb{R}_+]$ *and* $g(t, w)$ *is monotone nondecreasing in* w *for each* $t \in J$. *Suppose further that the maximal solution* $r(t, t_0, w_0)$ *of the scalar differential equation*

$$w' = g(t, w), \quad w(t_0) = w_0 \geq 0, \tag{2.2.5}$$

exists on J. *Then, if* $U(t),\ V(t)$ *are any two solutions through* $(t_0, U_0),\ (t_0, V_0)$ *respectively on* J, *it follows that*

$$D[U(t), V(t)] \leq r(t, t_0, w_0), \quad t \in J, \tag{2.2.6}$$

provided $D[U_0, V_0] \leq w_0$.

Proof Set $m(t) = D[U(t), V(t)]$, so that $m(t_0) = D[U_0, V_0] \leq w_0$. Then, in view of the properties of the metric D, we get

$$\begin{aligned} m(t) &= D\left[U_0 + \int_{t_0}^t F(s, U(s))ds,\ V_0 + \int_{t_0}^t F(s, V(s))ds\right] \\ &\leq D\left[U_0 + \int_{t_0}^t F(s, U(s))ds,\ U_0 + \int_{t_0}^t F(s, V(s))ds\right] \\ &\quad + D\left[U_0 + \int_{t_0}^t F(s, V(s))ds,\ V_0 + \int_{t_0}^t F(s, V(s))ds\right] \\ &= D\left[\int_{t_0}^t F(s, U(s))ds, \int_{t_0}^t F(s, V(s))ds\right] + D[U_0, V_0]. \end{aligned}$$

Now using the properties of the integrals and condition (2.2.4), we observe that,

$$\begin{aligned} m(t) &\leq m(t_0) + \int_{t_0}^t D\left[F(s, U(s)), F(s, V(s))\right] ds \\ &\leq m(t_0) + \int_{t_0}^t g(s, D[U(s), V(s)])ds \\ &= m(t_0) + \int_{t_0}^t g(s, m(s))ds, \quad t \in J. \end{aligned}$$

Now applying Theorem 1.9.2 given in Lakshmikantham and Leela [1], we conclude that

$$m(t) \leq r(t, t_0, w_0), \quad t \in J.$$

This establishes Theorem 2.2.1.

Remark 2.2.1 *If we employ the theory of differential inequalities instead of integral inequalities, we can dispense with the monotone character of $g(t, w)$ assumed in Theorem 2.2.1. This is the content of the next comparison principle.*

Theorem 2.2.2 *Let the assumptions of Theorem 2.2.1 hold except the nondecreasing property of $g(t, w)$ in w. Then the conclusion (2.2.6) is valid.*

Proof For small $h > 0$, the Hukuhara differences $U(t+h)-U(t)$, $V(t+h)-V(t)$ exist, and we have for $t \in J$,

$$m(t + h) - m(t) = D[U(t + h), V(t + h)] - D[U(t), V(t)].$$

Using the property (1.3.8) for D, we get

$$\begin{aligned} D[U(t + h), V(t + h)] &\leq D[U(t + h), U(t) + hF(t, U(t))] \\ &\quad + D[U(t) + hF(t, U(t)), V(t + h)], \end{aligned}$$

and

$$\begin{aligned}
&D[U(t)+hF(t,U(t)),V(t+h)] \\
&\quad\leq\ D[V(t)+hF(t,V(t)),V(t+h)] \\
&\qquad +D[U(t)+hF(t,U(t)),V(t)+hF(t,V(t))].
\end{aligned}$$

Also, we observe that

$$\begin{aligned}
&D[U(t)+hF(t,U(t)),V(t)+hF(t,V(t))] \\
&\quad\leq\ D[U(t)+hF(t,U(t)),U(t)+hF(t,V(t))] \\
&\qquad +D[U(t)+hF(t,V(t)),V(t)+hF(t,V(t))] \\
&\quad=\ D[hF(t,U(t)),hF(t,V(t))]+D[U(t),V(t)].
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
\frac{m(t+h)-m(t)}{h} \ \leq\ & \frac{1}{h}D[U(t+h),U(t)+hF(t,U(t))] \\
& +\frac{1}{h}D[V(t)+hF(t,V(t)),V(t+h)] \\
& +\frac{1}{h}D[hF(t,U(t)),hF(t,V(t))]
\end{aligned}$$

and consequently, in view of the properties of D and the fact that $U(t),\ V(t)$ are solutions of (2.2.1), we find that

$$\begin{aligned}
D^+m(t) \ \leq\ & \limsup_{h\to 0^+}\frac{1}{h}[m(t+h)-m(t)] \\
\leq\ & \limsup_{h\to 0^+} D\left[\frac{U(t+h)-U(t)}{h},F(t,U(t))\right] \\
& +\limsup_{h\to 0^+} D\left[F(t,V(t)),\frac{V(t+h)-V(t)}{h}\right] \\
& +D[F(t,U(t)),F(t,V(t))].
\end{aligned}$$

Here, we have used the fact that

$$\begin{aligned}
D[U(t+h),U(t)+hF(t,U(t))] &= D[U(t)+Z(t,h),U(t)+hF(t,U(t))] \\
&= D[Z(t,h)+U(t),U(t)+hF(t,U(t))] \\
&= D[Z(t,h),hF(t,U(t))] \\
&= D[U(t+h)-U(t),hF(t,U(t))].
\end{aligned}$$

The conclusion (2.2.6) follows from Theorem 1.4.1 in Lakshmikantham and Leela [1].

The next comparison result provides an estimate under weaker assumptions.

Theorem 2.2.3 *Assume that* $F\in C[J\times K_c(\mathbb{R}^n),K_c(\mathbb{R}^n)]$ *and*

$$\limsup_{h\to 0}\frac{1}{h}[D[U+hF(t,U),V+hF(t,V)]-D[U,V]]\leq g(t,D[U,V]),\ \ t\in J,$$

where $U, V \in K_c(\mathbb{R}^n)$, $g \in C[J \times \mathbb{R}_+, \mathbb{R}]$. The maximal solution $r(t, t_0, w_0)$ of (2.2.5) exists on J. Then the conclusion of Theorem 2.2.1 is valid.

Proof Proceeding as in the proof of Theorem 2.2.2, we see that

$$\begin{aligned} & m(t+h) - m(t) \\ &\quad = D[U(t+h), V(t+h)] - D[U(t), V(t)] \\ &\quad \leq D[U(t+h), U(t) + hF(t, U(t))] \\ &\qquad + D[V(t) + hF(t, V(t)), V(t+h)] \\ &\qquad + D[U(t) + hF(t, U(t)), V(t) + hF(t, V(t))] - D[U(t), V(t)]. \end{aligned}$$

$$\begin{aligned} D^+ m(t) &= \limsup_{h \to 0^+} \frac{1}{h}[m(t+h) - m(t)] \\ &\leq \limsup_{h \to 0^+} \frac{1}{h}[D[U(t) + hF(t, U(t)), \\ &\qquad\qquad V(t) + hF(t, V(t))] - D[U(t), V(t)]] \\ &\quad + \limsup_{h \to 0^+} D\left[\frac{U(t+h) - U(t)}{h}, F(t, U(t))\right] \\ &\quad + \limsup_{h \to 0^+} D\left[F(t, V(t)), \frac{V(t+h) - V(t)}{h}\right] \\ &\leq g(t, D[U(t), V(t)]) = g(t, m(t)), \;\; t \in J. \end{aligned}$$

The conclusion follows as before by Theorem 1.4.1 in Lakshmikantham and Leela [1] and the proof is complete.

We wish to remark that in Theorem 2.2.3, $g(t, w)$ need not be nonnegative and therefore the estimate in Theorem 2.2.3 would be finer than the estimates in Theorems 2.2.1 and 2.2.2.

As special cases of Theorems 2.2.1, 2.2.2 and 2.2.3, we have the following important corollaries.

Corollary 2.2.1 *Assume that $F \in C[J \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$ and either*

(a) $D[F(t, U), \theta] \leq g(t, D[U, \theta])$ or

(b) $\limsup_{h \to 0} \frac{1}{h}[D[U + hF(t, U), \theta] - D[U, \theta]] \leq g(t, D[U, \theta])$, where $g \in C[J \times \mathbb{R}_+, \mathbb{R}]$.

Then, if $D[U_0, \theta] \leq w_0$, we have

$$D[U(t), \theta] \leq r(t, t_0, w_0), \quad t \in J,$$

where $r(t, t_0, w_0)$ is the maximal solution of (2.2.5) on J.

Corollary 2.2.2 *The function* $g(t,w) = \lambda(t)w, \quad \lambda(t) \geq 0$ *and continuous is admissible in Theorem 2.2.1 to give*

$$m(t) \leq m(t_0) + \int_{t_0}^{t} \lambda(s)m(s)ds, \quad t \in J.$$

Then the Gronwall inequality implies

$$m(t) \leq m(t_0)\, exp\left[\int_{t_0}^{t} \lambda(s)ds\right], \quad t \in J,$$

which shows that (2.2.6) reduces to

$$D[U(t),V(t)] \leq D[U_0,V_0]\, exp\left[\int_{t_0}^{t} \lambda(s)ds\right], \quad t \in J.$$

Corollary 2.2.3 *The function* $g(t,w) = -\lambda(t)w$, *with* $\lambda(t)$ *as in Corollary 2.2.2, is also admissible in Theorem 2.2.3, and we get,*

$$D[U(t),V(t)] \leq D[U_0,V_0]\, exp\left[-\int_{t_0}^{t} \lambda(s)ds\right], \quad t \in J.$$

If $\lambda(t) = \lambda > 0$, *we find that*

$$D[U(t),V(t)] \leq D[U_0,V_0]\, e^{-\lambda(t-t_0)}, \quad t \in J.$$

If $J = [t_0, \infty)$, *we see that* $\lim_{t\to\infty} D[U(t),V(t)] = 0$, *showing the advantage of Theorem 2.2.3.*

2.3 Local Existence and Uniqueness

We shall begin by proving the existence and uniqueness result under assumptions more general than the Lipschitz type condition, which exhibits the idea of the comparison principle.

Theorem 2.3.1 *Assume that*

(a) $F \in C[R_0, K_c(\mathbb{R}^n)]$ *and* $D[F(t,U),\theta] \leq M_0$ *where* $R_0 = J \times B(U_0,b)$, $B(U_0,b) = [U \in K_c(\mathbb{R}^n) : D[U,U_0] \leq b]$ *on* R_0*;*

(b) $g \in C\left[J \times [0,2b], \mathbb{R}_+\right]$, $g(t,w) \leq M_1$ *on* $J \times [0,2b]$, $g(t,0) \equiv 0$, $g(t,w)$ *is nondecreasing in* w *for each* $t \in J$ *and* $w(t) \equiv 0$ *is the only solution of*

$$w' = g(t,w), \quad w(t_0) = 0, \tag{2.3.1}$$

on J*;*

(c) $D[F(t,U),F(t,V)] \leq g(t, D[U,V])$ *on* R_0.

Then the successive approximations defined by

$$U_{n+1}(t) = U_0 + \int_{t_0}^{t} F(s, U_n(s))ds, \quad n = 0, 1, 2, \ldots, \tag{2.3.2}$$

exist on $J_0 = [t_0, t + \eta)$, *where* $\eta = \min\{a, \frac{b}{M}\}$, $M = max\{M_0, M_1\}$, *as continuous functions and converge uniformly to the unique solution* $U(t)$ *of the IVP (2.2.1) on* J_0.

Proof Using the properties of Hausdorff metric, we get by induction,

$$\begin{aligned} D[U_{n+1}(t), U_0] &= D\left[U_0 + \int_{t_0}^{t} F(s, U_n(s))\, ds, U_0\right] \\ &= D\left[\int_{t_0}^{t} F(s, U_n(s))\, ds, \theta\right] \\ &\leq \int_{t_0}^{t} D\left[F(s, U_n(s)), \theta\right]\, ds \\ &\leq M_0(t - t_0) \leq M_0 a \leq b, \end{aligned}$$

and consequently, the successive approximations $\{U_n(t)\}$ are well defined on J_0. We shall next define the successive approximations of (2.3.1) as follows:

$$\begin{aligned} w_0(t) &= M(t - t_0), \\ w_{n+1}(t) &= \int_{t_0}^{t} g(s, w_n(s))\, ds, \quad t \in J_0, \quad n = 0, 1, 2, \ldots \end{aligned} \tag{2.3.3}$$

An easy induction, in view of the monotone character of $g(t, w)$ in w, proves that $\{w_n(t)\}$ are well defined and

$$0 \leq w_{n+1}(t) \leq w_n(t), \quad t \in J_0. \tag{2.3.4}$$

Since $|w'_n(t)| \leq g(t, w_{n-1}(t)) \leq M_1$, we conclude by Ascoli-Arzela Theorem and the monotonicity of the sequence $\{w_n(t)\}$ that

$$\lim_{n\to\infty} w_n(t) = w(t),$$

uniformly on J_0. It is also clear that $w(t)$ satisfies (2.3.1) and therefore by condition (b), $w(t) \equiv 0$ on J_0.

We observe that

$$D[U_1(t), U_0] \leq \int_{t_0}^{t} D[F(s, U_0), \theta]\, ds \leq M(t - t_0) \equiv w_0(t).$$

Assume that for some $k > 1$, we have

$$D[U_k(t), U_{k-1}(t)] \leq w_{k-1}(t), \quad \text{on} \quad J_0.$$

Since

$$D[U_{k+1}(t), U_k(t)] \leq \int_{t_0}^{t} D[F(s, U_k(s)), F(s, U_{k-1}(s))]\, ds,$$

using condition (c) and the monotone character of $g(t, w)$, we get

$$\begin{aligned} D[U_{k+1}(t), U_k(t)] &\leq \int_{t_0}^{t} g(s, D[U_k(s), U_{k-1}(s)])\, ds \\ &\leq \int_{t_0}^{t} g(s, w_{k-1}(s))\, ds = w_k(t). \end{aligned}$$

Thus by induction, the estimate

$$D[U_{n+1}(t), U_n(t)) \leq w_n(t), \quad t \in J_0, \tag{2.3.5}$$

is true for all n.

Letting $u(t) = D[U_{n+1}(t), U_n(t)]$, $t \in J_0$, the proof of Theorem of 2.2.2 shows that

$$D^+ u(t) \leq g\left(t, D[U_n(t), U_{n-1}(t)]\right) \leq g(t, w_{n-1}(t)), \quad t \in J_0.$$

Now let $n \leq m$. Setting $v(t) = D[U_n(t), U_m(t)]$, we obtain from (2.3.2)

$$\begin{aligned} D^+ v(t) &\leq D[D_H U_n(t), D_H U_m(t)] = D[F\left(t, U_{n-1}(t)\right), F\left(t, U_{m-1}(t)\right)] \\ &\leq D\left[F\left(t, U_n(t)\right), F\left(t, U_{n-1}(t)\right)\right] + D\left[F\left(t, U_n(t)\right), F\left(t, U_m(t)\right)\right] \\ &\quad + D[F\left(t, U_m(t)\right), F\left(t, U_{m-1}(t)\right)] \\ &\leq g\left(t, w_{n-1}(t)\right) + g\left(t, w_{m-1}(t)\right) + g\left(t, D[U_n(t), U_m(t)]\right) \\ &\leq g\left(t, v(t)\right) + 2g\left(t, w_{n-1}(t)\right), \qquad t \in J_0. \end{aligned}$$

Here we have used the arguments of the proof of Theorem 2.2.2, the monotone character of $g(t, w)$ and the fact that $w_{m-1} \leq w_{n-1}$, since $n \leq m$ and $w_n(t)$ is a decreasing sequence. The comparison Theorem 1.4.1 in Lakshmikantham and Leela [1] yields the estimate

$$v(t) \leq r_n(t), \quad t \in J_0,$$

where $r_n(t)$ is the maximal solution of

$$r_n' = g(t, r_n) + 2g(t, w_{n-1}(t)), \quad r_n(t_0) = 0, \tag{2.3.6}$$

for each n. Since as $n \to \infty$, $2g\left(t, w_{n-1}(t)\right) \to 0$ uniformly on J_0, it follows by Lemma 1.3.1 in Lakshmikantham and Leela [1] that $r_n(t) \to 0$ as $n \to \infty$ uniformly, on J_0. This implies from (2.2.5) and the definition of $v(t)$ that $U_n(t)$ converges uniformly to $U(t)$, and clearly $U(t)$ is a solution of (2.2.1).

To show uniqueness, let $V(t)$ be another solution of (2.2.1), on J_0. Then setting $m(t) = D[U(t), V(t)]$ and noting that $m(t_0) = 0$, we get, as before,

$$D^+ m(t) \leq g\left(t, m(t)\right), \quad t \in J_0,$$

and $m(t) \le r(t, t_0, 0),\ t \in J_0$, by Theorem 2.2.1. By assumption $r(t, t_0, 0) \equiv 0$, we get $U(t) \equiv V(t)$ on J_0, proving the theorem.

We shall discuss, in the next result, the continuous dependence of solutions with initial values. We need the following lemma before we proceed.

Lemma 2.3.1 *Let $F \in C[J \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$ and let*

$$G(t,r) = \max[D\,[F(t,U), \theta] : D[U, U_0] \le r].$$

Assume that $r^*(t, t_0, 0)$ is the maximal solution of

$$w' = G(t,w), \quad w(t_0) = 0, \ \text{on } J.$$

Let $U(t) = U(t, t_0, U_0)$ be the solution of (2.2.1). Then

$$D[U(t), U_0] \le r^*(t, t_0, 0), \quad t \in J.$$

Proof Define $m(t) = D[U(t), U_0], t \in J$. Then Corollary 2.2.1 shows that

$$\begin{aligned} D^+m(t) &\le D[D_H U(t), \theta] \\ &= D[F\,(t, U(t))\,, \theta] \\ &\le \max_{D[U,U_0] \le m(t)} D[F(t,U), \theta] \\ &= G(t, m(t)). \end{aligned}$$

This implies by Theorem 1.4.1 in Lakshmikantham and Leela [1] that

$$D[U(t), U_0] \le r^*(t, t_0, 0), t \in J,$$

proving the lemma.

Theorem 2.3.2 *Suppose that the assumptions $(a), (b), (c)$ of Theorem 2.3.1 hold. Assume further that the solutions $w(t, t_0, w_0)$ of (2.2.5) through every point (t_0, w_0) are continuous with respect to (t_0, w_0). Then the solutions $U(t) = U(t, t_0, U_0)$ of (2.2.1) are continuous relative to (t_0, U_0).*

Proof Let $U(t) = U(t, t_0, U_0), V(t) = V(t, t_0, V_0), U_0, V_0 \in K_c(\mathbb{R}^n)$, be the two solutions of (2.2.1). Then defining $m(t) = D[U(t), V(t)]$, we get from Theorem 2.2.1, the estimate

$$D[U(t), V(t)] \le r\,(t, t_0, D[U_0, V_0])\,, t \in J.$$

Since $\lim_{U_0 \to V_0} r\,(t, t_0, D[U_0, V_0]) = r(t, t_0, 0)$ uniformly on J and by hypothesis $r(t, t_0, 0) \equiv 0$, it follows that $\lim_{U_0 \to V_0} U(t, t_0, U_0) = V(t, t_0, V_0)$ uniformly and hence continuity of $U(t, t_0, U_0)$ relative to U_0 is valid.

To prove the continuity relative to t_0, we let $U(t) = U(t, t_0, U_0)$, $V(t) = V(t, \tau_0, U_0)$ be the two solutions of (2.2.1) and let $\tau_0 > t_0$. As before, setting $m(t) = D[U(t), V(t)]$, noting that $m(\tau_0) = D[U(\tau_0), U_0]$, we obtain from Lemma 2.3.1,

$$m(\tau_0) \le r^*(\tau_0, t_0, 0),$$

and consequently, by Theorem 2.2.1, we arrive at

$$m(t) \le \tilde{r}(t), \quad t \ge \tau_0,$$

where $\tilde{r}(t) = \tilde{r}(t, \tau_0, r^*(\tau_0, t_0, 0))$ is the maximal solution of (2.2.5) through $(\tau_0, r^*(\tau_0, t_0, 0))$. Since $r^*(t_0, t_0, 0) = 0$, we have

$$\lim_{\tau_0 \to t_0} \tilde{r}(t, \tau_0, r^*(\tau_0, t_0, 0)) = \tilde{r}(t, t_0, 0),$$

uniformly on J. By hypothesis $\tilde{r}(t, t_0, 0) \equiv 0$ which proves the continuity of $U(t, t_0, U_0)$, with respect to t_0 and the proof is complete.

2.4 Local Existence and Extremal Solutions

We begin by proving the local existence result corresponding to Peano's theorem for the IVP (2.2.1). For this purpose, we need the Ascoli-Arzela theorem suitably generalized in the present set up, which we state below, see Morales[1].

Since $K_c(\mathbb{R}^n)$ is a closed subset of $K(\mathbb{R}^n)$, and the family of compact sets included in a closed ball of $\mathbb{R}^n$ is compact, the following Ascoli-Arzela theorem holds.

Theorem 2.4.1 *If $\{U_n(t)\}$ is a sequence of equicontinuous and equibounded multimappings defined on an interval J, we can extract a subsequence that converges uniformly to a continuous multimapping $U(t)$ on J.*

Using Theorem 2.4.1, we can prove the local existence result for the IVP (2.2.1).

Theorem 2.4.2 *Assume that $F \in C[R_0, K_c(\mathbb{R}^n)]$ where $R_0 = J \times B[U_0, b]$, $B[U_0, b] = \{U \in K_c(\mathbb{R}^n) : D[U, U_0] \le b\}$ and $D[F(t, U), \theta] \le M$ on R_0. Then there exists at least one solution $U(t)$ for the IVP (2.2.1) on $J_0 = [t_0, t_0 + \alpha]$, where $\alpha = \min\{a, \frac{b}{M}\}$. Here, as before, $J = [t_0, t_0 + a]$, $a > 0$.*

Proof Let $U_0 \in C^1[[t_0 - \delta, t_0], K_c(\mathbb{R}^n)]$, $\delta > 0$ such that $U_0(t_0) = U_0$, $D[U_0(t), U_0] \le b$ and $D[D_H U_0(t), \theta] \le M$. Consider $0 < \epsilon \le \delta$ and define

$$\left[\begin{array}{lcl} U_\epsilon(t) & = & U_0(t), \quad t_0 - \delta \le t \le t_0, \\ U_\epsilon(t) & = & U_0 + \int_{t_0}^t F(s, U_\epsilon(s - \epsilon))ds, \;\; t_0 \le t \le t_0 + \alpha_1, \end{array} \right. \tag{2.4.1}$$

where $\alpha_1 = \min\{\alpha, \epsilon\}$. We then have, using the properties (1.3.8), and (1.7.11) of D,

$$\begin{aligned} D[U_\epsilon(t), U_0] &= D\left[U_0 + \int_{t_0}^t F(s, U_\epsilon(s-\epsilon))ds, \; U_0\right] \\ &= D\left[\int_{t_0}^t F(s, U_\epsilon(s-\epsilon))ds, \; \theta\right] \\ &\le \int_{t_0}^t D[F(s, U_\epsilon(s-\epsilon)), \theta]ds \\ &\le M(t - t_0) \le M\alpha_1 \le b. \end{aligned}$$

Further more,

$$D_H U_\epsilon(t) = F(t, U_\epsilon(t-\epsilon)) \text{ on } [t_0, t_0 + \alpha_1], \tag{2.4.2}$$

and therefore,

$$D[D_H U_\epsilon(t), \theta] \leq M. \tag{2.4.3}$$

If $\alpha_1 < \alpha$, we can use (2.4.1) to extend $U_\epsilon(t)$ satisfying the relations (2.4.2) and (2.4.3) on $[t_0 - \delta, t_0 + \alpha_2]$ where $\alpha_2 = \min\{\alpha_1, 2\epsilon\}$. Continuing this process we arrive at an a.e. differentiable function $U_\epsilon(t)$ such that (2.4.1), (2.4.2) and (2.4.3) hold on $[t_0 - \delta, t_0 + \alpha]$. Thus $U_\epsilon(t)$ forms a family of equicontinuous and uniformly bounded functions. By Theorem 2.4.1, we obtain a decreasing sequence $\{\epsilon_n\}$ such that $\epsilon_n \to 0$ as $n \to \infty$ and $U(t) = \lim_{n\to\infty} U_{\epsilon_n}(t)$ exists uniformly for $t_0 - \delta \leq t \leq t_0 + \alpha$. Since F is uniformly continuous, we show that $F(t, U_{\epsilon_n}(t - \epsilon_n)) \to F(t, U(t))$ uniformly, as $n \to \infty$ on J_0. This allows for term by term integration in (2.4.1) with $\epsilon = \epsilon_n$ and $\alpha_1 = \alpha$ which yields

$$U(t) = U_0 + \int_{t_0}^{t} F(s, U(s))ds, \quad t \in J_0.$$

Hence $U(t)$ is a solution of (2.2.1) on J_0 and the proof is complete.

In order to discuss the existence of extremal solutions for the IVP (2.2.1), we require a comparison result which demands introducing a partial order in the metric space $(K_c(\mathbb{R}^n), D)$.

We denote by $K(K^0)$ the subfamily of $K_c(\mathbb{R}^n)$ consisting of sets $U \in K_c(\mathbb{R}^n)$ such that any $u \in U$ is a nonnegative (positive) vector of $n-$components satisfying $u_i \geq 0 \;\; (u_i > 0)$ for $i = 1, 2, \cdots n$. Thus K is a cone in $K_c(\mathbb{R}^n)$ and K^0 is the nonempty interior of K. We can therefore induce a partial ordering in $K_c(\mathbb{R}^n)$ as follows.

Definition 2.4.1 *For any U and $V \in K_c(\mathbb{R}^n)$, if there exists a $Z \in K_c(\mathbb{R}^n)$ such that $Z \in K(K^0)$ and*

$$U = V + Z, \tag{2.4.4}$$

then, we write $U \geq V \;\; (U > V)$. Similarly, one can define $U \leq V \;\; (U < V)$.

We are now in a position to define the maximal and minimal solutions of (2.2.1).

Definition 2.4.2 *Let $R(t)$ be a solution of the set differential equation (2.2.1). Then we say that $R(t)$ is the maximal solution of (2.2.1) if, for every solution $U(t)$ of (2.2.1) existing on J_0, we have*

$$U(t) \leq R(t), \;\; t \in J_0. \tag{2.4.5}$$

We define the minimal solution of (2.2.1) similarly by reversing the inequality in (2.4.5).

We now state and prove the basic comparison result.

Theorem 2.4.3 *Assume that*

(i) $F \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$, *$F(t, U)$ is monotone nondecreasing in U for each $t \in \mathbb{R}_+$; that is, whenever $U \leq V$, we have* $F(t, U) \leq F(t, V)$, $t \in \mathbb{R}_+$;

(ii) $V, W \in C^1[\mathbb{R}_+, K_c(\mathbb{R}^n)]$,

$$D_H V < F(t, V) \text{ and } D_H W \geq F(t, W), \ t \in \mathbb{R}_+; \tag{2.4.6}$$

(iii) $V(t_0) < W(t_0)$.

Then,

$$V(t) < W(t), \quad t \geq t_0. \tag{2.4.7}$$

Proof Let $t_1 > 0$ be the supremum of all positive numbers $\delta > 0$ such that $V(t_0) < W(t_0)$ implies $V(t) < W(t)$ on $[t_0, \delta]$.

Clearly $t_1 > t_0$ and $V(t_1) \leq W(t_1)$. Now using the nondecreasing nature of $F(t, U)$ in U and the assumption (ii), we arrive at

$$D_H V(t_1) < F(t_1, V(t_1)) \leq F(t_1, W(t_1)) \leq D_H W(t_1).$$

It therefore follows that there exists an $\eta > 0$ satisfying

$$V(t) - W(t) > V(t_1) - W(t_1), \quad t_1 - \eta < t < t_1.$$

This implies that $t_1 > t_0$ cannot be the supremum due to the continuity of the functions involved, and hence the relation (2.4.7) holds, completing the proof.

Remark 2.4.1 *It is clear that the inequalities 2.4.6 can be replaced by*

$$D_H V \leq F(t, V) \text{ and } D_H W > F(t, W), \ t \in \mathbb{R}_+,$$

respectively, to get the conclusion of Theorem 2.4.3.

We are now ready to prove the existence of extremal solutions of (2.2.1).

Theorem 2.4.4 *Let the assumptions of Theorem 2.4.2 hold and suppose further $F(t, U)$ is nondecreasing in U for each $t \in J$. Then the IVP (2.2.1) possesses the extremal solutions on $J^0 = [t_0, t_0 + \alpha_0]$, where $\alpha_0 = \min\{a, \frac{b}{2M+b}\}$.*

Proof Let $\epsilon = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n) > 0$ be such that $\|\epsilon\| \leq \frac{b}{2}$. Then consider for each positive integer N, the following IVP for $t \in J$,

$$D_H U = F(t, U) + \frac{\epsilon}{N}, \quad U(t_0) = U_0 + \frac{\epsilon}{N}. \tag{2.4.8}$$

We observe that $F_N(t, U) = F(t, U) + \frac{\epsilon}{N}$ is defined and is continuous on $R_\epsilon = J \times B[U_0, \frac{b}{2}]$, where $B[U_0, \frac{b}{2}] = \{U \in K_c(\mathbb{R}^n) : D[U, U_0 + \frac{\epsilon}{N}] \leq \frac{b}{2}\}$. Also

$D[F_N(t,U),\theta] \leq M + \frac{b}{2}$ on R_ϵ. Hence we deduce from Theorem 2.4.2 that (2.4.8) has a solution $U_N(t,\epsilon) \in K_c(\mathbb{R}^n)$ on J^0. For $0 < \epsilon_2 < \epsilon_1 \leq \epsilon$, we see that

$$\begin{aligned} U_N(t_0,\epsilon_2) &< U_N(t_0,\epsilon_1), \\ D_H U_N(t,\epsilon_2) &\leq F(t,U_N(t,\epsilon_2)) + \frac{\epsilon_2}{N}, \\ D_H U_N(t,\epsilon_1) &> F(t,U_N(t,\epsilon_1)) + \frac{\epsilon_2}{N}, \quad \text{on} \quad J^0. \end{aligned}$$

We can apply the Theorem 2.4.3 (in fact Remark 2.4.1) to get

$$U_N(t,\epsilon_2) < U_N(t,\epsilon_1), \text{ on } J^0.$$

Since the family of functions $\{U_N(t,\epsilon)\}$ is equicontinuous and uniformly bounded on J^0, it follows by Theorem 2.4.1 that there exists a decreasing sequence $\{\frac{\epsilon}{N_k}\}$ such that $\frac{\epsilon}{N_k} \to 0$ as $k \to \infty$ and the uniform limit

$$R(t) = \lim_{k\to\infty} U_{N_k}(t,\epsilon), \tag{2.4.9}$$

exists on J^0. Obviously $R(t_0) = U_0$. The uniform continuity of F implies that $F(t, U_{N_k}(t,\epsilon))$ tends uniformly to $F(t,R(t))$ as $k \to \infty$, and thus term by term integration is applicable to

$$U_{N_k}(t,\epsilon) = U_0 + \frac{\epsilon}{N_k} + \int_{t_0}^{t} F(s, U_{N_k}(t,\epsilon))ds,$$

which in turn yields that the limit $R(t)$ is a solution of (2.2.1) on J^0.

We shall next show that $R(t)$ is the required maximal solution of IVP (2.2.1) on J^0. For this purpose, we observe that

$$\begin{aligned} U(t_0) &= U_0 < U_0 + \frac{\epsilon}{N} = U_N(t_0,\epsilon), \\ D_H U(t) &< F(t,U(t)) + \frac{\epsilon}{N}, \\ D_H U_N(t,\epsilon) &\geq F(t,U_N(t,\epsilon)) + \frac{\epsilon}{N}, \quad \text{on } J^0. \end{aligned}$$

We then obtain from Theorem 2.4.3 (or Remark 2.4.1), that

$$U(t) < U_N(t,\epsilon), \quad \text{on} \quad J^0.$$

The uniqueness of maximal solution $R(t)$ shows that $U_N(t,\epsilon)$ tends uniformly to $R(t)$ on J^0 as $N \to \infty$. This proves that $R(t)$ is the maximal solution of IVP (2.2.1). Similarly, one can prove the existence of the minimal solution $\rho(t)$ of IVP (2.2.1) by considering the IVP

$$D_H U = F(t,U) - \frac{\epsilon}{N}, \quad U(t_0) = U_0 - \frac{\epsilon}{N},$$

and proceeding with suitable change of arguments. Hence the proof is complete.

Having the notion of maximal solution and how to obtain it, one can prove the following comparison result analogous to the well known comparison result in ordinary differential system.

Theorem 2.4.5 *Assume that the conditions of Theorem 2.4.4 are satisfied. Suppose that* $M \in C^1[J, K_c(\mathbb{R}^n_+)]$ *and*

$$D_H M(t) \leq F(t, M(t)), \quad M(t_0) \leq U_0. \tag{2.4.10}$$

Then $M(t) \leq R(t)$ *on* J^0.

Proof Let $U_N(t,\epsilon)$ be a solution of (2.4.8) on J^0. Then

$$\begin{aligned} M(t_0) &< U_0 + \frac{\epsilon}{N}, \\ D_H U_N(t,\epsilon) &> F(t, U_N(t,\epsilon)) \text{ on } J^0. \end{aligned}$$

This together with (2.4.10), yields, by Theorem 2.4.3,

$$M(t) < U_N(t,\epsilon) \text{ on } J^0.$$

The last inequality, in view of (2.4.9) proves the assertion of the Theorem.

2.5 Monotone Iterative Technique

The method of lower and upper solutions coupled with the monotone iterative technique offers an effective and flexible mechanism to provide constructive existence results for nonlinear problems. In the development of this technique, one uses the fact that when the right hand side is not monotone, it can be made monotone by adding a suitable function. A generalization of this idea has been recently developed where one considers the situation when the right hand side can be split into the difference of two monotone functions. This unified setting provides very general results which cover several known cases of importance in addition to providing new results.

In this section, we develop the monotone iterative technique, in the same general set up. See Ladde, Lakshmikantham and Vatsala [1], Pao [1], and Köksal and Lakshmikantham [1] for details.

In the previous section we introduced a partial ordering in the metric space $(K_c(\mathbb{R}^n), D)$ and proved the comparison Theorem 2.4.3 for strict inequalities, which was essential for discussing the existence of extremal solutions for IVPs of set differential equations. We now prove first the following basic result on nonstrict set differential inequalities.

Theorem 2.5.1 *Assume that*

(i) $V, W \in C^1[\mathbb{R}_+, K_c(\mathbb{R}^n)],\ F \in C[R_+ \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)], F(t,X)$ *is monotone nondecreasing in* X *for each* $t \in \mathbb{R}_+$ *and*

$$D_H V \leq F(t,V),\ D_H W \geq F(t,W),\ t \in \mathbb{R}_+;$$

(ii) for any $X, Y \in K_c(\mathbb{R}^n)$ *such that* $X \geq Y,\ t \in \mathbb{R}_+$,

$$F(t,X) \leq F(t,Y) + L(X - Y)$$

for some $L > 0$.

Then $V(t_0) \leq W(t_0)$ *implies*

$$V(t) \leq W(t),\ t \geq t_0. \tag{2.5.1}$$

Proof Let $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n) > 0$ and define $\tilde{W} = W + \epsilon e^{2Lt}$. Since $V(t_0) \leq W(t_0) < \tilde{W}(t_0)$, it is enough to prove that

$$V(t) < \tilde{W}(t),\ t \geq t_0, \tag{2.5.2}$$

to arrive at the conclusion (2.5.1) in view of the fact $\epsilon > 0$ is arbitrary.

Let $t_1 > 0$ be the supremum of all positive numbers $\delta > 0$ such that $V(t_0) < \tilde{W}(t_0)$ implies $V(t) < \tilde{W}(t)$ on $[t_0, \delta]$. It is clear that $t_1 > t_0$ and $V(t_1) \leq \tilde{W}(t_1)$. From this follows, using the nondecreasing nature of F and condition (ii), that

$$\begin{aligned} D_H V(t_1) &\leq F(t_1, V(t_1)) \\ &\leq F(t_1, \tilde{W}(t_1)) \\ &\leq F(t_1, W(t_1)) + L(\tilde{W} - W) \\ &\leq D_H W(t_1) + L\varepsilon e^{2Lt_1} \\ &< D_H W(t_1) + 2L\varepsilon e^{2Lt_1} \\ &= D_H \tilde{W}(t_1). \end{aligned}$$

Consequently, it follows that there exists an $\eta > 0$ satisfying

$$V(t) - \tilde{W}(t) > V(t_1) - \tilde{W}(t_1),\ t_1 - \eta < t < t_1.$$

This implies that $t_1 > t_0$ cannot be the supremum in view of the continuity of the functions involved and therefore the relation (2.5.2) is true, which, in turn, leads to the conclusion (2.5.1). The proof is complete.

The following corollary is useful.

Corollary 2.5.1 *Let* $V, W \in C^1[\mathbb{R}_+, K_c(\mathbb{R}^n)]$, $\sigma \in C[\mathbb{R}_+, K_c(\mathbb{R}^n)]$. *Suppose that*

$$D_H V \leq \sigma,\ D_H W \geq \sigma,\ \textit{for}\ \ t \geq t_0.$$

Then $V(t) \leq W(t),\ t \geq t_0,$ *provided* $V(t_0) \leq W(t_0)$.

In order to develop the monotone iterative technique, we shall consider the following set differential equation,

$$D_H U = F(t, U) + G(t, U),\ U(0) = U_0 \in K_c(\mathbb{R}^n), \tag{2.5.3}$$

where $F, G \in C[J \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$ and $J = [0, T]$.

We need the following definition which gives various possible notions of lower and upper solutions relative to (2.5.3).

Definition 2.5.1 *Let* $V, W \in C^1[J, K_c(\mathbb{R}^n)]$. *Then* V, W *are said to be*

(a) natural lower and upper solutions of (2.5.3) if

$$D_H V \leq F(t, V) + G(t, V),\ D_H W \geq F(t, W) + G(t, W),\ t \in J; \tag{2.5.4}$$

(b) coupled lower and upper solutions of type I of (2.5.3) if

$$D_H V \le F(t,V) + G(t,W),\ D_H W \ge F(t,W) + G(t,V),\ t \in J; \quad (2.5.5)$$

(c) coupled lower and upper solutions of type II of (2.5.3) if

$$D_H V \le F(t,W) + G(t,V),\ D_H W \ge F(t,V) + G(t,W),\ t \in J; \quad (2.5.6)$$

(d) coupled lower and upper solutions of type III of (2.5.3) if

$$D_H V \le F(t,W) + G(t,W),\ D_H W \ge F(t,V) + G(t,V),\ t \in J. \quad (2.5.7)$$

We observe that whenever we have $V(t) \le W(t),\ t \in J$, if $F(t,X)$ is nondecreasing in X for each $t \in J$ and $G(t,Y)$ is nonincreasing in Y for each $t \in J$, the lower and upper solutions defined by (2.5.4) and (2.5.7) reduce to (2.5.6) and consequently, it is sufficient to investigate the cases (2.5.5) and (2.5.6).

We are now in a position to prove the following result.

Theorem 2.5.2 *Assume that*

(A_1) $V, W \in C^1[J, K_c(\mathbb{R}^n)]$ are coupled lower and upper solutions of type I relative to (2.5.3) with $V(t) \le W(t),\ t \in J$;

(A_2) $F, G \in C[J \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$, $F(t,X)$ is nondecreasing in X and $G(t,Y)$ is nonincreasing in Y, for each $t \in J$;

(A_3) F and G map bounded sets into bounded sets in $K_c(\mathbb{R}^n)$.

Then there exist monotone sequences $\{V_n(t)\}$, $\{W_n(t)\}$ in $K_c(\mathbb{R}^n)$ such that $V_n(t) \to \rho(t)$, $W_n(t) \to R(t)$ in $K_c(\mathbb{R}^n)$ and (ρ, R) are the coupled minimal and maximal solutions of (2.5.3) respectively, that is, they satisfy

$$D_H \rho = F(t,\rho) + G(t,R), \qquad \rho(0) = U_0,$$
$$D_H R = F(t,R) + G(t,\rho), \qquad R(0) = U_0, \ \textit{on} \ \ J.$$

Proof For each $n \ge 0$, define the unique solutions $V_{n+1}(t),\ W_{n+1}(t)$ by

$$D_H V_{n+1} = F(t,V_n) + G(t,W_n), \quad V_{n+1}(0) = U_0, \quad (2.5.8)$$
$$D_H W_{n+1} = F(t,W_n) + G(t,V_n), \quad W_{n+1}(0) = U_0,\ t \in J, \quad (2.5.9)$$

where $V(0) \le U_0 \le W(0)$. We set $V_0 = V,\ W_0 = W$.

Our aim is to prove

$$V_0 \le V_1 \le V_2 \le ... \le V_n \le W_n \le ... \le W_2 \le W_1 \le W_0,\ t \in J. \quad (2.5.10)$$

Since V_0 is the coupled lower solution of type I of (2.5.3), we have using the fact $V_0 \le W_0$ and the nondecreasing character of F,

$$D_H V_0 \le F(t,V_0) + G(t,W_0).$$

Also from (2.5.8), we get for $n = 0$,

$$D_H V_1 = F(t, V_0) + G(t, W_0).$$

Consequently, following the proof of Theorem 2.5.1, we arrive at $V_0 \leq V_1$ on J. A similar argument shows that $W_1 \leq W_0$ on J. We next prove $V_1 \leq W_1$ on J. For this purpose consider

$$\begin{aligned} D_H V_1 &= F(t, V_0) + G(t, W_0) \\ D_H W_1 &= F(t, W_0) + G(t, V_0), \\ V_1(0) &= W_1(0) = U_0. \end{aligned}$$

Then, the monotone nature of F and G respectively yield

$$D_H V_1 \leq F(t, W_0) + G(t, W_0),\ D_H W_1 \geq F(t, W_0) + G(t, W_0),\ t \in J.$$

We therefore have, by Corollary 2.5.1, $V_1 \leq W_1$ on J. As a result, we obtain

$$V_0 \leq V_1 \leq W_1 \leq W_0 \ \text{ on } J. \tag{2.5.11}$$

Assume that for some $j > 1$, we have

$$V_{j-1} \leq V_j \leq W_j \leq W_{j-1} \text{ on } J. \tag{2.5.12}$$

Then we show that

$$V_j \leq V_{j+1} \leq W_{j+1} \leq W_j \text{ on } J. \tag{2.5.13}$$

To do this, consider

$$\begin{aligned} &D_H V_j = F(t, V_{j-1}) + G(t, W_{j-1}), \quad V_j(0) = U_0, \\ &D_H V_{j+1} = F(t, V_j) + G(t, W_j) \geq F(t, V_{j-1}) + G(t, W_{j-1}), \quad t \in J. \end{aligned}$$

Here we have employed (2.5.12) and the monotone nature of F and G. Corollary 2.5.1 now gives $V_j \leq V_{j+1}$ on J. Similarly, we can get $W_{j+1} \leq W_j$ on J.

Next we show that $V_{j+1} \leq W_{j+1},\ t \in J$. We have from (2.5.8) and (2.5.9)

$$\begin{aligned} D_H V_{j+1} &= F(t, V_j) + G(t, W_j),\ V_{j+1}(0) = U_0, \\ D_H W_{j+1} &= F(t, W_j) + G(t, V_j),\ W_{j+1}(0) = U_0,\ t \in J. \end{aligned}$$

Using (2.5.12) and the monotone character of F and G, we arrive at

$$\begin{aligned} D_H V_{j+1} &\leq F(t, W_j) + G(t, W_j), \\ D_H W_{j+1} &\geq F(t, W_j) + G(t, W_j),\ t \in J, \end{aligned}$$

and therefore Corollary 2.5.1 implies that $V_{j+1} \leq W_{j+1},\ t \in J$. Hence (2.5.13) follows and consequently, by induction (2.5.10) is valid for all n. Clearly the sequences $\{V_n\}$, $\{W_n\}$ are uniformly bounded on J.

To show that they are equicontinuous, consider for any $s < t$, where $t, s \in J$,

$$D[V_n(t), V_n(s)] = D\left[U_0 + \int_0^t \{F(\xi, V_{n-1}(\xi)) + G(\xi, W_{n-1}(\xi))\}d\xi,\right.$$
$$\left. U_0 + \int_0^s \{F(\xi, V_{n-1}(\xi)) + G(\xi, W_{n-1}(\xi))\}d\xi\right]$$
$$= D\left[\int_0^t \{F(\xi, V_{n-1}(\xi)) + G(\xi, W_{n-1}(\xi)\}d\xi,\right.$$
$$\left.\int_0^s \{F(\xi, V_{n-1}(\xi) + G(\xi, W_{n-1}(\xi))\}d\xi\right]$$
$$\leq \int_s^t D[F(\xi, V_{n-1}(\xi)) + G(\xi, W_{n-1}(\xi)), \theta]d\xi \leq M|t-s|.$$

Here we have utilized the properties of integral and the metric D, together with the fact $F + G$ are bounded since $\{V_n\}, \{W_n\}$ are uniformly bounded. Hence $\{V_n(t)\}$ is equicontinuous on J. The corresponding Ascoli's Theorem 2.4.1 now gives a subsequence $\{V_{n_k}(t)\}$ which converges uniformly to $\rho(t) \in K_c(\mathbb{R}^n),\ t \in J$, and since $\{V_n(t)\}$ is a monotone nondecreasing sequence, the entire sequence $\{V_n(t)\}$ converges uniformly to $\rho(t)$ on J.

Similar arguments apply to the sequence $\{W_n(t)\}$ and $W_n(t) \to R(t)$ uniformly on J. It therefore follows, using the integral representation of (2.5.8) and (2.5.9) that $\rho(t), R(t)$ satisfy

$$\begin{aligned} D_H\rho(t) &= F(t, \rho(t)) + G(t, R(t)),\ \rho(0) = U_0, \\ D_H R(t) &= F(t, R(t)) + G(t, \rho(t)),\ R(0) = U_0,\ t \in J, \end{aligned} \tag{2.5.14}$$

and that

$$V_0 \leq \rho \leq R \leq W_0,\ t \in J. \tag{2.5.15}$$

Next we claim that (ρ, R) are coupled minimal and maximal solutions of (2.5.3), that is, if $U(t)$ is any solution of (2.5.3) such that

$$V_0 \leq U \leq W_0,\ t \in J, \tag{2.5.16}$$

then

$$V_0 \leq \rho \leq U \leq R \leq V_0,\ t \in J. \tag{2.5.17}$$

Suppose that for some n,

$$V_n \leq U \leq W_n \text{ on } J. \tag{2.5.18}$$

Then we have, using the monotone nature of $F,\ G$ and (2.5.18),

$$D_H U = F(t, U) + G(t, U) \geq F(t, V_n) + G(t, W_n), \quad U(0) = U_0,$$
$$D_H V_{n+1} = F(t, V_n) + G(t, W_n), \quad V_{n+1}(0) = U_0.$$

Corollary 2.5.1 yields $V_{n+1} \leq U$ on J. Similarly $W_{n+1} \geq U$ on J. Hence by induction (2.5.18) is true for all $n \geq 1$. Now taking the limit as $n \to \infty$, we get (2.5.17), proving the claim. The proof is therefore complete.

Corollary 2.5.2 *If, in addition to the assumptions of Theorem 2.5.2, F and G satisfy, whenever $X \geq Y,\ X, Y \in K_c(\mathbb{R}^n)$,*

$$F(t, X) \leq F(t, Y) + N_1(X - Y)$$

and

$$G(t, X) + N_2(X - Y) \geq G(t, Y),$$

where $N_1, N_2 > 0$. Then $\rho = R = U$ is the unique solution of (2.5.3).

Proof Since $\rho \leq R$, we have $R = \rho + m$ or $m = R - \rho$. Now

$$\begin{aligned} D_H\rho + D_H m &= D_H R = F(t, R) + G(t, \rho), \\ &\leq F(t, \rho) + N_1 m + G(t, R) + N_2 m, \\ &= D_H\rho + (N_1 + N_2)m, \end{aligned}$$

which means

$$D_H m \leq (N_1 + N_2)m, \ \ m(0) = 0,$$

which by Theorem 2.5.1 leads to $R \leq \rho$ on J, proving the uniqueness of $\rho = R = U$, completing the proof.

Several remarks are now in order.

Remark 2.5.1 *(1) In Theorem 2.5.2, if $G(t, Y) \equiv 0$, then we get a result when F is nondecreasing.*

(2) In (1) above, suppose that F is not nondecreasing, but $\tilde{F}(t, X) = F(t, X) + MX$ is nondecreasing in X for each $t \in J$, for some $M > 0$, then one can consider the IVP

$$D_H U + MU = \tilde{F}(t, U), \ U(0) = U_0,$$

where $\tilde{F}(t, X) = F(t, X) + MX$ to obtain the same conclusion as in (1). To see this, use the transformation $\tilde{U}(t) = U(t)e^{Mt}$ so that

$$D_H\tilde{U} = [D_H U + MU]e^{Mt} = \tilde{F}(t, \tilde{U}e^{-Mt})e^{Mt} \equiv F_0(t, \tilde{U}), \ \tilde{U}(0) = U_0. \tag{2.5.19}$$

Clearly (2.5.19) has $\tilde{V}(t) = V(t)e^{Mt}$ as a lower solution and $\tilde{W}(t) = W(t)e^{Mt}$ as an upper solution, and therefore we have the same conclusion as in (1). Here we assume that $D_H\tilde{U}$ exists.

(3) If $f(t, X) \equiv 0$ in Theorem 2.5.2, then we obtain the result for G nonincreasing.

(4) *If in* (3) *above, G is not monotone but there exists a function $\tilde{G}(t, U)$ that is nonincreasing in U for each $t \in J$, and a constant $M > 0$ such that*

$$G(t, U) = MU + \tilde{G}(t, U) \textit{and}$$

$$\tilde{G}(t, U) = G(t, U) - MU.$$

Then setting $U(t) = \tilde{U}(t)e^{Mt}$*, we obtain*

$$\begin{aligned} D_H\tilde{U} &= G_0(t, \tilde{U}), \\ \tilde{U}(0) &= U_0, \end{aligned} \tag{2.5.20}$$

where $G_0(t, \tilde{U}) = \tilde{G}(t, \tilde{U}e^{Mt})e^{-Mt}$*. In this case, we need to assume that* (2.5.20) *has coupled lower and upper solutions to get the same conclusion as in* (3)*.*

(5) *Suppose that in Theorem 2.5.2,* $G(t, Y)$ *is nonincreasing in* Y *and* $F(t, X)$ *is not monotone but* $\tilde{F}(t, X) = F(t, X) + MX, M > 0$ *is nondecreasing in* X*. Then we consider the IVP*

$$D_H\tilde{U} + MU = \tilde{F}(t, U) + G(t, U), \;\; U(0) = U_0. \tag{2.5.21}$$

The transformation in (2) *yields the conclusion by Theorem 2.5.2 in this case as well.*

(6) *If in Theorem 2.5.2,* F *is nondecreasing and* G *is not monotone then we suppose that there exists a function* $\tilde{G}(t, U)$ *and a constant* $M > 0$ *as in* (4) *and consider the IVP*

$$D_H\tilde{U} = F_0(t, \tilde{U}) + \tilde{G}(t, \tilde{U}), \;\; U(0) = U_0, \tag{2.5.22}$$

where $F_0(t, \tilde{U}) = F(t, \tilde{U}e^{Mt})e^{-Mt}$ *and* $G_0(t, \tilde{U}) = \tilde{G}(t, \tilde{U}e^{Mt})e^{-Mt}$*.*

(7) *If both* F *and* G *are not monotone in Theorem 2.5.2, then suppose that there are functions* $\tilde{F}(t, U), \tilde{G}(t, U)$*, and a constant* $M > 0$ *such that* $\tilde{F}(t, U) + \tilde{G}(t, U) \; + MU = F(t, U) + G(t, U)$*, where* $\tilde{F}(t, U)$ *is nondecreasing in* U *and* $\tilde{G}(t, U)$ *is nonincreasing in* U*. Now the transformation* $U(t) = \tilde{U}(t)e^{Mt}$ *gives,*

$$D_H\tilde{U} = F_0(t, \tilde{U}) + G_0(t, \tilde{U}), \;\; U(0) = U_0 \tag{2.5.22*}$$

where $F_0(t, \tilde{U}) = \tilde{F}(t, \tilde{U}e^{Mt})e^{-Mt}$*,* $G_0(t, \tilde{U}) = \tilde{G}(t, \tilde{U}e^{Mt})e^{-Mt}$*. Assuming that* (2.5.22*) *has coupled lower and upper solutions of type* I*, one gets the same conclusion by Theorem 2.5.2.*

Let us next consider utilizing the coupled lower and upper solutions of type II. In this case, we don't need to assume the existence of coupled lower and upper solutions of type II of (2.5.3) since one can construct them under the given assumptions. However, we have to pay a price to get monotone flows, by assuming certain conditions on the second iterates. Also, we get alternative sequences which are monotone but complicated.

Theorem 2.5.3 *Assume that* (A_2) *and* (A_3) *of Theorem 2.5.2 hold. Then for any solution* $U(t)$ *of (2.5.3) with* $V_0 \leq U \leq W_0$ *on* J*, we have the iterates* $\{V_n\}, \{W_n\}$ *satisfying*

$$V_0 \leq V_2 \leq ... \leq V_{2n} \leq U \leq V_{2n+1} \leq ... \leq V_3 \leq V_1 \;\text{ on } J, \tag{2.5.23}$$

$$W_1 \le W_3 \le ... \le W_{2n+1} \le U \le W_{2n} \le ... \le W_2 \le W_0 \ \text{on}\ J, \tag{2.5.24}$$

provided $V_0 \le V_2, W_2 \le W_0$ *on* J, *where the iterative schemes are given by*

$$D_H V_{n+1} = F(t, W_n) + G(t, V_n),\ V_{n+1}(0) = U_0, \tag{2.5.25}$$

$$D_H W_{n+1} = F(t, V_n) + G(t, W_n),\ W_{n+1}(0) = U_0,\ \text{on}\ J. \tag{2.5.26}$$

Moreover, the monotone sequences $\{V_{2n}\}, \{V_{2n+1}\}, \{W_{2n}\}, \{W_{2n+1}\}$ *in* $K_c(\mathbb{R}^n)$ *converge to* ρ, R, ρ^*, R^* *in* $K_c(\mathbb{R}^n)$ *respectively and verify*

$$\begin{aligned}
&D_H R = F(t, R^*) + G(t, \rho),\ R(0) = U_0,\\
&D_H \rho = F(t, \rho^*) + G(t, R),\ \rho(0) = U_0,\\
&D_H R^* = F(t, R) + G(t, \rho^*),\ R^*(0) = U_0,\\
&D_H \rho^* = F(t, \rho) + G(t, R^*),\ \rho^*(0) = U_0,\ \text{on}\ J.
\end{aligned}$$

Proof We shall first show that coupled lower and upper solutions V_0, W_0 of type II of (2.5.3) exist on J satisfying $V_0 \le W_0$ on J. For this purpose, consider the IVP

$$D_H Z = F(t, \theta) + G(t, \theta),\ Z(0) = U_0. \tag{2.5.27}$$

Let $Z(t)$ be the unique solution of (2.5.27) which exists on J. Define V_0, W_0 by

$$R_0 + V_0 = Z \text{ and } W_0 = Z + R_0,$$

where the positive vector $R_0 = (R_{01}, R_{02}, ..., R_{0n})$ is chosen sufficiently large so that we have $V_0 \le \theta \le W_0$ on J. Then using the monotone character of F and G, we arrive at

$$D_H V_0 = D_H Z = F(t, \theta) + G(t, \theta) \le F(t, W_0) + G(t, V_0),$$

$$V_0(0) = Z(0) - R_0 \le Z(0) = U_0.$$

Similarly, $D_H W_0 \ge F(t, V_0) + G(t, W_0),\ W_0(0) \ge U_0$. Thus V_0, W_0 are the coupled lower and upper solutions of type II of (2.5.3).

Let $U(t)$ be any solution of (2.5.3) such that $V_0 \le U \le W_0$ on J. We shall show that

$$V_0 \le V_2 \le U \le V_3 \le V_1,\ W_1 \le W_3 \le U \le W_2 \le W_0 \text{ on } J. \tag{2.5.28}$$

Since U is a solution of (2.5.3), we have, using the monotone character of F and G, and the fact $V_0 \le U \le W_0$,

$$D_H U = F(t, U) + G(t, U) \le F(t, W_0) + G(t, V_0),\ U(0) = U_0,$$

and V_1 satisfies

$$D_H V_1 = F(t, W_0) + G(t, V_0),\ V_1(0) = U_0,\ \text{on}\ J. \tag{2.5.29}$$

Hence Corollary (2.5.1) yields $U \le V_1$ on J. Similarly, $W_1 \le U$ on J.

Next we show that $V_2 \le U$ on J. Note that

$$D_H V_2 = F(t, W_1) + G(t, V_1),\ V_2(0) = U_0,$$

and because of monotonicity of F and G, we get

$$D_H U = F(t, U) + G(t, U) \ge F(t, W_1) + G(t, V_1),\ U(0) = U_0 \text{ on } J.$$

Corollary 2.5.1 therefore gives $V_2 \le U$ on J. A similar argument shows that $U \le W_2$ on J. Next we find utilizing the assumption $V_0 \le V_2, W_2 \le W_0$ on J and monotonicity of F and G,

$$D_H V_3 = F(t, W_2) + G(t, V_2) \le F(t, W_0) + G(t, V_0),\ V_3(0) = U_0 \text{ on } J.$$

This together with (2.5.29) shows by Corollary 2.5.1 that $V_3 \le V_1$, on J. In the same way one can show that $W_1 \le W_3$ on J. Also, employing similar reasoning, one can prove that $U \le V_3$ and $W_3 \le U$ on J, proving the relations (2.5.28).

Now assuming for some $n > 2$, the inequalities

$$V_{2n-4} \le V_{2n-2} \le U \le V_{2n-1} \le V_{2n-3},$$

$$W_{2n-3} \le W_{2n-1} \le U \le W_{2n-2} \le W_{2n-4}, \text{ on } J,$$

hold, it can be shown, employing similar arguments that

$$V_{2n-2} \le V_{2n} \le U \le V_{2n+1} \le V_{2n-1},$$

$$W_{2n-1} \le W_{2n+1} \le U \le W_{2n} \le W_{2n-2}, \text{ on } J.$$

Thus by induction (2.5.23) and (2.5.24) are valid for all $n = 0, 1, 2, \cdots$.

Since $V_n, W_n \in K_c(\mathbb{R}^n)$ for all n, employing a similar reasoning as in Theorem 2.5.2, we conclude that the limits $\lim_{n\to\infty} V_{2n} = \rho,\ \lim_{n\to\infty} V_{2n+1} = R,\ \lim_{n\to\infty} W_{n+1} = \rho^*$, and $\lim_{n\to\infty} W_{2n} = R^*$, exist, in $K_c(\mathbb{R}^n)$, uniformly on J. It therefore follows by suitable use of the integral representation (2.5.25)and (2.5.26) that $\rho, \rho^*,\ R,\ R^*$ satisfy corresponding set differential equations given in Theorem 2.5.3 on J. Also, from (2.5.23)and 2.5.24), we arrive at

$$\rho \le U \le R,\ \rho^* \le U \le R^* \text{ on } J.$$

The proof is therefore complete.

Corollary 2.5.3 *Under the assumptions of Theorem 2.5.3 if F and G satisfy the assumptions of Corollary 2.5.2, then $\rho = \rho^* = R = R^* = U$ is the unique solution of (2.5.3).*

Proof Let $q_1 + \rho = R,\ q_2 + \rho^* = R^*$ so that $q_1, q_2 \ge 0$ on J, since $\rho \le R$ and $\rho^* \le R^*$ on J. It then follows using the assumptions, that

$$D_H(q_1 + q_2) \le (N_1 + N_2)(q_1 + q_2),\ q_1(0) + q_2(0) = 0 \text{ on } J.$$

This implies that $q_1 + q_2 \leq 0$ on J and consequently, we get

$$U = \rho = R \text{ and } \rho^* = R^* = U \text{ on } J,$$

and this proves the claim of Corollary 2.5.2.

Theorem 2.5.3 also has several remarks which correspond to the remarks of Theorem 2.5.2. To repetition we do not list them again. For similar results which unify monotone iterative technique refer to Lakshmikantham and Köksal [1].

2.6 Global Existence

We consider the set differential equation

$$D_H U = F(t, U), \quad U(t_0) = U_0 \in K_c(\mathbb{R}^n), \tag{2.6.1}$$

where $F \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$. In this section, we shall investigate the global existence of solutions for $t \geq t_0$. Assuming local existence, we shall prove the following global existence result.

Theorem 2.6.1 *Assume that $F \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$ and*

$$D[F(t,U), \theta] \leq g(t, D[U, \theta]), \quad (t, U) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n),$$

where $g \in C[\mathbb{R}_+^2, \mathbb{R}_+]$, $g(t,w)$ is nondecreasing in w for each $t \in \mathbb{R}_+$ and the maximal solution $r(t, t_0, w_0)$ of (2.2.5) exists on $[t_0, \infty)$. Suppose further that F is smooth enough to guarantee local existence of solutions of (2.6.1) for any $(t_0, U_0) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n)$. Then the largest interval of existence of any solution $U(t, t_0, U_0)$ of (2.6.1) such that $D[U_0, \theta] \leq w_0$ is $[t_0, \infty)$.

Proof Let $U(t) = U(t, t_0, U_0)$ be any solution of (2.6.1) with $D[U_0, \theta] = w_0$, which exists on $[t_0, \beta)$, $t_0 < \beta < \infty$ and the value of β cannot be increased. Define $m(t) = D[U(t), \theta]$. Then Corollary 2.2.1 shows that

$$m(t) \leq r(t, t_0, D[U_0, \theta]) \quad t_0 \leq t < \beta. \tag{2.6.2}$$

For any $t_1,\ t_2$ such that $t_0 < t_1 < t_2 < \beta$, we have

$$\begin{aligned} D[U(t_1), U(t_2)] &= D\left[U_0 + \int_{t_0}^{t_1} F(s, U(s))ds,\ U_0 + \int_{t_0}^{t_2} F(s, U(s))ds\right] \\ &= D\left[\int_{t_1}^{t_2} F(s, U(s))ds, \theta\right] \\ &\leq \int_{t_1}^{t_2} D[F(s, U(s)), \theta]ds \\ &\leq \int_{t_1}^{t_2} g(s, D[U(s), \theta])ds. \end{aligned}$$

The relation (2.6.2) and the nondecreasing nature of $g(t,w)$ now yields

$$\begin{aligned} D[U(t_1),U(t_2)] &\leq \int_{t_1}^{t_2} g(s,r(s,t_0,w_0))ds \\ &= r(t_2,t_0,w_0) - r(t_1,t_0,w_0). \end{aligned} \tag{2.6.3}$$

Since $\lim_{t\to\beta^-} r(t,t_0,w_0)$ exists and is finite by hypothesis, taking the limit as $t_1,t_2 \to \beta^-$ and using the Cauchy criterion for convergence, it follows from (2.6.3) that $\lim_{t\to\beta^-} U(t,t_0,U_0)$ exists.

We define

$$U(\beta,t_0,U_0) = \lim_{t\to\beta^-} U(t,t_0,U_0)$$

and consider the initial value problem

$$D_H U = F(t,U), \quad U(\beta) = U(\beta,t_0,U_0).$$

By the assumed local existence, we see that $U(t,t_0,U_0)$ can be continued beyond β, contradicting our assumption that β cannot be continued. Hence every solution $U(t,t_0,U_0)$ of (2.6.1) such that $D[U_0,\theta] \leq w_0$ exists globally on $[t_0,\infty)$ and the proof is complete.

Remark 2.6.1 *Since $r(t,t_0,w_0)$ is nondecreasing because of the fact that $g(t,w) \geq 0$, if we assume that $r(t,t_0,w_0)$ is bounded on $[t_0,\infty)$ it follows that $\lim_{t\to\infty} r(t,t_0,w_0)$ exists and is finite. This, together with (2.6.2) which now holds for $t \in [t_0,\infty)$, implies that $\lim_{t\to\infty} U(t,t_0,U_0) = Y \in K_c(\mathbb{R}^n)$ exists.*

2.7 Approximate Solutions

We shall obtain an error estimate between the solutions and approximate solutions of IVP (2.6.1). Let us define the notion of approximate solutions.

Definition 2.7.1 *A function $V(t) = V(t,t_0,V_0,\epsilon)$, $\epsilon > 0$, is said to be an $\epsilon-$approximate solutions of IVP (2.6.1) if $V \in C[\mathbb{R}_+,K_c(\mathbb{R}^n)]$, $V(t_0,t_0,V_0,\epsilon) = V_0$ and*

$$D[D_H V(t), F(t,V(t))] \leq \epsilon, \quad t \geq t_0.$$

In case $\epsilon = 0$, $V(t)$ is a solution of (2.6.1).

Theorem 2.7.1 *Assume that $F \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$ and for $t \geq t_0$, $U,V \in K_c(\mathbb{R}^n)$,*

$$D[F(t,U),F(t,V)] \leq g(t,D[U,V]), \tag{2.7.1}$$

where $g \in C[\mathbb{R}_+^2,\mathbb{R}_+]$. Suppose that $r(t) = r(t,t_0,w_0,\epsilon)$ is the maximal solution of

$$w' = g(t,w) + \epsilon, \quad w(t_0) = w_0 \geq 0, \tag{2.7.2}$$

existing for $t \geq t_0$. *Let* $U(t) = U(t, t_0, U_0)$ *be any solution of (2.6.1) and* $V(t) = V(t, t_0, V_0, \epsilon)$ *be an* $\epsilon-$*approximate solution of IVP 2.6.1 existing for* $t \geq t_0$. *Then*

$$D[U(t), V(t)] \leq r(t, t_0, w_0, \epsilon), \quad t \geq t_0, \tag{2.7.3}$$

provided $D[U_0, V_0] \leq w_0$.

Proof We proceed, as in the proof of Theorem 2.2.3, with $m(t) = D[U(t), V(t)]$, until we arrive at

$$\begin{aligned} D^+m(t) \quad &\leq \quad \limsup_{h \to 0^+} D\left[\frac{U(t+h) - U(t)}{h}, F(t, U(t))\right] \\ &\quad + \limsup_{h \to 0^+} D\left[F(t, V(t)), \frac{V(t+h) - V(t)}{h}\right] \\ &\quad + D[F(t, U(t)), F(t, V(t))], \quad t \geq t_0. \end{aligned}$$

This implies, using the definition of approximate solution and (2.7.2), the differential inequality

$$D^+m(t) \leq g(t, m(t)) + \epsilon, \quad t \geq t_0,$$

and $m(t_0) \leq w_0$. The stated estimate follows from Theorem 1.4.1 in Lakshmikantham and Leela [1].

The following corollary provides the well-known error estimate between the solution and an $\epsilon-$approximate solution of (2.6.1).

Corollary 2.7.1 *The function* $g(t, w) = Lw, \ L > 0$, *is admissible in Theorem 2.7.1 to yield*

$$\begin{aligned} &D[U(t, t_0, U_0), V(t, t_0, V_0, \epsilon)] \\ &\leq \quad D[U_0, V_0] e^{L(t-t_0)} + \frac{\epsilon}{L}\left(e^{L(t-t_0)} - 1\right), \quad t \geq t_0. \end{aligned} \tag{2.7.4}$$

Proof Since (2.7.2) in this case reduces to

$$w' = Lw + \epsilon, \quad w(t_0) = D[U_0, V_0] \tag{2.7.5}$$

it is easy to obtain the estimate (2.7.4) by solving the linear differential equation (2.7.5).

2.8 Existence of Euler Solutions

We consider the initial value problem (IVP) for set differential equation

$$D_H U = F(t, U), \quad U(t_0) = U_0 \in K_c(\mathbb{R}^n), \tag{2.8.1}$$

where F is any function from $[t_0, T] \times K_c(\mathbb{R}^n) \to K_c(\mathbb{R}^n)$. Let

$$\pi = [t_0, t_1, ..., t_N \ = \ T] \tag{2.8.2}$$

be a partition of $[t_0, T]$.

Consider the interval $[t_0, t_1]$. Observe that the right hand side of the set differential equation

$$D_H U(t) = F(t_0, U_0), \quad U(t_0) = U_0,$$

on $[t_0, t_1]$ is a constant. Therefore, this IVP clearly has a unique solution $U(t) = U(t, t_0, U_0)$ on $[t_0, t_1]$.

Define the node $U_1 = U(t_1)$ and iterate next by considering on $[t_1, t_2]$ the IVP

$$D_H U = F(t_1, U_1) \quad U(t_1) = U_1 \in K_c(\mathbb{R}^n).$$

The next node is $U_2 = U(t_2)$ and we proceed this way till an arc $U_\pi = U_\pi(t)$ has been defined on all $[t_0, T]$. We employ the notation U_π to emphasize the role played by the particular partition π in determining U_π which is the Euler polygonal arc corresponding to the partition π. The diameter μ_π of the partition π is given by

$$\mu_\pi \ = \ \max\{t_i - t_{i-1} : \ 1 \leq i \leq N\}. \tag{2.8.3}$$

Definition 2.8.1 *By an Euler solution of (2.8.1) we mean any arc $U = U(t)$ which is the uniform limit of Euler polygonal arcs U_{π_J}, corresponding to some sequence π_J such that $\pi_J \to 0$, where this means the convergence of the diameters $\mu_{\pi_J} \to 0$ as $J \to \infty$.*

Clearly the corresponding number N_J of the partition points in π_J must then go to infinity. Obviously, the Euler arc satisfies the initial condition $U(t_0) = U_0$.

We can now prove the following result on existence of an Euler solution for (2.8.1).

Theorem 2.8.1 *Assume that*

(i) $D[F(t, A), \theta] \leq g(t, D[A, \theta])$, $(t, A) \in [t_0, T] \times K_c(\mathbb{R}^n)$, where $g \in C[[t_0, T] \times \mathbb{R}_+, \mathbb{R}_+]$ $g(t, u)$ is nondecreasing in (t, u);

(ii) the maximal solution $r(t) = r(t, t_0, u_0)$ of the scalar differential equation

$$u' \ = \ g(t, u), \quad u(t_0) = u_0 \ \geq 0, \tag{2.8.4}$$

exists on $[t_0, T]$.

Then,

(a) there exists at least one Euler solution $U(t) = U(t, t_0, U_0)$ to the IVP (2.8.1), which satisfies a Lipschitz condition;

(b) any Euler solution $U(t)$ of (2.8.1) satisfies the relation

$$D[U(t), U_0] \leq r(t, t_0, u_0) - u_0, \ \ t \in [t_0, T], \tag{2.8.5}$$

where $u_0 = D[U_0, \theta]$.

Proof Let π be the partition of $[t_0, T]$ defined by (2.8.2) and let $U_\pi = U_\pi(t)$ denote the corresponding arc with nodes of U_π represented by $U_0, U_1, ..., U_N$.

Let us set $U_\pi(t) = U_i(t)$ on $t_i \le t \le t_{i+1}, \quad i = 0, 1....., N-1$, and observe that $U_i(t_i) = U_i, \ i = 0, 1, ..., N-1$.

On the interval (t_i, t_{i+1}) we have

$$D[D_H U_\pi(t), \theta] = D[F(t_i, U_i), \theta] \le \ g(t_i, D[U_i, \theta]). \tag{2.8.6}$$

On the interval $[t_0, t_1]$, we obtain

$$\begin{aligned} D[U_1(t), U_0] &= D\left[U_0 + \int_{t_0}^{t} F(t_0, U_0)\, ds, \ U_0\right] \\ &= D\left[\int_{t_0}^{t} F(t_0, U_0)\, ds, \ \theta\right] \\ &\le \int_{t_0}^{t} D[F(t_0, U_0), \theta]\, ds \\ &\le \int_{t_0}^{t} g(t_0, D[U_0, \theta])\, ds \\ &\le \int_{t_0}^{t} g(s, r(s))\, ds \\ &= r(t, t_0, D[U_0, \theta]) - D[U_0, \theta] \\ &\le r(T, t_0, D[U_0, \theta]) - D[U_0, \theta] = M(\text{say}). \end{aligned}$$

Here we have employed the properties of the metric D and the integral, monotone character of $g(t, u)$ in (t, u) and the fact that $r(t, t_0, U_0) \ge 0$ is nondecreasing in t.

Similarly on $[t_1, t_2]$, we get

$$\begin{aligned} D[U_2(t), U_0] &= D\left[U_1 + \int_{t_1}^{t} F(t_1, U_1)\, ds, \ U_0\right] \\ &= D\left[U_0 + \int_{t_0}^{t_1} F(t_0, U_0)\, ds + \int_{t_1}^{t} F(t_1, U_1)\, ds, \ U_0)\right] \\ &= D\left[\int_{t_0}^{t_1} F(t_0, U_0)\, ds + \int_{t_1}^{t} F(t_1, U_1)\, ds, \theta\right] \\ &\le \int_{t_0}^{t_1} D[F(t_0, U_0), \theta]\, ds + \int_{t_1}^{t} D[F(t_1, U_1), \theta]\, ds \\ &\le \int_{t_0}^{t_1} g(s, r(s))\, ds + \int_{t_1}^{t} g(s, r(s)\, ds = \int_{t_0}^{t} g(s, r(s))\, ds \\ &\le r(T, t_0, D[U_0, \theta]) - D[U_0, \theta] = M(say). \end{aligned}$$

Proceeding in this way, we arrive at

$$D[U_i(t), U_0] \le r(T, t_0, \ D[U_0, \ \theta]) - D[U_0, \ \theta] = M, \text{ on } [t_i, t_{i+1}].$$

Hence it follows that

$$D[\, U_\pi(t),\; U_0] \le M, \text{ on } [t_0, T].$$

Also, from (2.8.6) we obtain,

$$D[D_H U_\pi(t), \theta] \le g(T, r(T)) = r'(T, t_0, D[U_0, \theta]) = k, \text{ say.}$$

Consequently, using similar arguments, we can find for $t_0 \le s \le t \le T$,

$$\begin{aligned}
D[U_\pi(t), U_\pi(s)] &\le \int_{t_0}^{t} D[F(\tau, U_\pi(\tau), \theta]\; d\tau + \int_{t_0}^{s} D[F(\tau, U_\pi(\tau)), \theta]\; d\tau \\
&\le \int_{t_0}^{t} g(\tau, r(\tau))\; d\tau + \int_{t_0}^{s} g(\tau, r(\tau))\; d\tau \\
&= \int_{s}^{t} g(\tau, r(\tau))\; d\tau \\
&= r(t) - r(s) = r'(\sigma) \mid t - s \mid \le k \mid t - s \mid
\end{aligned}$$

for some σ such that $s \le \sigma \le t$, proving $U_\pi(t)$ is Lipschitz of rank k on $[t_0, T]$.

Now, let π_J be a sequence of partitions of $[t_0, T]$ such that $\pi_J \to 0$, that is such that $\mu_{\pi_J} \to 0$ and therefore $N_J \to \infty$. Then the corresponding polygonal arcs U_{π_J} on $[t_0, T]$ all satisfy

$$U_{\pi_J}(t_0) = U_0,\; D[U_{\pi_J}(t), U_0] \le M \text{ and } D[D_H U_{\pi_J}(t),\; \theta] \;\le\; k \text{ on } [t_0, T].$$

Hence the family $\{U_{\pi_J}\}$ is equicontinuous and uniformly bounded, and, as a consequence, Ascoli-Arzela Theorem 2.4.1 guarantees the existence of a subsequence which converges uniformly to a continuous function $U(t)$ on $[t_0, T]$ and thus absolutely continuous on $[t_0, T]$. Thus, by definition, $U(t)$ is an Euler solution of the IVP (2.8.1) on $[t_0, T]$ and the claim of the theorem follows. The inequality (2.8.5) in part (b) is inherited by $U(t)$ from the sequence of polygonal arcs generating it when we identify T with t. Hence the proof is complete.

If $F(t, U)$ in (2.8.1) is assumed to be continuous, then one can show that $U(t)$ actually satisfies the IVP (2.8.1).

Theorem 2.8.2 *Under the assumptions of Theorem 2.8.1, if we suppose in addition that* $F \in C[[t_0, T] \times\; K_c(\mathbb{R}^n),\; K_c(\mathbb{R}^n)]$, *then* $U(t)$ *is a solution of (2.8.1).*

Proof Let $U_{\pi_J}(t)$ denote a sequence of polygonal arcs for IVP (2.8.1) converging uniformly to an Euler solution $U(t)$ on $[t_0, T]$. Clearly, the arcs $U_{\pi_J}(t)$ all lie in $\overline{B}(U_0, M)$ and satisfy a Lipschitz condition of the same rank k. Since a continuous function is uniformly continuous on compact sets, for any given $\epsilon > 0$, one can find a $\delta > 0$ such that

$$t, t^* \in [t_0, T],\;\; U, U^* \in \{U_{\pi_J}\},\;\; \mid t - t^* \mid < \delta,\; D[U, U^*] < \delta$$

$$\text{implies } D[F(t,U), F(t^*,U^*)] \ < \ \epsilon.$$

Let J be sufficiently large so that the partition diameter μ_{π_J} satisfies $\mu_{\pi_J} < \delta$ and $k\mu_{\pi_J} < \delta$. For any t, which is not one of the finitely many points at which $U_{\pi_J}(t)$ is a node, we have $D_H U_{\pi_J}(t) = F(\tilde{t}, U_{\pi_J}(\tilde{t}))$ for some $\tilde{t}$ within $\mu_{\pi_J} < \delta$ of t.

Since

$$D[U_{\pi_J}(t), U_{\pi_J}(\tilde{t})] \leq k\mu_{\pi_J} < \delta, \ \text{ we get}$$

$$D[D_H U_{\pi_J}(t), F(t, U_{\pi_J}(t))] = D[F(\tilde{t}, U_{\pi_J}(\tilde{t})), F(t, U_{\pi_J}(t))] \ < \ \epsilon.$$

It follows that for any $t \in [t_0, T]$, we obtain,

$$\begin{aligned}
& D\left[U_{\pi_J}(t), U_{\pi_J}(t_0) + \int_{t_0}^{t} F(\tau, U_{\pi_J}(\tau))d\tau\right] \\
&= D\left[U_{\pi_J}(t_0) + \int_{t_0}^{t} D_H U_{\pi_J}(\tau)\ d\tau, U_{\pi_J}(t_0) + \int_{t_0}^{t} F(\tau, U_{\pi_J}(\tau))\ d\tau\ \right] \\
&= D\left[\int_{t_0}^{t} D_H U_{\pi_J}(\tau)\ d\tau, \int_{t_0}^{t} F(\tau, U_{\pi_J}(\tau))d\tau\right] \\
&\leq \int_{t_0}^{t} D\left[D_H U_{\pi_J}(\tau), F(\tau, U_{\pi_J}(\tau))\right]\ d\tau \\
&\leq \epsilon(t - t_0) \leq \epsilon(T - t_0).
\end{aligned}$$

Letting $J \to \infty$, we have from this,

$$D[U(t),\ U_0 + \int_{t_0}^{t} F(\tau, U(\tau))\ d\tau] < \epsilon(T - t_0).$$

Since ϵ is arbitrary, it follows that

$$U(t) = U_0 + \int_{t_0}^{t} F(\tau, U(\tau))\ d\tau, \ \ t \in [t_0, T],$$

which implies that $U(t)$ is C^1 and therefore

$$D_H U(t) = F(t, U(t)), \ \ U(t_0) = U_0, \ \ t \in [t_0, T].$$

The proof is therefore complete.

Remark 2.8.1 *We can extend the notion of an Euler solution of (2.8.1) from the interval* $[t_0, T]$ *to* $[t_0, \infty)$, *if we define* F *and* g *on* $[t_0, \infty)$ *instead of* $[t_0, T]$ *and assume that the maximal solution* $r(t)$ *exists on* $[t_0, \infty)$ *and show that an Euler solution exists on every* $[t_0, T]$ *where* $T \in (t_0, \infty)$.

2.9 Proximal Normal and Flow Invariance

Let $\Omega \subset K_c(\mathbb{R}^n)$ be a nonempty, closed set. Assume that for any $U \in K_c(\mathbb{R}^n)$ such that U and Ω are disjoint, and for any $S \in \Omega$, there exists a $Z \in K_c(\mathbb{R}^n)$ such that $U = S + Z$. Then $U - S$ is called the Hukuhara difference. Suppose now that, for any $U \in K_c(\mathbb{R}^n)$ there is an element $S \in \Omega$ whose distance to U is minimal, that is,

$$D_0[U, \Omega] = \|U - S\| = \inf_{S_0 \in \Omega} \|U - S_0\|. \tag{2.9.1}$$

Then S is called a projection of U onto Ω. The set of all such elements is denoted by $\text{proj}_\Omega(U)$. The element $U - S$ will be called the proximal normal direction to Ω at S. Any nonnegative multiple $\xi = t(U - S),\ t \geq 0$, is called proximal normal to Ω at S. The set of all ξ obtained in this way is said to be proximal normal cone to Ω at S and is denoted by $N_\Omega^P(S)$.

Definition 2.9.1 *The system (Ω, F) is said to be weakly invariant provided that for all $U_0 \in \Omega$, there exists an Euler solution $U(t)$ of (2.8.1) on $[t_0, \infty)$ such that $U(t_0) = U_0$ and $U(t) \in \Omega,\ \ t \geq t_0$.*

Before proceeding further we introduce the following notation.

For any $A \in K_c(\mathbb{R}^n)$, we get $D[A, \theta] = \|A\| = \sup_{a \in A} \|a\|$ and we define for any $A, B \in K_c(\mathbb{R}^n)$,

$$< A, B >= \sup\{(a \cdot b) : a \in A, b \in B\}$$

so that we obtain the relation

$$\|A + B\|^2 \leq \|A\|^2 + \|B\|^2 + 2 < A, B > .$$

We can now prove the following result which offers sufficient conditions in terms of proximal normal for the weak invariance of the system (Ω, F).

Theorem 2.9.1 *Let F and g satisfy the conditions of Theorem 2.8.1 on $[t_0, \infty)$, $t_0 \geq 0$. Suppose that $U(t) = U(t, t_0, U_0)$ is an Euler solution of (2.8.1) on $[t_0, \infty)$, which lies in an open set $Q \subset K_c(\mathbb{R}^n)$. Suppose also that for every $(t, Z) \in [t_0, \infty) \times Q$, the proximal aiming condition is satisfied: namely, there exists an $S \in proj_\Omega(Z)$ such that*

$$2 < F(t, Z), Z - S > \ \leq \ q(t, D_0^2[Z, \Omega]), \tag{2.9.2}$$

where $q \in C[[t_0, \infty) \times \mathbb{R}_+, R]$. Assume that $r(t) = r(t, t_0, u_0)$ is the maximal solution of the scalar differential equation

$$u' = q(t, u),\ u(t_0) = u_0 \geq 0,$$

existing on $[t_0, \infty)$. Then we have

$$D_0^2[U(t), \Omega] \ \leq \ r(t, t_0, D_0^2[U_0, \Omega]),\ \ t_0 \leq t < \infty. \tag{2.9.3}$$

If, in addition $r(t, t_0, 0) \equiv 0$ then $U_0 \in \Omega$ implies $U(t) \in \Omega,\ \ t \geq t_0$, that is, the system (Ω, F) is weakly invariant.

Proof Let U_π be one polygonal arc in the sequence converging uniformly to U as per the definition of Euler solution of (2.8.1). We denote as before, its nodes at t_i by U_i, $i = 0, 1, ..., N$ and hence $U_0 = U(t_0)$. Let $U_\pi(t)$ be in Q for all $t_0 \le t \le T$, where $T \in (t_0, \infty)$. Accordingly, there exists for each i, an element $S_i \in \text{proj}_\Omega(U_i)$ such that

$$2 < F(t_i, U_i), U_i - S_i > \quad \le \quad q(t_i,\ D_0^2[U_i, \Omega]).$$

As in Theorem 2.8.1, letting $D[D_H U_\pi(t), \theta] \le k$, we find

$$D_0^2[U_1, \Omega] \quad \le \quad \|U_1 - S_0\|^2, \ \text{ since } \ S_0 \in \Omega.$$

We note that $U_1 = U_0 + Z_1$, where $Z_1 = F(t_0, U_0)(t_1 - t_0)$ and $U_0 = S_0 + Z_0$ and therefore, we get successively

$$\begin{aligned} D_0^2[U_1, \Omega] \ &\le \ \|Z_1 + Z_0\|^2 \ \le \ \|Z_1\|^2 + \|Z_0\|^2 + 2 < Z_1, Z_0 > \\ &\le \ k^2(t_1 - t_0)^2 + D_0^2[U_0, \Omega] + 2\int_{t_0}^{t_1} < D_H U_\pi(t_0), Z_0 > \ dt \\ &\le \ k^2(t_1 - t_0)^2 + D_0^2[U_0, \Omega] + 2\int_{t_0}^{t_1} < F(t_0, U_0), U_0 - S_0 > \ dt \\ &\le \ k^2(t_1 - t_0)^2 + D_0^2[U_0, \Omega] + q(t_0, D_0^2[U_0, \Omega])(t_1 - t_0). \end{aligned}$$

Since similar estimates hold at any node, we obtain,

$$D_0^2[U_i, \Omega] \le k^2(t_i - t_{i-1})^2 + D_0^2[U_{i-1}, \Omega] + q(t_{i-1}, D_0^2[U_{i-1}, \Omega])(t_i - t_{i-1}).$$

and therefore it follows that

$$\begin{aligned} D_0^2[U_i, \Omega] \ &\le \ D_0^2[U_0, \Omega] + k^2 \sum_{J=1}^{i} (t_J - t_{J-1})^2 \\ &\qquad + \sum_{J=1}^{i} q(t_{J-1}, D_0^2[U_{J-1}, \Omega])(t_J - t_{J-1}) \\ &\le \ D_0^2[U_0, \Omega] + k^2 \mu_\pi \sum_{J=1}^{i} (t_J - t_{J-1}) \\ &\qquad + \sum_{J=1}^{i} q(t_{J-1}, D_0^2[U_{J-1}, \Omega])(t_J - t_{J-1}) \\ &\le \ D_0^2[U_0, \Omega] + k^2 \mu_\pi (T - t_0) + \sum_{J=1}^{i} q(t_{J-1}, D_0^2[U_{J-1}, \Omega])(t_J - t_{J-1}). \end{aligned}$$

Consider now, the sequence $U_{\pi_J}(t)$ of polynomial arcs converging to $U(t)$. Since the last estimate is true at every node, $\mu_{\pi_J} \to 0$, as $J \to \infty$, and the same k applies to each U_{π_J}, we deduce in the limit the integral inequality

$$D_0^2[U(t), \Omega] \ \le \ D_0^2[U_0, \Omega] + \int_{t_0}^{t} q(s, D_0^2[U(s), \Omega])\, ds, \quad t_0 \le t \le T, \qquad (2.9.4)$$

for every $T \in (t_0, \infty)$. If we know that $q(t,u)$ is nondecreasing in u, then we can apply the theory of integral inequalities (see Theorem 1.9.2 in Lakshmikantham and Leela [1]), to arrive at

$$D_0^2[U(t),\Omega] \;\; \leq \;\; r(t,t_0,D_0^2[U_0,\Omega]), \;\; t \geq t_0. \tag{2.9.5}$$

If, on the other hand, $q(t,u)$ is not nondecreasing in u, we can obtain instead of (2.9.4), the following integral inequality for any $t_0 \leq t \leq t+h \leq T, \; h > 0$, employing similar reasoning,

$$D_0^2[U(t+h),\Omega] \;\; \leq \;\; D_0^2[U(t),\Omega] + \int_t^{t+h} q(s, D_0^2[U(s),\Omega])\, ds, \tag{2.9.6}$$

from which we obtain, setting $m(t) = D_0^2[U(t),\Omega]$, the differential inequality

$$D^+m(t) \leq q(t,m(t)), \; m(t_0) = D_0^2[U_0,\Omega], \tag{2.9.7}$$

where $D^+m(t)$ is a Dini derivative.

Applying now the theory of Differential inequalities (see Theorem 1.4.1 Lakshmikantham and Leela [1]), we arrive at the same estimate (2.9.5). If $r(t,t_0,0) \equiv 0$, then, supposing that $U_0 \in \Omega$ implies that $U(t) \in \Omega$ for $t \geq t_0$ and therefore the system (Ω, F) is weakly invariant as claimed. The proof is hence complete.

We shall next discuss the strong invariance of the system (Ω, F).

Definition 2.9.2 *The system (Ω, F) is said to be strongly invariant if every Euler solution $U(t)$ of (2.8.1) existing on $[t_0,\infty)$ for which $U(t_0) = U_0 \in \Omega$, satisfies $U(t) \in \Omega, \;\; t \geq t_0$.*

We can now prove the following result for strong invariance of Euler solutions of (2.8.1).

Theorem 2.9.2 *Let the assumptions of Theorem 2.8.1 hold. Suppose that F satisfies the generalized Lipschitz condition*

$$< F(t,A), C > \leq < F(t,B), C > \; + L\, \|C\|^2, \tag{2.9.8}$$

where there exists a $C \in K_c(\mathbb{R}^n)$ for those $A, B \in K_c(\mathbb{R}^n)$ such that $A = B + C$ and $L > 0$. Then, if the proximal normal condition

$$< F(t,B), C > \;\; \leq \;\; 0, \tag{2.9.9}$$

holds, we have the strong invariance of the system (Ω, F).

Proof Let $V(t)$ be any Euler solution of (2.8.1) on $[t_0,\infty)$ with $V(t_0) = U_0 \in \Omega$. By Theorem 2.8.1, there exists a $M > 0$ such that

$$D[V(t),U_0] \;\leq\; M \;\text{ on }\; [t_0,T], \text{ for any } \; T \in (t_0,\infty).$$

If U lies in $\overline{B}[U_0,M]$ and $S \in \mathrm{proj}_\Omega(U)$ then we have

$$D[S,U_0] \le D[S,U] + D[U,U_0] \le 2D[U,U_0] \le 2M,$$

which implies that $S \in \overline{B}[U_0, 2M]$. Let L be the Lipschitz constant for F in $\overline{B}[U_0, 2M]$ and consider any $U \in \overline{B}[U_0, M]$ and $S \in \text{proj}_\Omega(U)$. Then $U - S \in N_\Omega^P(S)$.

Consequently, using (2.9.8), we get

$$< F(t,U), U-S > \ \le \ \frac{1}{2}\, D_0^2[U,\Omega]. \tag{2.9.10}$$

The relation (2.9.10) is a special case of Theorem 2.9.1, with $q(t,w) = \frac{L}{2}w$ and therefore we obtain the conclusion from Theorem 2.9.1,

$$D_0^2[V(t),\Omega] \le D_0^2[U_0,\Omega]\ \ e^{\frac{L}{2}(t-t_0)},\ \ t \ge t_0. \tag{2.9.11}$$

Since $U_0 \in \Omega$ is assumed, we get readily from (2.9.11), $V(t) \in \Omega, t \ge t_0$ and the proof is complete.

2.10 Existence, Upper Semicontinuous Case

We shall consider the IVP for set differential equation

$$D_H U = F(t,U), \quad U(t_0) = U_0 \in K_c(\mathbb{R}^n), \tag{2.10.1}$$

where F is a function from $J \times K_c(\mathbb{R}^n)$ to $K_c(\mathbb{R}^n)$. By a solution of (2.10.1) we mean an absolutely continuous function $U : J \to K_c(\mathbb{R}^n)$, $U(t_0) = U_0$, whose derivative $D_H U(t)$, in the sense of Hukuhara, satisfies (2.10.1) almost everywhere (a.e.) on $J = [t_0, b]$, $t_0 \ge 0$, $b \in (t_0, \infty)$.

In what follows, by $F : J \times K_c(\mathbb{R}^n) \to K_c(\mathbb{R}^n)$, we mean that F is a single-valued function and when we write $F : J \times K_c(\mathbb{R}^n) \to \mathbb{R}^n$, it means that F is a multifunction defined on metric space $J \times K_c(\mathbb{R}^n)$ with values in $\mathbb{R}^n$. From the context it would be clear when we consider F as a single-valued function or a multifunction.

Let $F : J \times K_c(\mathbb{R}^n) \to \mathbb{R}^n$ be a multifunction with compact, convex values and V be a compact convex subset of $C(J, \mathbb{R}^n)$. Then a function $V(t) = \{x(t) : x(\cdot) \in V\}$, $t \in J$, is continuous from J to $K_c(\mathbb{R}^n)$. If the multifunction $t \to (t, V(t))$ is measurable then there exists a measurable selector of $F(t, V(t))$. For a compact convex subset $V \subset C(T, \mathbb{R}^n)$ we denote by $T(V_0, F, V)$, $V_0 = V(t_0)$ the collection of all functions $x : J \to \mathbb{R}^n$ representable as

$$x(t) = x_0 + \int_{t_0}^{t} v(s)ds,\ t \in J,\ x_0 \in V \tag{2.10.2}$$

where $v(s)$ is a Bochner integrable selector of $F(s, V(s))$. Let us list the following assumptions:

(i) $F : J \times K_c(\mathbb{R}^n) \to \mathbb{R}^n$ is a multifunction with compact, convex values and $F(t, A)$ is monotone nondecreasing with respect to $A \in K_c(\mathbb{R}^n)$;

(ii) the map $(t,U) \to F(t,U)$ is $\mathcal{L} \oplus \mathcal{B}(K_c(\mathbb{R}^n))$ is measurable;

(iii) the map $U \to F(t,U)$ is usc for almost all $t \in J$;

(iv) $\|F(t,U)\| \leq g(t,\|U\|)$, where $g: J \times \mathbb{R}_+ \to \mathbb{R}_+$ is a Caratheodory function integrally bounded on bounded subsets of $J \times \mathbb{R}_+$, $g(t,r)$ is monotone nondecreasing in r, a.e. in $t \in J$ and $r(t) = r(t,t_0,r_0)$ is the maximal solution of the scalar differential equation

$$r' = g(t,r), \quad r(t_0) = r_0 \geq 0, \tag{2.10.3}$$

existing on J.

We are now in a position to prove the following existence result.

Theorem 2.10.1 *Assume that conditions* (i) *to* (iv) *hold. Then for any* $U_0 \in K_c(\mathbb{R}^n)$*, there exists a solution* $U: J \to K_c(\mathbb{R}^n)$ *of the IVP* (2.10.1) *on* J.

Proof According to (ii), (iii) and Theorem 2.2 and Remark 2.1 in Tolstonogov [2] for any $\varepsilon > 0$, there exists a compact set $T_\varepsilon \subset J$, $\mu(J \backslash T_\varepsilon) \leq \varepsilon$ such that the restriction of $F(t,A)$ on $T_\varepsilon \times K_c(\mathbb{R}^n)$ is usc. Then for any continuous function $V: J \to K_c(\mathbb{R}^n)$ the restriction of $F(t,U(t))$ on T_ε is usc. Hence the multifunction $t \to F(t,U(t))$ is measurable.

Let $V \subset C(T,\mathbb{R}^n)$ be a compact set of $C(J,\mathbb{R}^n)$ then we can define the multivalued operator $T(V_0,F,V)$, $V_0 = V(0)$ by using (2.10.2). We note also that $T(V_0,F,V)$ is monotone relative to V in view of the monotonicity of $F(t,A)$ with respect to A assumed in (i).

Now let $r_0 = \max\{\|x\| : \ x \in V_0\}$ and $r(t) = r(t,t_0,r_0)$ be the maximal solution of (2.10.3) on J. We denote by U_0 a set of all absolutely continuous functions $x: J \to \mathbb{R}^n$, $x(t_0) \in V_0$, whose derivatives $x'(t)$ satisfy the estimate $\|\dot{x}(t)\| \leq r'(t)$ a.e. on J.

This implies that $\|x(t)\| \leq r(t)$, $t \in J$, for any $x(\cdot) \in U_0$ and therefore U_0 is a convex, bounded and equicontinuous subset of $C(T,\mathbb{R}^n)$.

Using Theorem 1.5 in Tolstonogov [1] we obtain that U_0 is a closed subset of $C(T,\mathbb{R}^n)$. Hence U_0 is a convex compact subset of $C(T,\mathbb{R}^n)$, $U_0(t_0) = V_0$ and a multifunction $U_0(t)$ is continuous from J to $K_c(\mathbb{R}^n)$.

Set $U_1 = T(V_0,F,U_0)$. By assumption (iv) we have

$$\|F(t,U_0(t))\| \leq g(t,\|U_0(t)\|) \leq g(t,r(t)) = r'(t) \text{ a.e.} \tag{2.10.4}$$

and for any $x(\cdot) \in T(V_0,F,U_0)$, it follows, using (2.10.4) that

$$\|x'(t)\| = \|v(t)\| \leq \|F(t,U_0(t))\| \leq r'(t), \text{ a.e.}$$

Hence $U_1 = T(V_0,F,U_0) \subset U_0$. By analogy with U_0 we can prove that U_1 is compact convex subset of $C(J,\mathbb{R}^n)$ and $U_1(t_0) = V_0$.

We now define $U_2 = T(V_0,F,U_1)$, and, since $U_1 \subset U_0$, it follows because of the monotone nature of $T(V_0,F,V)$ in V, that $T(V_0,F,U_1) \subset T(V_0,F,U_0)$. Thus $U_2 \subset U_1$.

Continuing this process, we obtain a sequence $\{U_k\},\ k \geq 1$, of compact convex sets of $C(J, \mathbb{R}^n)$ decreasing relative to the inclusion.

Hence $U = \bigcap_{k=0}^{\infty} U_k$ is a nonempty compact convex subset of $C(J, \mathbb{R}^n)$ and the sequence $\{U_k\}$ converges to U in Hausdorff metric on the space of all nonempty, closed, bounded sets of $C(J, \mathbb{R}^n)$. It is clear that $U(t_0) = V_0$. Since $U \subset U_k,\ k \geq 0$, we have

$$T(V_0, F, U) \subset T(V_0, F, U_{k-1}) \subset U_{k-1},\ k \geq 1,$$

and therefore

$$T(V_0, F, U) \subset \bigcap_{k=0}^{\infty} V_k = U. \tag{2.10.5}$$

It is easy to prove that $U(t) = \bigcap_{k=0}^{\infty} U_k(t), t \in J$, and the sequence $U_k(t), k \geq 1$, converges pointwise on J to $U(t)$.

Let $x(\cdot) \in U$. Then $x(\cdot) \in T(V_0, F, U_{k-1}), k \geq 1$. Hence $x(t)$ is absolutely continuous and

$$x'(t) \in F(t, U_{k-1}(t)) \text{ a.e.}, k \geq 1.$$

Then

$$x'(t) \in F(t, U(t)) \text{ a.e.} \tag{2.10.6}$$

due to (iii) and the convergence of $U_k(t),\ k \geq 1$, to $U(t)$ in $K_c(\mathbb{R}^n)$.

As a consequence of (2.10.6) we have

$$x(\cdot) \in T(V_0, F, U). \tag{2.10.7}$$

Combining (2.10.5), (2.10.7) we obtain

$$T(V_0, F, U) = U. \tag{2.10.8}$$

It is easy to see from (2.10.8) and (1.8.2) that

$$U(t) = V_0 + \int_0^t F(s, U(s))ds,\ t \in J. \tag{2.10.9}$$

Taking into consideration (1.8.3), (1.8.4) and (2.10.9) we obtain

$$D_H U(t) = F(t, U(t)) \text{ a.e. on } J$$

and $U(0) = V_0$ and the proof is complete.

Let $\Gamma : J \times \mathbb{R}^n \to \mathbb{R}^n$ be a multifunction. We need the following hypotheses $H(\Gamma) : \Gamma : J \times \mathbb{R}^n \to \mathbb{R}^n$ is a multifunction with compact values such that

(i) $(t, x) \to \Gamma(t, x)$ is $\mathcal{L} \oplus \mathcal{B}_{\mathbb{R}^n}$ measurable;

(ii) $x \to \Gamma(t, x)$ is usc a.e. on J;

(iii)
$$\|\Gamma(t, x)\| = \sup\{\|y\| :\ y \in \Gamma(t, x)\} \leq g(t, \|x\|). \tag{2.10.10}$$

Consider a multifunction $\Phi : J \times K_c(\mathbb{R}^n) \to \mathbb{R}^n$ defined by

$$\Phi(t,U) = \overline{co}\,\Gamma(t,U), \;\; U \in K_c(\mathbb{R}^n), \tag{2.10.11}$$

where $\overline{co}$ denotes closed convex hull.

Lemma 2.10.1 *Suppose that hypotheses $H(\Gamma)$ hold. Then there exists a multifunction $F : J \times K_c(\mathbb{R}^n) \to \mathbb{R}^n$ such that assumptions (i) to (iv) hold and*

$$F(t,U) = \Phi(t,U), \;\; U \in K_c(\mathbb{R}^n) \;\; \textit{a.e. on } J,$$

where $\Phi(t,U)$ is defined by (2.10.11).

Proof Let $T_k,\; k \geq 1$, be a sequence of closed subsets of J increasing with respect to inclusion such that $\mu(J \setminus \bigcup_{k=1}^{\infty} T_k) = 0$. For every $t \in T_k$ the multifunction $x \to \Gamma(t,x)$ is usc. Fix $k \geq 1$. Then the restriction of $\Gamma(t,x)$ on $T_k \times E$ is $\mathcal{L} \oplus \mathcal{B}_{\mathbb{R}^n}$ measurable.

From Theorem 2.2 in Tolstonogov [2] we know that for every $\varepsilon > 0$ there exists a closed set $\mathcal{T}_\varepsilon \subset T_n,\; \mu(T_k \setminus \mathcal{T}_\varepsilon) \leq \varepsilon$ such that the restriction of multifunction on $\mathcal{T}_\varepsilon \times \mathbb{R}^n$ is usc with respect to $(t,x) \in \mathcal{T}_\varepsilon \times \mathbb{R}^n$. Since the multifunction $x \to \Gamma(t,x)$ is usc for every $t \in \mathcal{T}_\varepsilon$, the set $\Gamma(t,A),\; A \in K_c(\mathbb{R}^n),\; t \in \mathcal{T}_\varepsilon$ is compact subset of $\mathbb{R}^n$.

Let us show that the restriction of $\Gamma(t,A)$ to $\mathcal{T}_\varepsilon \times K_c(\mathbb{R}^n)$ is usc. To this end we have to prove that restriction of $\Gamma(t,U)$ on $\mathcal{T}_\varepsilon \times \mathcal{M}$ is usc for any compact set $\mathcal{M} \subset K_c(\mathbb{R}^n)$, Tolstonogov [2].

Let $\mathcal{M} \subset K_c(\mathbb{R}^n)$ be a compact set. Then the set $M = \{\bigcup U;\; U \in \mathcal{M}\}$ is compact set of $\mathbb{R}^n$. Since $\Gamma(t,x)$ is usc on $\mathcal{T}_\varepsilon \times M$, there exists a compact set $Q \subset \mathbb{R}^n$ such that

$$\Gamma(t,x) \subset Q, \; t \in \mathcal{T}_\varepsilon, \; x \in M. \tag{2.10.12}$$

From (2.10.12) it follows that upper semicontinuity of restriction $\Gamma(t,U)$ on $\mathcal{T}_\varepsilon \times \mathcal{M}$ is equivalent to closedness of graph of restriction $\Gamma(t,U)$ on $\mathcal{T}_\varepsilon \times \mathcal{M}$.

Let $t_m \to t,\; t_m \in \mathcal{T}_\varepsilon,\; U_m \to U$ in $K_c(\mathbb{R}^n),\; U_m \in \mathcal{M}$, and $y_m \to y, y_m \in \Gamma(t,U_m)$. Then there exists $x_m \in U_m$ such that $y_m \in \Gamma(t,x_m)$. Since $x_m \in M,\; m \geq 1$, without loss of generality we can suppose that $x_m \to x$. It is clear that $x \in U$.

From the upper semicontinuity of $\Gamma(t,x)$ on $\mathcal{T}_\varepsilon \times M$ it follows that $y \in \Gamma(t,x) \subset \Gamma(t,U)$. It means that the restriction of $\Gamma(t,U)$ on $\mathcal{T}_\varepsilon \times \mathcal{M}$ has closed graph in $\mathcal{T}_\varepsilon \times K_c(\mathbb{R}^n) \times \mathbb{R}^n$. Hence the restriction of $\Gamma(t,U)$ on $\mathcal{T}_\varepsilon \times \mathcal{M}$ is usc. Then, restriction $\Phi(t,U) = \overline{co}\Gamma(t,U)$ on $\mathcal{T}_\varepsilon \times \mathcal{M}$. Therefore the restriction of $\Phi(t,U)$ on $\mathcal{T}_\varepsilon \times K_c(\mathbb{R}^n)$ is usc.

By using similar arguments we can prove that for every $t \in T_k,\; k \geq 1$, the multifunction $\Phi(t,U)$ is usc.

In this case Theorem 2.2 in Tolstonogov [2] tells us that the restriction of the multifunction $\Phi(t,U)$ on $T_k \times K_c(\mathbb{R}^n),\; k \geq 1$, is $\mathcal{L} \oplus \mathcal{B}_{K_c(\mathbb{R}^n)}$ measurable. Hence for every $t \in \bigcup_{k=1}^{\infty} T_k$ the multifunction $U \to \Phi(t,U)$ is usc and the restriction of the multifunction $\Phi(t,U)$ on $\bigcup_{k=1}^{\infty} T_k \times K_c(\mathbb{R}^n)$ is $\mathcal{L} \oplus \mathcal{B}_{K_c(\mathbb{R}^n)}$ measurable.

Set

$$F(t,U) = \Phi(t,U),\ t \in \bigcup_{k=1}^{\infty} T_k,\ U \in K_c(\mathbb{R}^n),$$

$$F(t,U) = \Theta,\ t \in J \setminus \bigcup_{k=1}^{\infty} T_k,\ U \in K_c(\mathbb{R}^n),$$

where Θ is the zero element of E, which is regarded as an one-point set. It is clear that the multifunction $U \to F(t,U)$ is usc for every $t \in J$ and is monotone nondecreasing with respect to $U \in K_c(\mathbb{R}^n)$.

Since $(T \setminus \bigcup_{k=1}^{\infty} T_k) \times K_c(\mathbb{R}^n)$ is a Borel subset of $T \times K_c(\mathbb{R}^n)$, the multifunction $F(t,A)$ is $\mathcal{L} \oplus \mathcal{B}_{K_c(\mathbb{R}^n)}$ measurable. From (2.10.10) it follows that multifunction $F(t,U)$ satisfies the assumption (iv). The theorem is proved.

Remark 2.10.1 *If multifunction $\Gamma(t,x)$ is $\mathcal{L} \oplus \mathcal{B}_{\mathbb{R}^n}$ measurable and the multifunction $x \to \Gamma(t,x)$ is usc for every $t \in J$, then the multifunction $\overline{co}\,\Gamma(t,A)$ is $\mathcal{L} \oplus \mathcal{B}_{K_c(R)}$ measurable and the multifunction $U \to \overline{co}\,\Gamma(t,U)$ is usc for every $t \in J$.*

Corollary 2.10.1 *Assume that the multifunction Γ satisfies hypotheses $H(\Gamma)$. Then there exists, for any $U_0 \in K_c(\mathbb{R}^n)$, a solution $U(t) = U(t,t_0,U_0) \in K_c(\mathbb{R}^n)$ of the IVP (2.10.1) with the multifunction $\Phi(t,U)$ defined by (2.10.11).*

By Lemma 2.10.1 without loss of generality we can consider that multifunction $\Phi(t,U)$ satisfies condition (i) to (iv). Then the Corollary follows from Theorem 2.10.1.

2.11 Notes and Comments

For the formulation of SDEs in the metric space $(K_c(\mathbb{R}^n), D)$ and the initiation of preliminary results of existence, uniqueness and extremal solutions with a suitable partial order, see Brandao Lopes Pinto, De Blasi, and Iervolino [1], and De Blasi and Iervolino [1]. For the case of SDEs in the metric space $(K_c(E), D)$, E being a Banach space, as a tool to prove existence results of multivalued differential inclusions without compactness and convexity and for several general results refer Tolstonogov [1]. The results of Section 2.2 and 2.3 are taken from Lakshmikantham, Leela and Vatsala [2]. For the contents of Section 2.4, see Lakshmikantham and Vasundhara Devi[1], which are analogous to the results of Brandao Lopes Pinto, De Blasi, and Iervolino [1] in the present framework.

Monotone Iterative Technique of Section 2.5 is from Lakshmikantham and Vatsala [1]. For earlier results on monotone iterative technique for ordinary and partial differential equations see Ladde, Lakshmikantham and Vatsala [1] and more recent general results Lakshmikantham and Köksal [1]. Sections 2.6 and 2.7 contain new results parallel to the corresponding theorems in ODE. For the existence of Euler solutions and flow invariance in Sections 2.8 and 2.9 in terms of nonsmooth analysis, see Gnana Bhaskar and Lakshmikantham [1]. For more information on nonsmooth analysis see Clarke, Ledyaev, Stern, and Wolenski [1]. For some generalizations refer to Gnana Bhaskar and Lakshmikantham [2, 4, 5]. The existence of solutions in USC case covered in 2.9 are adopted from Lakshmikantham and Tolstonogov [1].

Chapter 3

Stability Theory

3.1 Introduction

In this chapter, we investigate stability theory via Lyapunov-like functions. We shall also initiate the development of set differential systems using generalized metric spaces.

In Section 3.2, we prove a comparison theorem in terms of Lyapunov-like functions which serves as a vehicle for the discussion of the stability theory of Lyapunov. Some special cases of the comparison result are given which are useful later. Section 3.3 considers a global existence result for solutions of SDE, in terms of Lyapunov-like functions using Zorn's lemma. Simple stability results are established in Section 3.4. Here an example is worked out to demonstrate the problems encountered in the study of stability theory of SDE, in view of the fact diam $\|U(t)\|$ is nondecreasing as t increases. A way to avoid the problems generated is suggested, which leads in certain cases to choosing an appropriate subset of the solution. The stability criteria is obtained in the suggested format throughout. Section 3.5 discusses non-uniform stability criteria under less restrictive conditions employing perturbing Lyapunov-like functions. The criteria for boundedness of solutions is dealt with in Section 3.6, where various definitions of boundedness are also given. The results proved also include the method of perturbing Lyapunov functions.

In Section 3.7, we embark on initiating the study of set differential systems, the consideration of which leads to generalized metric spaces, in terms of which are proved comparison results utilizing vector Lyapunov-like functions. Section 3.8 develops the method of vector Lyapunov-like functions and stability criteria in this set up. Since we get a comparison differential system in this situation, the study of which is sometimes difficult, we provide certain results to reduce the study of comparison systems to a single comparison equation.

We begin to utilize, in Section 3.9, the tools of nonsmooth analysis to investigate the stability results via lower semicontinuous Lyapunov-like functions. Employing the connection between proximal normal theory and subdifferentials of lower semicontinuous functions, we provide the necessary framework in this section. In Section 3.10, we prove stability criteria for Euler solutions of SDE, utilizing the tools provided in Section 3.9. Notes and comments are given in section 3.10.

3.2 Lyapunov-like Functions

We consider the IVP for set differential equations

$$D_H U = F(t, U), \quad U(t_0) = U_0 \in K_c(\mathbb{R}^n), \tag{3.2.1}$$

where $F \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$. To investigate the qualitative behaviour of solutions of (3.2.1), the following comparison result in terms of a Lyapunov-like function is very important and can be proved via the theory of ordinary differential inequalities. Here the Lyapunov -like function serves as a vehicle to transform the set differential equation into a scalar comparison differential equation. Therefore, it is enough to consider the qualitative properties of the simpler comparison equation, under suitable conditions for the Lyapunov-like function.

We also require the IVP for scalar differential equation

$$w' = g(t, w), \quad w(t_0) = w_0 \geq 0, \tag{3.2.2}$$

where $g \in C[\mathbb{R}_+^2, \mathbb{R}]$.

Theorem 3.2.1 *Assume that*

(i) $V \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$ *and* $\mid V(t, A) - V(t, B) \mid \leq L\, D[A, B]$, *where* L *is the local Lipschitz constant , for* $A,\ B \in K_c(\mathbb{R}^n),\ t \in \mathbb{R}_+$*;*

(ii) $g \in C[\mathbb{R}_+^2, \mathbb{R}]$ *and for* $t \in \mathbb{R}_+,\ A \in K_c(\mathbb{R}^n)$,

$$D^+V(t, A) \equiv \limsup_{h \to 0+} \frac{1}{h}[V(t + h, A + hF(t, A)) - V(t, A)] \leq g(t, V(t, A)). \tag{3.2.3}$$

Then, if $U(t) = U(t, t_0, U_0)$ *is any solution of (3.2.1) existing on* $[t_0, T)$ *such that* $V(t_0, U_0) \leq w_0$, *we have*

$$V(t, U(t)) \leq r(t, t_0, w_0), \quad t \in [t_0, T), \tag{3.2.4}$$

where $r(t, t_0, w_0)$ *is the maximal solution of* (3.2.2) *existing on* $[t_0, T)$.

Proof Let $U(t) = U(t, t_0, U_0)$ be any solution of (3.2.1) existing on $[t_0, T)$. Define $m(t) = V(t, U(t))$ so that $m(t_0) = V(t_0, U_0) \leq w_0$. Now for small $h > 0$,

$$\begin{aligned} m(t + h) - m(t) &= V(t + h, U(t + h)) - V(t, U(t)) \\ &= V(t + h, U(t + h)) - V(t + h, U(t) + hF(t, U(t))) \\ &\quad + V(t + h, U(t) + hF(t, U(t))) - V(t, U(t)) \\ &\leq LD[U(t + h), U(t) + hF(t, U(t))] \\ &\quad + V(t + h, U(t) + hF(t, U(t))) - V(t, U(t)), \end{aligned}$$

using the Lipschitz condition given in (i). Thus

$$\begin{aligned} D^+ m(t) &= \limsup_{h \to 0^+} \frac{1}{h}[m(t+h) - m(t)] \\ &\leq D^+ V(t, U(t)) + L \limsup_{h \to 0^+} \frac{1}{h}[D[U(t+h), U(t) + hF(t, U(t))]]. \end{aligned}$$

Since

$$\frac{1}{h} D[U(t+h), U(t) + hF(t, U(t))] = \left[\frac{U(t+h) - U(t)}{h}, F(t, U(t))\right],$$

we find that,

$$\begin{aligned} &\limsup_{h \to 0^+} \frac{1}{h}[D[U(t+h), U(t) + hF(t, U(t))]] \\ &\quad = \limsup_{h \to 0^+} D\left[\frac{U(t+h) - U(t)}{h}, F(t, U(t))\right], \\ &\quad = D[D_H U(t), F(t, U(t))] \equiv 0, \end{aligned}$$

since $U(t)$ is a solution of (3.2.1). We therefore have the scalar differential inequality

$$D^+ m(t) \leq g(t, m(t)), \quad m(t_0) \leq w_0,$$

which yields, as before, the estimate

$$m(t) \leq r(t, t_0, w_0), \quad t \in [t_0, T).$$

This proves the claimed estimate of the Theorem.

The following Corollaries are useful.

Corollary 3.2.1 *The function $g(t, w) = 0$ is admissible in Theorem 3.2.1 to yield the estimate*

$$V(t, U(t)) \leq V(t_0, U_0), \quad t \in [t_0, T).$$

Corollary 3.2.2 *If, in Theorem 3.2.1, we assume that $g(t, w) = -\alpha w, \alpha > 0$, we get the relation*

$$V(t, U(t)) \leq V(t_0, U_0) \;\; exp(-\alpha(t - t_0)), \quad t \in [t_0, T).$$

Corollary 3.2.3 *If, in Theorem 3.2.1, we strengthen the assumption on $D^+ V(t, U)$ to*

$$D^+ V(t, U) + c[w(t, U)] \leq g(t, V(t, U)),$$

where $w \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$, $c \in \mathcal{K}$ and $g(t, w)$ is nondecreasing in w for each $t \in [t_0, T)$, then we get

$$V(t, U(t)) + \int_{t_0}^{t} c[w(s, U(s))]\, ds \; \leq r(t, t_0, V(t_0, U_0)), \quad t \in [t_0, T), \qquad (3.2.5)$$

whenever $w_0 = V(t_0, U_0)$.

Proof Set $L(t, U(t)) = V(t, U(t)) + \int_{t_0}^{t} c[w(s, U(s))]\, ds$ and note that

$$\begin{aligned} D^+L(t, U(t)) &\leq D^+V(t, U(t)) + c[w(t, U(t))] \\ &\leq g(t, V(t, U(t))) \leq g(t, L(t, U(t))), \end{aligned}$$

using the monotone character of $g(t, w)$. We then get immediately by Theorem 3.2.1 the desired estimate (3.2.4).

3.3 Global Existence

Employing the comparison Theorem 3.2.1, we shall prove the following global existence result.

Theorem 3.3.1 *Assume that*

(i) $F \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$, *F maps bounded sets onto bounded sets, and there exists a local solution of (3.2.1) for every* (t_0, U_0), $t_0 \geq 0$ *and* $U_0 \in K_c(\mathbb{R}^n)$*;*

(ii) $V \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$; $\mid V(t, A) - V(t, B) \mid \leq L\, D[A, B]$ *where L is the local Lipschitz constant, for* $A, B \in K_c(\mathbb{R}^n)$, $t \in \mathbb{R}_+$, $V(t, A) \to \infty$ *as* $D[A, \theta] \to \infty$ *uniformly for* $[0, T]$ *for every* $T > 0$ *and for* $t \in \mathbb{R}_+$, $A \in K_c(\mathbb{R}^n)$,

$$D^+V(t, A) \leq g(t, V(t, A)),$$

where $g \in C[\mathbb{R}_+^2, \mathbb{R}]$, $D^+V(t, A)$ *is as defined in Theorem 3.2.1.;*

(iii) The maximal solution $r(t) = r(t, t_0, w_0)$ *of (3.2.2) exists on* $[t_0, \infty)$*, and is positive whenever* $w_0 > 0$.

Then, for every $U_0 \in K_c(\mathbb{R}^n)$ *such that* $V(t_0, U_0) \leq w_0$*, the IVP (3.2.1) has a solution* $U(t)$ *on* $[t_0, \infty)$ *which satisfies*

$$V(t, U(t)) \ \leq\ r(t), \quad t \geq t_0.$$

Proof Let S denote the set of all functions U defined on $I_U = [t_0, c_U)$ with values in $K_c(\mathbb{R}^n)$ such that $U(t)$ is a solution of (3.2.1) on I_U and

$$V(t, U(t)) \leq r(t) \text{on } I_U.$$

Define a partial order $\leq$ on S as follows:

the relation $U \leq V$ implies $I_U \leq I_V$ and $V(t) = U(t)$ on I_U.

We shall first show that S is nonempty. By (i) there exists a solution $U(t)$ of (3.2.1) defined on $I_U = [t_0, c_U)$. Setting $m(t) = V(t, U(t)), t \in I_U$ and using assumption (ii), we get by Theorem 3.2.1, that

$$V(t, U(t)) \leq r(t), \quad t \in I_U,$$

where $r(t)$ is the maximal solution of (3.2.2). This proves that $U \in S$ and hence S is not empty.

If $(U_\beta)_\beta$ is a chain in $(S, \leq)$, then there is uniquely defined map V on $I_V = [t_0, \sup_\beta c_{U_\beta})$ that coincides with U_β on I_{U_β}. Clearly, $V \in S$ and therefore V is an upperbound of $(U_\beta)_\beta$ in $(S, \leq)$. Then Zorn's lemma assures the existence of a maximal element Z in $(S, \leq)$. The proof of the Theorem is complete if we show that $c_Z = \infty$.

Suppose that it is not true, so that $c_Z < \infty$. Since $r(t)$ is assumed to exist on $[t_0, \infty)$, $r(t)$ is bounded on I_Z. Since $V(t, A) \to \infty$ as $D[A, \theta] \to \infty$ uniformly in t on $[t_0, c_Z]$, the relation $V(t, Z(t)) \leq r(t)$ on I_Z implies that $D[Z(t), \theta]$ is bounded on I_Z. By (i), this shows that there is a constant $M > 0$ such that

$$D[F(t, Z(t)), \theta] \leq M \quad \text{on } \; I_Z.$$

We then have, for all $t_1, t_2 \in I_Z, \quad t_1 \leq t_2$,

$$D[Z(t_2), Z(t_1)] \leq \int_{t_1}^{t_2} D[F(s, Z(s)), \theta] \, ds \quad \leq \quad M(t_2 - t_1),$$

which proves that Z is Lipschitizian on I_Z and consequently has a continuous extension $Z_0(t)$ on $[t_0, c_Z]$.

By continuity, we get

$$Z_0(c_Z) = U_0 + \int_{t_0}^{c_Z} F(s, Z_0(s)) \, ds.$$

This implies that $Z_0(t)$ is a solution of (3.2.1) on $[t_0, c_Z]$ and obviously $V(t, Z_0(t)) \leq r(t), \quad t \in [t_0, c_Z]$.

Consider the IVP

$$D_H U = F(t, U), \quad U(c_Z) = Z_0(c_Z).$$

By the assumed local existence there is a solution $U_0(t)$ on $[c_Z, c_Z + \delta)$, $\delta > 0$.

Define

$$Z_1(t) = \left[\begin{array}{ll} Z_0(t) \text{ for } & t_0 \leq t \leq c_Z, \\ U_0(t) \text{ for } & c_Z \leq t \leq c_Z + \delta. \end{array} \right.$$

Clearly, $Z_1(t)$ is a solution of (3.2.1) on $[t_0, c_Z + \delta)$ and repeating the argument, we get

$$V(t, Z_1(t)) \leq r(t), \quad t \in [t_0, c_Z + \delta).$$

This contradicts the maximality of Z and hence $c_Z = \infty$. The proof is complete.

3.4 Stability Criteria

Having necessary comparison results in terms of Lyapunov-like functions, it is easy to establish the stability results for the set differential equations (SDE) (3.2.1) analogous to the original Lyapunov results and their extensions. However, in order to investigate the stability criteria for the trivial solution of (3.2.1), we need to employ, in a natural way, the measure $D[U(t), \theta] = \|U(t)\| =$

diam$U(t)$ for $t \geq t_0$, where $U(t) = U(t, t_0, U_0)$ is the solution of (3.2.1). This implies by Proposition 1.6.1 that the diam $U(t)$ is nondecreasing in $t \geq t_0$. This is due to the fact that, in the generation of the SDE from ordinary differential equations(ODEs), certain undesirable elements may enter the solution $U(t)$ and the measure to be employed, namely, $\|U(t)\|$ is therefore unsuitable to develop stability theory without some adjustment. Recall that SDE (3.2.1) reduces to ODE when $U(t)$ is single valued and SDE (3.2.1) can be generated from ODE as well. The latter is done as follows.

We let $F(t, A) = \overline{co}\, f(t, A),\ A \in K_c(\mathbb{R}^n)$, where $f \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ arising from the ODE

$$u' = f(t, u), \quad u(t_0) = u_0 \in \mathbb{R}^n. \tag{3.4.1}$$

Consequently, the solutions of $u(t)$ of ODE (3.4.1) are imbedded in the solution $U(t)$ of the SDE (3.2.1).

Since the cause of the problem in SDE is the requirement of the existence of Hukuhara differences in formulating SDE, we need to incorporate the Hukuhara difference in the initial conditions also, in order to match the behavior of solutions of SDE with the corresponding ODE. This is precisely what we plan to do.

Suppose that the Hukuhara difference exists for any given initial values $U_0,\ V_0 \in K_c(\mathbb{R}^n)$ so that $U_0 - V_0 = W_0$ is defined. Then we consider the stability of the solution $U(t, t_0, U_0 - V_0) = U(t, t_0, W_0)$ of (3.2.1). Before presenting the stability results, in this new set up, let us present some examples to illustrate our approach.

Consider the ODE

$$u' = -u, \;\; u(0) = u_0 \in \mathbb{R}, \tag{3.4.2}$$

and the corresponding SDE

$$D_H U = -U, \;\; U(0) = U_0 \in K_c(\mathbb{R}). \tag{3.4.3}$$

Since the values of the solution (3.4.3) are interval functions, the equation (3.4.3) can be written as,

$$[u_1',\ u_2'] = (-1)U = [-u_2, -u_1], \tag{3.4.4}$$

where $U(t) = [u_1(t), u_2(t)]$ *and* $U(0) = [u_{10}, u_{20}]$. The relation (3.4.4) is equivalent to the system of equations

$$\begin{aligned} u_1' &= -u_2, \qquad u_1(0) = u_{10}, \\ u_2' &= -u_1, \qquad u_2(0) = u_{20}, \end{aligned}$$

whose solution for $t \geq 0$, is

$$\begin{aligned} u_1(t) &= \tfrac{1}{2}[u_{10} + u_{20}]e^{-t} + \tfrac{1}{2}[u_{10} - u_{20}]e^{t}, \\ u_2(t) &= \tfrac{1}{2}[u_{20} + u_{10}]e^{-t} + \tfrac{1}{2}[u_{20} - u_{10}]e^{t}. \end{aligned} \tag{3.4.5}$$

Given $U_0 \in K_c(\mathbb{R})$, if there exists $V_0, W_0 \in K_c(\mathbb{R})$ such that $U_0 = V_0 + W_0$, then the Hukuhara difference $U_0 - V_0 = W_0$, exists. Let us choose

$$U_0 = [u_{10}, u_{20}], \quad V_0 = \frac{1}{2}[(u_{10} - u_{20}), (u_{20} - u_{10})],$$

so that

$$W_0 = \frac{1}{2}[(u_{10} + u_{20}), (u_{20} + u_{10})].$$

If $u_{10} \neq -u_{20}$, then we have for $t \geq 0$,

$$\begin{aligned} U(t, U_0) &= \frac{1}{2}[-(u_{20} - u_{10}),\ (u_{20} - u_{10})]e^t + \frac{1}{2}[(u_{10} + u_{20}),\ (u_{10} + u_{20})]e^{-t} \\ U(t, V_0) &= \frac{1}{2}[(u_{10} - u_{20}),\ (u_{20} - u_{10}]e^t, \text{ and} \\ U(t, W_0) &= \frac{1}{2}[(u_{10} + u_{20}),\ (u_{10} + u_{20})]e^{-t}. \end{aligned}$$

If on the other hand, $u_{10} = -u_{20}$, implies we choose $U_0 = [-d, d]$ with $d = u_{20}$. Then $U_0 = V_0$ and $W_0 = [0, 0]$. This choice eliminates the term with e^{-t} and we have only undesirable part of the solution. The other situation is to choose $U_0 = [c, c]$ for some c, which eliminates the term with e^t and retains only the desirable part of the solution compared with the ODE. Even when U_0 is chosen as $U_0 = [-d, d]$, we can find $V_0 = [c - d, c + d]$ for some c so that we have $V_0 = U_0 + W_0$ where $W_0 = [c, c]$.

We note that for any general initial value U_0, the solution of SDE (3.4.2) contains both desired and undesired parts compared to the solution of the ODE (3.4.2). In order to isolate the desired part of the solution $U(t, U_0)$ of (3.4.2) that matches the solution of the ODE (3.4.2), we need to use the initial values satisfying the desired Hukuhara difference of the given two initial values.

If, on the other hand, we have the SDE as

$$D_H U = \lambda(t)U, \quad U(0) = U_0, \tag{3.4.6}$$

which is generated by

$$u' = \lambda(t)u, \ \ u(0) = u_0 \tag{3.4.7}$$

where $\lambda(t) > 0$ is a real valued function from $\mathbb{R}_+ \to \mathbb{R}_+$ such that $\lambda \in L^1(\mathbb{R}_+)$, then we see that, with similar computation,

$$U(t, U_0) \ = \ U_0 \; exp \left[\int_0^t \ \lambda(s) \; ds \right] \ , \quad t \geq 0,$$

for any $U_0 \in K_c(\mathbb{R}^n)$.

Hence we get the stability of the trivial solution of (3.4.6). In this case, we note that the solutions of both equations (3.4.6) and (3.4.7) match, providing the same stability results. There is no necessity, therefore, to choose the initial values as before, since the undesirable part of the solution does not exist among solutions of (3.4.6). Consequently, it does not matter, whether we use the Hukuhara difference or not, we get the same conclusion. In order to be consistent and to take care of all the situations, most of the results in this chapter are formulated in terms of Hukuhara differences of initial values.

In order to consider the stability of the trivial solution of (3.2.1), let us assume that $F(t, \theta) = \theta$, the solutions are unique and exist for all $t \geq t_0$. Also, we assume, as a standard hypothesis, that the Hukuhara difference $U_0 - V_0 = W_0$ exists, since we suppose that $U_0 = V_0 + W_0$. Consequently, we utilize the solutions $U(t) = U(t, t_0, U_0 - V_0) = U(t, t_0, W_0)$, throughout.

Let us start with the following result on equi-stability.

Theorem 3.4.1 *Assume that the following hold:*

(i) $V \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$, $|V(t, U_1) - V(t, U_2)| \leq L\ D[U_1, U_2]$, $L > 0$ *and for* $(t, U) \in \mathbb{R}_+ \times S(\rho)$, *where* $S(\rho) = [U \in K_c(\mathbb{R}^n) : D[U, \theta] < \rho]$,

$$D^+V(t, U) \equiv \limsup_{h \to 0^+} \frac{1}{h}[V(t + h, U + hF(t, U)) - V(t, U)] \leq 0; \quad (3.4.8)$$

(ii) $b(\|U\|) \leq V(t, U) \leq a(t, \|U\|)$, *for* $(t, U) \in \mathbb{R}_+ \times S(\rho)$ *where* $b, a(t, .) \in \mathcal{K} = \{\sigma \in C[[0, \rho), \mathbb{R}_+] : \sigma(0) = 0$ *and* $\sigma(\omega)$ *is increasing in* $\omega\}$.

Then, the trivial solution of (3.2.1) is equi-stable.

Proof Let $0 < \varepsilon < \rho$ and $t_0 \in \mathbb{R}_+$, be given. Choose a $\delta = \delta(t_0, \varepsilon)$ such that

$$a(t_0, \delta) < b(\varepsilon). \quad (3.4.9)$$

We claim that with this δ, equi-stability holds. If not, there would exist a solution $U(t) = U(t, t_0, W_0)$ of (3.2.1) and $t_1 > t_0$ such that

$$\|U(t_1)\| = \varepsilon \text{ and } \|U(t)\| \leq \varepsilon < \rho, \quad t_0 \leq t \leq t_1, \quad (3.4.10)$$

whenever $\|W_0\| < \delta$. By Corollary 3.2.1, we then have

$$V(t, U(t)) \leq V(t_0, W_0), \quad t_0 \leq t \leq t_1.$$

Consequently, using (ii), (3.4.9) and (3.4.10), we arrive at the following contradiction:

$$b(\varepsilon) = b(\|U(t_1)\|) \leq V(t_1, U(t_1)) \leq V(t_0, W_0) \leq a(t_0, \|W_0\|) \leq a(t_0, \delta) < b(\varepsilon).$$

Hence equi-stability holds, completing the proof.

The next result provides sufficient conditions for equi-asymptotic stability. In fact, it gives exponential asymptotic stability.

Theorem 3.4.2 *Let the assumptions of Theorem 3.4.1 hold, except that the estimate (3.4.8) be strengthened to*

$$D^+V(t, U) \leq -\beta V(t, U), \quad (t, U) \in \mathbb{R}_+ \times S(\rho). \quad (3.4.11)$$

Then the trivial solution of (3.2.1) is equi-asymptotically stable.

Proof Clearly, the trivial solution of (3.2.1) is equi-stable. Hence taking $\varepsilon = \rho$ and designating $\delta_0 = \delta(t_0, \rho)$, we have by Theorem 3.4.1,

$$\|W_0\| < \delta_0 \text{ implies } \|U(t)\| < \rho, \quad t \geq t_0,$$

where $U(t) = U(t, t_0, W_0)$ as before.

Consequently, we get from the assumption (3.4.11), the estimate

$$V(t, U(t)) \leq V(t_0, W_0)\, exp[-\beta(t - t_0)], \quad t \geq t_0.$$

Given $\varepsilon > 0$, we choose $T = T(t_0, \varepsilon) = \frac{1}{\beta}\, ln \frac{a(t_0,\delta_0)}{b(\varepsilon)} + 1$. Then it is easy to see that,

$$b(\|U(t)\|) \leq V(t, U(t)) \leq a(t_0, \delta)\, e^{-\beta(t-t_0)} \; < \; b(\varepsilon), \quad t \geq t_0 + T.$$

The proof is complete.

We shall next consider the uniform stability criteria.

Theorem 3.4.3 *Assume that, for* $(t, U) \in \mathbb{R}_+ \times S(\rho) \cap S^c(\eta)$ *for each* $0 < \eta < \rho$, $V \in C[\mathbb{R}_+ \times S(\rho) \cap S^c(\eta), \mathbb{R}_+]$, *we have,*

$$|V(t, U_1) - V(t, U_2)| \leq L\, D[U_1, U_2], \;\; L > 0,$$

$$D^+V(t, U) \; \leq \; 0, \tag{3.4.12}$$

and

$$b(\|U\|) \leq V(t, U) \leq a(\|U\|), \quad a, b \in \mathcal{K}. \tag{3.4.13}$$

Then the trivial solution of (3.2.1) is uniformly stable.

Proof Let $0 < \varepsilon < \rho$ and $t_0 \in \mathbb{R}_+$ be given. Choose $\delta = \delta(\varepsilon) > 0$ such that $a(\delta) < b(\varepsilon)$. Then we claim that with this δ, uniform stability follows. If not, there would exist a solution $U(t)$ of (3.2.1), and a $t_2 > t_1 > t_0$ satisfying

$$\|U(t_1)\| = \delta, \;\; \|U(t_2)\| = \varepsilon \quad \text{and} \;\; \delta \leq \|U(t)\| \leq \epsilon < \rho, \;\; t_1 \; \leq \; t \; \leq \; t_2. \tag{3.4.14}$$

Taking $\eta = \delta$, we get from (3.4.12), the estimate

$$V(t_2, U(t_2)) \leq V(t_1, U(t_1)),$$

and therefore, (3.4.13) and (3.4.14), together with the definition of δ, yield

$$\begin{aligned} b(\varepsilon) \;\; &= \;\; b(\|U(t_2)\|) \leq V(t_2, U(t_2)) \leq V(t_1, U(t_1)) \\ &\leq \;\; a(\|U(t_1)\|) = a(\delta) < b(\varepsilon). \end{aligned}$$

This contradiction proves uniform stability, completing the proof.

Finally, we shall prove uniform asymptotic stability.

Theorem 3.4.4 *Let the assumptions of Theorem 3.4.3 hold except that (3.4.12) is strengthened to*

$$D^+V(t, U) \; \leq -c(\|U\|), \quad c \in \mathcal{K}. \tag{3.4.15}$$

Then the trivial solution of (3.2.1) is uniformly asymptotically stable.

Proof By Theorem 3.4.3, uniform stability follows. Now, for $\varepsilon = \rho$, we designate $\delta_0 = \delta_0(\rho)$. This means,

$$\|W_0\| \;<\; \delta_0 \;\text{ implies }\; \|U(t)\| \;<\; \rho, \quad t \geq t_0.$$

In view of the uniform stability, it is enough to show that there exists a t^* such that for $t_0 \leq t^* \leq t_0 + T$, where $T = 1 + \frac{a(\delta_0)}{c(\delta)}$,

$$\|U(t^*)\| \;<\; \delta. \tag{3.4.16}$$

If this is not true, $\delta \leq \|U(t)\|$, for $t_0 \leq t \leq t_0 + T$. Then, (3.4.15) gives,

$$V(t, U(t)) \leq V(t_0, W_0) - \int_{t_0}^{t} c(\|U(s)\|)\, ds, \quad t_0 \leq t \leq t_0 + T.$$

As a result, we have, in view of the choice of T,

$$0 \leq V(t_0 + T, U(t_0 + T)) \leq a(\delta_0) - c(\delta)T < 0$$

a contradiction. Hence there exists a t^* satisfying (3.4.16) and uniform stability then shows that

$$\|W_0\| < \delta_0 \;\text{ implies }\; \|U(t)\| < \varepsilon, \quad t \geq t_0 + T,$$

and the proof is complete.

3.5 Nonuniform Stability Criteria

In section 3.4, we discussed stability results parallel to Lyapunov's original theorems for set differential equations. We note that in proving nonuniform stability concepts, one needs to impose assumptions throughout $\mathbb{R}_+ \times S(\rho)$, whereas to investigate uniform stability notions it is enough to assume conditions in $\mathbb{R}_+ \times S(\rho) \cap S^c(\eta)$ for $0 < \eta < \rho$, where $S^c(\eta)$ denotes the complement of $S(\eta)$. The question therefore arises whether one can prove nonuniform stability notions under less restrictive assumptions. The answer is yes and one needs to employ the method of perturbing Lyapunov functions to achieve this. This is what we plan to do in this section.

We begin with the following result which provides nonuniform stability criteria under weaker assumptions.

Theorem 3.5.1 *Assume that*

(i) $V_1 \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$, $\;|\, V_1(t, U_1) - V_1(t, U_2) \,| \leq LD[U_1, U_2]$, $L_1 > 0$, $V_1(t, U) \leq a_0(t, \|U\|)$, *where* $a_0 \in C[\mathbb{R}_+ \times [0, \rho), \mathbb{R}_+]$ *and* $a_0(t, .) \in \mathcal{K}$ *for each* $t \in \mathbb{R}_+$.

(ii) $D^+V_1(t, U) \leq g_1(t, V_1(t, U))$, $(t, U) \in \mathbb{R}_+ \times S(\rho)$, *where* $g_1 \in C[\mathbb{R}_+^2, \mathbb{R}]$ *and* $g_1(t, 0) \equiv 0$;

(iii) for every $\eta > 0$*, there exists a* $V_\eta \in C[\mathbb{R}_+ \times S(\rho) \cap S^c(\eta), \mathbb{R}_+]$,

$$| V_\eta(t, U_1) - V_\eta(t, U_2) | \leq L_\eta D[U_1, U_2]$$

$$b(\|U\|) \leq V_\eta(t, U) \leq a(\|U\|), \; a, b \in \mathcal{K};$$

and

$$D^+V_1(t, U) + D^+V_\eta(t, U) \leq g_2(t, V_1(t, U) + V_\eta(t, U))$$

for $(t, U) \in \mathbb{R}_+ \times S(\rho) \cap S^c(\eta)$*, where* $g_2 \in C[\mathbb{R}_+^2, \mathbb{R}]$ *and* $g_2(t, 0) \equiv 0$*;*

(iv) the trivial solution $w_1 \equiv 0$ *of*

$$w_1' = g_1(t, w_1), \; w_1(t_0) = w_{10} \geq 0, \tag{3.5.1}$$

is equistable.

(v) the trivial solution $w_2 \equiv 0$ *of*

$$w_2' = g_2(t, w_2), \; w_2(t_0) = w_{20} \geq 0, \tag{3.5.2}$$

is uniformly stable.

Then, the trivial solution of (3.2.1) is equi-stable.

Proof Let $0 < \varepsilon < \rho$ and $t_0 \in \mathbb{R}_+$, be given. Since the trivial solution of (3.5.2) is uniformly stable, given $b(\varepsilon) > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta_0 = \delta_0(\varepsilon) > 0$ satisfying

$$0 < w_{20} < \delta_0 \text{ implying } w_2(t, t_0, w_{20}) < b(\varepsilon), \quad t \geq t_0, \tag{3.5.3}$$

where $w_2(t, t_0, w_{20})$ is any solution of (3.5.2). In view of the hypothesis on $a(w)$, there is a $\delta_2 = \delta_2(\varepsilon) > 0$ such that

$$a(\delta_2) < \frac{\delta_0}{2}. \tag{3.5.4}$$

Since the trivial solution of (3.5.1) is equi-stable, given $\frac{\delta_0}{2} > 0$ and $t_0 \in \mathbb{R}_+$, we can find a $\delta^* = \delta^*(t_0, \varepsilon) > 0$ such that

$$0 < w_{10} < \delta^* \text{ implies } w_1(t, t_0, w_{10}) < \frac{\delta_0}{2}, \quad t \geq t_0, \tag{3.5.5}$$

where $w_1(t, t_0, w_{10})$ is any solution of (3.5.1). Choose $w_{10} = V_1(t_0, W_0)$. Since $V_1(t, U) \leq a_0(t, \|U\|)$, we see that there exists $\delta_1 = \delta_1(t_0, \varepsilon) > 0$ satisfying

$$\|W_0\| < \delta_1 \text{ and } a_0(t_0, \|W_0\|) < \delta^*, \tag{3.5.6}$$

simultaneously. Define $\delta = min(\delta_1, \delta_2)$.

Then, we claim that

$$\|W_0\| < \delta \text{ implies } \|U(t)\| < \varepsilon, \; t \geq t_0, \tag{3.5.7}$$

for any solution $U(t) = U(t, t_0, W_0)$ of (3.2.1). If this is false, there would exist a solution $U(t)$ of (3.2.1) with $\|W_0\| < \delta$ and $t_1, t_2 > t_0$ such that

$$\|U(t_1)\| = \delta_2, \quad \|U(t_2)\| = \varepsilon,$$

and

$$\delta_2 \le \|U(t)\| \le \varepsilon < \rho, \tag{3.5.8}$$

for $t_1 \le t \le t_2$. We let $\eta = \delta_2$ so that the existence of a V_η satisfying hypothesis (iii) is assured. Hence setting

$$m(t) = V_1(t, U(t)) + V_\eta(t, U(t)), \quad t \in [t_1, t_2],$$

we obtain the differential inequality

$$D^+ m(t) \le g_2(t, m(t)), \quad t_1 \le t \le t_2,$$

which yields

$$V_1(t_2, U(t_2)) + V_\eta(t_2, U(t_2)) \le r_2(t_2, t_1, w_{20}), \tag{3.5.9}$$

where $w_{20} = V_1(t_1, U(t_1)) + V_\eta(t_1, U(t_1))$, and $r_2(t, t_1, w_{20})$ is the maximal solution of (3.5.2). We also have, because of assumptions (i) and (ii),

$$V_1(t_1, U(t_1)) \le r_1(t_1, t_0, w_{10}),$$

with $w_{10} = V_1(t_0, W_0)$ where $r_1(t, t_0, w_{10})$ is the maximal solution of (3.5.1).

By (3.5.5) and (3.5.6), we get

$$V_1(t_1, U(t_1)) < \frac{\delta_0}{2}. \tag{3.5.10}$$

Also, by (3.5.4), (3.5.8) and assumption (iii) , we arrive at

$$V_\eta(t_1, U(t_1)) \le a(\delta_2) < \frac{\delta_0}{2}. \tag{3.5.11}$$

Thus, (3.5.10) and (3.5.11) and the definition of w_{20} shows that $w_{20} < \delta_0$ which, in view of (3.5.3), shows that $w_2(t_2, t_1, w_{20}) < b(\varepsilon)$. It then follows from (3.5.9), $V_1(t, U) \ge 0$ and assumption (iii),

$$b(\varepsilon) = b(\|U(t_2)\|) \le V_1(t_2, U(t_2)) \le r_2(t_2, t_1, w_{20}) < b(\varepsilon).$$

This contradiction proves equi-stability of the trivial solution of (3.2.1) since (3.5.7) is then true. The proof is complete.

The next result offers conditions for equi-asymptotic stability.

Theorem 3.5.2 *Let the assumptions of Theorem 3.5.1 hold except that condition (ii) is strengthened to*

$$D^+ V_1(t, U) \le -c(w(t, U)) + g_1(t, V_1(t, U)), \ c \in \mathcal{K}, \tag{ii^*}$$

$w \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$,

$$| \, w(t, U_1) - w(t, U_2) \, | \leq \; N \; D[U_1, U_2], \;\; N > 0,$$

and $D^+ w(t, U)$ is bounded above or below. Then, the trivial solution of (3.2.1) is equi-asymptotically stable, if $g_1(t, w)$ is monotone nondecreasing in w and

$$w(t, U) \geq b_0(\|U\|), \;\; b_0 \in \mathcal{K}. \tag{3.5.12}$$

Proof By Theorem 3.5.1, the trivial solution of (3.2.1) is equi-stable. Hence letting $\varepsilon = \rho$ so that $\delta_0 = \delta(\rho, t_0)$, we get, by equi-stability

$$\|W_0\| < \delta_0 \text{ implies } \|U(t_0)\| < \rho, \;\; t \geq t_0.$$

We shall show that, for any solution $U(t)$ of (3.2.1) with $\|W_0\| < \delta_0$, we have $\lim_{t\to\infty} w(t, U(t)) = 0$. This and (3.5.12) imply $\lim_{t\to\infty} \|U(t)\| = 0$, completing the proof.

Suppose that $\lim_{t\to\infty} \sup w(t, U(t)) \neq 0$. Then there would exist two divergent sequences $\{t_i'\}, \; \{t_i''\}$ and a $\sigma > 0$ satisfying

$$(a) \;\; w(t_i', U(t_i')) = \frac{\sigma}{2}, \;\; w(t_i'', U(t_i'')) = \sigma \text{ and } w(t, U(t)) \geq \frac{\sigma}{2}, \;\; t \in (t_i'.t_i''),$$

or

$$(b) \;\; w(t_i', U(t_i')) = \sigma, \;\; w(t_i'', U(t_i'')) = \frac{\sigma}{2} \text{ and } w(t, U(t)) \geq \frac{\sigma}{2}, \;\; t \in (t_i', t_i'').$$

Suppose that $D^+ w(t, U(t)) \; \leq \; M$. Then using (a) we obtain

$$\frac{\sigma}{2} = \sigma - \frac{\sigma}{2} = w(t_i'', U(t_i'')) - w(t_i', U(t_i')) \; \leq \; M(t_i'' - t_i'),$$

which shows that $t_i'' - t_i' \; \geq \; \frac{\sigma}{2M}$ for each i. Hence by (ii^*) and Corollary 3.2.3 we have

$$V_1(t, U(t)) \; \leq \; r_1(t, t_0, w_{10}) - \sum_{i=1}^{n} \int_{t_i'}^{t_i''} \; c[w(s, U(s))] \, ds, \quad t \geq t_0.$$

Since $w_{10} = V_1(t_0, W_0) \; \leq \; a_0(t_0, \|W_0\|) \; \leq \; a_0(t_0, \delta_0) \; < \; \delta^*(\rho)$, we get from (3.5.5), $w_1(t, t_0, w_{10}) < \frac{\delta_0(\rho)}{2}, \; t \geq t_0$. we thus obtain

$$0 \; \leq \; V_1(t, U(t)) \; \leq \; \frac{\delta_0(\rho)}{2} - c(\frac{\sigma}{2}) \, \frac{\sigma}{2M} \, n.$$

For sufficiently large n, we get a contradiction and therefore $\lim \sup_{t\to\infty} w(t, U(t)) = 0$. Since $w(t, U) \geq b_0(\|U(t)\|)$ by assumption, it follows that $\lim_{t\to\infty} \|U(t)\| = 0$ and the proof is complete.

The following remarks are relevant.

Remark 3.5.1 *The functions $g_1(t,w) = g_2(t,w) \equiv 0$ are admissible in Theorem 3.5.1, and so the same conclusion can be reached. If $V_1(t,U) \equiv 0$ and $g_1(t,w) \equiv 0$, then we get uniform stability from Theorem 3.5.1. If, on the other hand, $V_\eta(t,U) \equiv 0$, $g_2(t,w) \equiv 0$ and $V_1(t,U) \geq b(\|U\|), b \in \mathcal{K}$, then Theorem 3.5.1 yields equi-stability. We note that known results on equi-stability require the assumption to hold everywhere in $S(\rho)$ and Theorem* 3.5.1 *relaxes such a requirement considerably by the method of perturbing Lyapunov functions.*

Remark 3.5.2 *The functions $g_1(t,w) \equiv g_2(t,w) \equiv 0$ are admissible in Theorem 3.5.2 to yield equi-asymptotic stability. Similarly, if $V_\eta(t,U) \equiv 0$, $g_2(t,w) \equiv 0$ with $V_1(t,U) \geq b(\|U\|)$, $b \in \mathcal{K}$, implies the same conclusion. If $V_1(t,U) \equiv 0$ and $g_1(t,w) \equiv 0$ in Theorem 3.5.1, to get uniform asymptotic stability, one needs to strengthen the estimate on $D^+V_\eta(t,U)$. This we state as a corollary.*

Corollary 3.5.1 *Suppose that the assumptions of Theorem 3.5.1 hold with $V_1(t,U) \equiv 0, g(t,w) \equiv 0$. Suppose further that*

$$D^+V_\eta(t,U) \leq -c[w(t,U)] + g_2(t, V_\eta(t,U)),\ (t,U) \in \mathbb{R}_+ \times S(\rho) \cap S^c(\eta), \quad (3.5.13)$$

where $w \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$, $w(t,U) \geq b(\|U\|)$, $c, b \in \mathcal{K}$ and $g_2(t,w)$ is nondecreasing in w. Then, the trivial solution of (3.2.1) is uniformly asymptotically stable.

Proof The trivial solution of (3.2.1) is uniformly stable by Remark 3.5.1 in the present case. Hence taking $\varepsilon = \rho$ and designating $\delta_0 = \delta(\rho)$, we have

$$\|W_0\| < \delta_0 \text{ implies } \|U(t)\| < \rho, \quad t \geq t_0.$$

To prove uniform attractivity, let $0 < \varepsilon < \rho$ be given. Let $\delta = \delta(\varepsilon) > 0$ be the number relative to ε in uniform stability. Choose $T = \frac{b(\rho)}{c(\delta)} + 1$. Then we shall show that there exists a $t^* \in [t_0, t_0 + T]$ such that $w(t^*, U(t^*)) < b(\delta)$ for any solution $U(t)$ of (3.2.1) with $\|W_0\| < \delta_0$.

If this is not true, $w(t, U(t)) \geq b(\delta)$, $t \in [t_0, t_0 + T]$. Now using the assumption (3.5.13) and arguing as in Corollary 3.2.3, we get

$$0 \leq V_\eta(t_0 + T, U(t_0 + T)) \leq r_2(t_0 + T, t_0, w_{20}) - \int_{t_0}^{t_0+T} w(s, U(s))\, ds.$$

This yields, since $r_2(t, t_0, w_{20}) < b(\rho)$ and the choice of T,

$$0 \leq V_\eta(t_0 + T, U(t_0 + T)) \leq b(\rho) - c(\delta)T < 0,$$

which is a contradiction. Hence there exists a $t^* \in [t_0, t_0 + T]$ satisfying $w(t^*, U(t^*)) < b(\delta)$, which implies $\|U(t)\| < \delta$. Consequently, it follows, by uniform stability that

$$\|W_0\| < \delta_0 \text{ implies } \|U(t)\| < \varepsilon, \quad t \geq t_0 + T,$$

and the proof is complete.

3.6 Criteria for Boundedness

We shall, in this section, investigate the boundedness of solutions of the set differential equation

$$D_H U = F(t, U),\ U(t_0) = U_0 \in K_c(\mathbb{R}^n), \tag{3.6.1}$$

where $F \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$. Corresponding to the definitions of various stability notions given in section 3.4, we also have boundedness concepts, which we define below.

Definition 3.6.1 *The solution of (3.6.1) is said to be*

(B1) equi-bounded*, if for any $\alpha > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\beta = \beta(t_0, \alpha) > 0$ such that*

$$\|W_0\| < \alpha \ \textit{implies}\ \|U(t)\| < \beta, \ \ t \geq t_0;$$

(B2) uniform-bounded*, if β in (B1)does not depend on t_0;*

(B3) quasi-equi-ultimately bounded for a bound B *if for each $\alpha \geq 0$, $t_0 \in \mathbb{R}_+$, there exists a $B > 0$ and a $T = T(t_0, \alpha) > 0$ such that*

$$\|W_0\| < \alpha \ \textit{implies}\ \|U(t)\| < B, \ \ t \geq t_0 + T.$$

(B4) quasi-uniform ultimately bounded *if T in (B3) is independent of t_0;*

(B5) equi-ultimately bounded, *if (B1) and (B3) hold simultaneously;*

(B6) uniform ultimately bounded *if (B2) and (B4) hold simultaneously;*

(B7) equi-Lagrange stable *if (B1) and (S3) hold;*

(B8) uniformly Lagrange stable *if (B2) and (S4) hold.*

Using the comparison results of section 3.2, we shall prove simple boundedness results.

Theorem 3.6.1 *Assume that*

(i) $V \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$, $\mid V(t, U_1) - V(t, U_2) \mid \leq LD[U_1, U_2]$, $L > 0$, *and for* $(t, U) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n), D^+V(t, U) \leq 0$*;*

(ii) $b(\|U\|) \leq V(t, U) \leq a(t, \|U\|)$ *, for* $(t, U) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n)$ *where* $b, a(t, .) \in \mathcal{K} = [\sigma \in C[\mathbb{R}_+, \mathbb{R}_+] : \sigma(\omega)$ *is increasing in* ω *and* $\sigma(w) \to \infty$ *as* $w \to \infty]$.

Then, (B1) holds.

Proof Let $0 < \alpha$ and $t_0 \in \mathbb{R}_+$, be given. Choose $\beta = \beta(t_0, \alpha)$ such that

$$a(t_0, \alpha) < b(\beta). \tag{3.6.2}$$

With this β, (B1) holds. If this is not true, there would exist a solution $U(t) = U(t, t_0, W_0)$ of (3.6.1) and a $t_1 > t_0$ such that

$$\|U(t_1)\| = \beta \quad \text{and} \quad \|U(t)\| \leq \beta, \;\; t_0 \leq t \leq t_1.$$

Assumption (i) and Corollary 3.2.1 show that

$$V(t, U(t)) \leq V(t_0, W_0), \quad t_0 \leq t \leq t_1.$$

As a result, condition (ii) and (3.6.2) yield

$$\begin{aligned} b(\beta) &= b(\|U(t_1)\|) \leq V(t_1, U(t_1)) \leq V(t_0, W_0) \\ &\leq a(t_0, \|W_0\|) < a(t_0, \alpha) < b(\beta). \end{aligned}$$

This contradiction proves (B1) and we are done.

For uniform boundedness, the following result is obtained under weaker assumptions.

Theorem 3.6.2 *Assume that*

(i) $V \in C[\mathbb{R}_+ \times S^c(\rho), \mathbb{R}_+]$, *where* ρ *may be large;* $|V(t, U_1) - V(t, U_2)| \leq LD[U_1, U_2]$, *and for* $(t, U) \in \mathbb{R}_+ \times S^c(\rho)$, $D^+V(t, U) \leq 0$*;*

(ii) $b(\|U\|) \leq V(t, U) \leq a(\|U\|)$, *for* $(t, U) \in \mathbb{R}_+ \times S^c(\rho)$ *where* $a, b \in \mathcal{K}$, *which are defined only on* $[\rho, \infty)$.

Then, (B2) holds.

Proof The proof is similar to the proof of Theorem 3.6.1 except that the choice of β is now made so that $a(\alpha) < b(\beta)$ and consequently β is independent of t_0. Also, $\alpha > \rho$ for the proof since the assumptions are only for $S^c(\rho)$. However, if $0 < \alpha \leq \rho$, we can take $\beta = \beta(\rho)$ and again the proof follows.

We shall give a typical result that offers conditions for equi-ultimate boundedness, that is for (B5).

Theorem 3.6.3 *Let the assumptions of Theorem 3.6.1 hold except that we strengthen the estimate on* $D^+V(t, U)$ *as*

$$D^+V(t, U) \;\leq\; -\eta V(t, U), \; \eta > 0, \; (t, U) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n), \tag{3.6.3}$$

and suppose that condition (ii) holds for $\|U\| \geq B$. *Then (B5) holds.*

Proof Clearly (B1) is obtained from Theorem 3.6.1. Hence

$$\|W_0\| < \alpha \;\text{ implies }\; \|U(t)\| < \beta, t \geq t_0.$$

Now (3.6.3) yields the estimate

$$V(t, U(t)) \leq V(t_0, W_0)e^{-\eta(t - t_0)}, \quad t \geq t_0. \tag{3.6.4}$$

Let $T = \frac{1}{\eta} \ln \frac{a(t_0,\alpha)}{b(B)}$ and suppose that for $t \geq t_0 + T,\ \|U(t)\| \geq B$. Then, we get from (3.6.4)

$$b(B) \leq b(\|U(t)\|) \leq V(t, U(t)) < a(t_0, \alpha)e^{-\eta T} = b(B).$$

This contradiction proves (B5) and the proof is complete.

Finally, we shall provide a result on nonuniform boundedness property using the method of perturbing Lyapunov functions.

Theorem 3.6.4 *Assume that*

(i) $\rho > 0,\ V_1 \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+],\ V_1$ *is bounded for* $(t, U) \in \mathbb{R}_+ \times \partial S(\rho)$, *and*

$$|V_1(t, U_1) - V(t, U_2)| \leq L_1 D[U_1, U_2], \quad L_1 > 0,$$

$$\begin{aligned} D^+V_1(t, U) &= \limsup_{h \to 0+} \frac{1}{h}[V_1(t + h, U + hF(t, U)) - V_1(t, U)] \\ &\leq g_1(t, V_1), \ \ (t, U) \in \mathbb{R}_+ \times S^c(\rho), \end{aligned}$$

where $g_1 \in C[\mathbb{R}_+^2, \mathbb{R}]$;

(ii) $V_2 \in C[\mathbb{R}_+ \times S^c(\rho), \mathbb{R}_+]$,

$$b(\|U\|) \leq V_2(t, U) \leq a(\|U\|), \quad a, b \in \mathcal{K},$$

$$D^+V_1(t, U) + D^+V_2(t, U) \leq g_2(t, V_1(t, U) + V_2(t, U)), \quad g_2 \in C[\mathbb{R}_+^2, \mathbb{R}],$$

(iii) the scalar differential equations

$$w_1' = g_1(t, w_1), \quad w_1(t_0) = w_{10} \geq 0, \tag{3.6.5}$$

and

$$w_2' = g_2(t, w_2), \quad w_2(t_0) = w_{20} \geq 0, \tag{3.6.6}$$

are equi-bounded and uniformly bounded respectively.

Then the system (3.6.1) is equi-bounded.

Proof Let $B_1 > \rho$ and $t_0 \in \mathbb{R}_+$, be given. Let

$$\begin{aligned} \alpha_1 &= \alpha_1(t_0, B_1) = max\{\alpha_0, \alpha^*\}, \\ \text{where } \alpha_0 &= max\{V_1(t_0, W_0) : W_0 \in cl\{S(B_1) \cap S^c(\rho)\}\} \\ \text{and} \alpha^* &\geq V_1(t, U) \text{ for } (t, U) \in \mathbb{R}_+ \times \partial S(\rho). \end{aligned}$$

Since equation (3.6.5) is equi-bounded, given $\alpha_1 > 0$, and $t_0 \in \mathbb{R}_+$, there exist a $\beta_0 = \beta(t_0, \alpha_1)$ such that

$$w_1(t, t_0, w_{10}) < \beta_0, \ \ t \geq t_0, \tag{3.6.7}$$

provided $w_{10} < \alpha_1$, where $w_1(t, t_0, w_{10})$ is any solution (3.6.5). Let $\alpha_2 = a(B_1) + \beta_0$, the uniform boundedness of equation (3.6.6) yields that

$$w_2(t, t_0, w_{20}) < \beta_1(\alpha_2), \quad t \geq t_0, \tag{3.6.8}$$

provided $w_{20} < \alpha_2$, where $w_2(t, t_0, w_{20})$ is any solution of (3.6.6). Choose B_2 satisfying

$$b(B_2) > \beta_1(\alpha_2). \tag{3.6.9}$$

We now claim that $W_0 \in S(B_1)$ implies that

$$U(t, t_0, W_0) \in S(B_2),$$

for $t \geq t_0$, where $U(t, t_0, W_0)$ is any solution of (3.6.1).

If it is not true, there exists a solution $U(t, t_0, W_0)$ of (3.6.7) with $W_0 \in S(B_1)$, such that for some $t^* > t_0$, $\|U(t*, t_0, W_0)\| = B_2$. Since $B_1 > \rho$, there are two possibilities to consider:

(1) $U(t, t_0, W_0) \in S^c(\rho)$ for $t \in [t_0, t^*]$;

(2) there exists a $\bar{t} \geq t_0$ such that $U(\bar{t}, t_0, W_0) \in \partial S(\rho)$ and $U(t, t_0, W_0) \in S^c(\rho)$ for $t \in [\bar{t}, t^*]$.

If (1) holds, we can find $t_1 > t_0$ such that

$$U(t_1, t_0, W_0) \in \partial S(B_1),$$

$$U(t^*, t_0, W_0) \in \partial S(B_2), \text{ and} \tag{3.6.10}$$

$$U(t, t_0, W_0) \in S^c(B_1), \quad t \in [t_1, t^*].$$

Setting $m(t) = V_1(t, U(t, t_0, W_0)) + V_2(t, U(t, t_0, W_0))$ for $t \in [t_1, t^*]$, and then using Theorem 3.2.1, we can obtain the differential inequality

$$D^+ m(t) \leq g_2(t, m(t)), \quad t \in [t_1, t^*],$$

and so,

$$m(t) \leq \gamma_2(t, t_1, m(t_1)), \quad t \in [t_1, t^*],$$

where $\gamma_2(t, t_1, v_0)$ is the maximal solution of (3.6.6) with $\gamma_2(t_1, t_1, v_0) = v_0$.

Thus,

$$\begin{aligned} &V_1(t^*, U(t^*, t_0, W_0)) + V_2(t^*, U(t^*, t_0, W_0)) \\ &\qquad \leq \gamma_2(t^*, t_1, V_1(t_1, U(t_1, t_0, W_0)) + V_2(t_1, U(t_1, t_0, W_0))). \end{aligned} \tag{3.6.11}$$

Similarly, we also have

$$V_1(t_1, U(t_1, t_0, W_0)) \leq \gamma_1(t_1, t_0, V_1(t_0, W_0)), \tag{3.6.12}$$

where $\gamma_1(t, t_0, u_0)$ is the maximal solution of (3.6.5).

Set $w_{10} = V_1(t_0, W_0) < \alpha_1$. Then

$$V_1(t_1, U(t_1, t_0, W_0)) \leq \gamma_1(t_1, t_0, V(t_0, W_0)) \leq \beta_0,$$

since (3.6.7) holds.

Furthermore, $V_2(t_1, U(t_1, t_0, W_0)) \leq a(B_1)$. Consequently, we have

$$w_{20} = V_1(t_1, U(t_1, t_0, W_0)) + V_2(t_1, U(t_1, t_0, W_0)) \leq \beta_0 + a(B_1) = \alpha_2. \quad (3.6.13)$$

Combining (3.6.8),(3.6.9),(3.6.10) and (3.6.13), we obtain

$$b(B_2) \leq m(t^*) \leq \gamma(t^*) \leq \beta_1(\alpha_2) < b(B_2), \quad (3.6.14)$$

which is a contradiction.

If case (2) holds, we also arrive at the inequality (3.6.11), where $t_1 > \bar{t}$ satisfies (3.6.10). We then have, in place of (3.6.12), the relation

$$V_1(t_1, U(t_1, t_0, W_0)) \leq \gamma_1(t_1, \bar{t}, V_1(\bar{t}, U(\bar{t}, t_0, W_0))).$$

Since $U(\bar{t}, t_0, W_0) \in \partial S(\rho)$ and $V_1(\bar{t}, U(\bar{t}, t_0, W_0)) \leq \alpha^* \leq \alpha_1$, arguing as before, we get the contradiction (3.6.14). This proves that for any given $B_1 > \rho,\ t_0 > 0$, there exists a B_2 such that

$$W_0 \in S(B_1) \text{ implies} U(t, t_0, W_0) \in S(B_2), t \geq t_0.$$

For $B_1 < \rho$, we set $B_2(t_0, B_1) = B_2(t_0, \rho)$ and hence the proof is complete.

3.7 Set Differential Systems

In this section we shall attempt to study the set differential system, given by

$$D_H U = F(t, U), \quad U(t_0) = U_0, \quad (3.7.1)$$

where $F \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n)^N, K_c(\mathbb{R}^n)^N],\ U \in K_c(\mathbb{R}^n)^N, K_c(\mathbb{R}^n)^N = (K_c(\mathbb{R}^n) \times K_c(\mathbb{R}^n) \times ..... \times K_c(\mathbb{R}^n), N$ times), $U = (U_1, ..., U_N)$ such that for each $i, 1 \leq i \leq N,\ U_i \in K_c(\mathbb{R}^n)$. Note also $U_0 \in K_c(\mathbb{R}^n)^N$.

We have the following two possibilities to measure the new variables U, U_0, F.

(1) Define $D_0[U, V] = \sum_{i=1}^{N} D[U_i, V_i]$, where $U, V \in K_c(\mathbb{R}^n)^N$ and employ the metric space $(K_c(\mathbb{R}^n)^N, D_0)$.

(2) Define $\tilde{D} : K_c(\mathbb{R}^n)^N \times K_c(\mathbb{R}^n)^N \to \mathbb{R}_+^N$ such that

$$\tilde{D}[U, V] = (D[U_1, V_1], D[U_2, V_2], ...D[U_N, V_N]),$$

and employ the generalized metric space $(K_c(\mathbb{R}^n)^N, \tilde{D})$.

If we utilize option (1) above, the assumption (2.2.4) of Theorem 2.2.1 appears as

$$D_0[F(t,U),F(t,V)] = \sum_{i=1}^{N} D(f_i(t,U),f_i(t,V) \le g(t,D_0[U,V]). \tag{3.7.2}$$

On the other hand, if we choose option (2) then the assumption (2.2.4) takes the form

$$\tilde{D}[F(t,U),F(t,V)] \le G(t,\tilde{D}[U,V]), \tag{3.7.3}$$

where $G \in C[\mathbb{R}_+ \times \mathbb{R}^N, \mathbb{R}^N]$.

In this case, condition (2.2.4) reduces to

$$\tilde{D}[F(t,U),F(t,V)] \le S\tilde{D}[U,V]), \tag{3.7.4}$$

where $S = (S_{ij})$ is an $N \times N$ matrix with $S_{ij} \ge 0$, for all i, j, which corresponds to the generalized contractive condition. Of course, the matrix S needs to satisfy a suitable condition, that is , for some $k > 1, S^k$ must be an $A-$matrix, which means $I - S^k$ is positive definite, where I is the identity matrix. For details of generalized spaces and contraction mapping theorem in this set up, see Bernfeld and Lakshmikantham [1].

Moreover, in order to arrive at the corresponding estimate (3.2.4) of Theorem 3.2.1, for example , one is required to utilize the corresponding theory of systems of differential inequalities, which demands that $G(t,w)$ have the quasi-monotone property, which is defined as follows:

$$w_1 \le w_2 \;\text{ and }\; w_{1i} = w_{2i} \;\text{ for some }\; i, 1 \le i \le N,$$

implies

$$G_i(t,w_1) \le G_i(t,w_2), \quad w_1, w_2 \in \mathbb{R}^N.$$

If $G(t,w) = Aw$, where A is an $N \times N$ matrix then the quasi-monotone property reduces to requiring $a_{ij} \ge 0, i \ne j$.

The method of vector Lyapunov-like functions has been very effective in the investigation of the qualitative properties of large-scale differential systems.

We shall extend this technique to set differential systems (3.7.1) where, as we shall see, both metrics described above are very useful. For this purpose, let us prove the following comparison result in terms of vector Lyapunov-like functions relative to the set differential system (3.7.1). We note that the inequalities between vectors in $\mathbb{R}^N$ are to be understood as componentwise.

Theorem 3.7.1 *Assume that* $V \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n)^N, \mathbb{R}_+^N]$,

$$|V(t,U_1) - V(t,U_2)| \le A\,\tilde{D}[U_1,U_2],$$

where A is on $N \times N$ matrix with nonnegative elements, and for $(t,U) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n)^N$,

$$D^+V(t,U) \le G(t,V(t,U)), \tag{3.7.5}$$

where $G \in C[\mathbb{R}_+ \times \mathbb{R}^N_+, \mathbb{R}^N]$. Suppose further that $G(t,w)$ is quasi-monotone in w for each $t \in \mathbb{R}_+$ and $r(t) = r(t,t_0,w_0)$ is the maximal solution of

$$w' = G(t,w), \quad w(t_0) = w_0 \geq 0, \tag{3.7.6}$$

existing for $t \geq t_0$. Then

$$V(t,U(t)) \leq r(t), \quad t \geq t_0, \tag{3.7.7}$$

where $U(t) = U(t,t_0,W_0)$ is any solution of (3.7.1) existing for $t \geq t_0$.

Proof Let $U(t)$ be any solution of (3.7.1) existing for $t \geq t_0$.

Define $m(t) = V(t,U(t))$ so that $m(t_0) = V(t_0,W_0) \leq w_0$. Now for small $h > 0$, we have, in view of Lipschitz conditions,

$$\begin{aligned} m(t+h) - m(t) &= V(t+h,U(t+h)) - V(t,U(t)) \\ &\leq A\tilde{D}[U(t+h),U(t)+hF(t,U(t))] \\ &\quad +V(t+h,U(t)+hF(t,U(t))) - V(t,U(t)). \end{aligned}$$

It follows therefore that

$$\begin{aligned} D^+m(t) &= \limsup_{h\to 0^+} \frac{1}{h}[m(t+h)-m(t)] \leq D^+V(t,U(t)) \\ &\quad +A \limsup_{h\to 0^+} \frac{1}{h}[\tilde{D}[U(t+h),U(t)+hF(t,U(t)))]. \end{aligned}$$

Since $D_H U$ is assumed to exist, we see that $U(t+h) = U(t) + Z(t)$ where $Z(t) = Z(t,h)$ is the Hukuhara difference for small $h > 0$. Hence utilizing the properties of $\tilde{D}[U,V]$ we obtain,

$$\begin{aligned} \tilde{D}[U(t+h),U(t)+hF(t,U(t))] &= \tilde{D}[U(t)+Z(t),U(t)+hF(t,U(t))] \\ &= \tilde{D}[Z(t),hF(t,U(t)))] \\ &= \tilde{D}[U(t+h)-U(t),hF(t,U(t)))]. \end{aligned}$$

As a result, we get

$$\frac{1}{h}\tilde{D}[U(t+h),U(t)+hF(t,U(t))] = \tilde{D}\left[\frac{U(t+h)-U(t)}{h},F(t,U(t))\right]$$

and consequently

$$\begin{aligned} &\limsup_{h\to 0^+} \frac{1}{h}\tilde{D}[U(t+h),U(t)+hF(t,U(t))] \\ &= \limsup_{h\to 0^+} \frac{1}{h}\left[\tilde{D}\left[\frac{U(t+h)-U(t)}{h},F(t,U(t))\right]\right] = \tilde{D}[D_H U(t),F(t,U(t))] = 0, \end{aligned}$$

since $U(t)$ is a solution of (3.7.1). We therefore have the vectorial differential inequality,

$$D^+m(t) \leq G(t,m(t)), \;\; m(t_0) \leq w_0, \;\; t \geq t_0,$$

which by the theory of differential inequalities for systems (Lakshmikantham and Leela [1]) yields

$$m(t) \leq r(t), \quad t \geq t_0,$$

proving the claimed estimate (3.7.7).

The following corollary of Theorem 3.7.1 is interesting.

Corollary 3.7.1 *The function $G(t,w) = Aw$, where A is an $N \times N$ matrix satisfying $a_{ij} \geq 0$, $i \neq j$, is admissible in Theorem 3.7.1 and yields the estimate*

$$V(t,U(t)) \leq V(t_0,W_0)e^{A(t-t_0)}, \quad t \geq t_0.$$

3.8 The Method of Vector Lyapunov Functions

We shall prove a typical result that gives sufficient conditions in terms of vector Lyapunov-like functions for the stability properties of the trivial solution of the set differential system (3.7.1).

Theorem 3.8.1 *Assume that*

(i) $G \in C[\mathbb{R}_+ \times \mathbb{R}^N_+, \mathbb{R}^N]$, $G(t,0) \equiv 0$ *and* $G(t,w)$ *is quasi-monotone nondecreasing in* w *for each* $t \in \mathbb{R}_+$*;*

(ii) $V \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}^N_+]$, $|V(t,U_1) - V(t,U_2)| \leq A\,\tilde{D}[U_1,U_2]$, *where* A *is a nonnegative* $N \times N$ *matrix and the function*

$$V_0(t,U) = \sum_{i=1}^{N} V_i(t,U) \tag{3.8.1}$$

satisfies

$$b(D_0[U,\theta]) \leq V_0(t,U) \leq a(D_0[U,\theta]), \quad a,b \in \mathcal{K};$$

(iii) $F \in C[\mathbb{R}_+ \times S(\rho), K_c(\mathbb{R}^n)^N]$, $F(t,\theta) \equiv \theta$ *and*

$$D^+V(t,U) \leq G(t,V(t,U)), \quad (t,U) \in \mathbb{R}_+ \times S(\rho),$$

where $S(\rho) = [U \in K_c(\mathbb{R}^n)^N : D_0[U,\theta] < \rho]$.

Then, the stability properties of the trivial solution of (3.7.6) imply the corresponding stability properties of the trivial solution of (3.7.1).

Proof We shall prove only equi-asymptotic stability of the trivial solution of (3.7.1). For this purpose, let us first prove equi-stability . Let $0 < \varepsilon < \rho$ and $t_0 \in \mathbb{R}_+$, be given. Assume that the trivial solution of (3.7.6) is equi-asymptotically stable. Then, it is equi-stable. Hence given $b(\varepsilon) > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta_1 = \delta_1(t_0,\varepsilon) > 0$ such that

$$\sum_{i=1}^{N} w_{i0} < \delta_1 \text{ implies } \sum_{i=1}^{N} w_i(t,t_0,w_0) < b(\varepsilon), \quad t \geq t_0, \tag{3.8.2}$$

where $w(t, t_0, w_0)$ is any solution of (3.7.6). Choose $w_0 = V(t_0, W_0)$ and a $\delta = \delta(t_0, \varepsilon) > 0$ satisfying

$$a(\delta) < \delta_1. \tag{3.8.3}$$

Let $D_0[W_0, \theta] < \delta$. Then, we claim that $D_0[U(t), \theta] < \varepsilon,\ t \geq t_0$, for any solution $U(t) = U(t, t_0, W_0)$ of (3.7.1). If this is not true, there would exist a solution $U(t)$ of (3.7.1) with $D_0[W_0, \theta] < \delta$ and a $t_1 > t_0$ such that

$$D_0[U(t_1), \theta] = \varepsilon \text{ and } D_0[U(t), \theta] \leq \varepsilon < \rho,\ t_0 \leq t \leq t_1. \tag{3.8.4}$$

Hence we have by Theorem 3.7.1,

$$V(t, U(t)) \leq r(t, t_0, w_0), \quad t_0 \leq t \leq t_1, \tag{3.8.5}$$

where $r(t, t_0, w_0)$ is the maximal solution of (3.7.6). Since

$$V_0(t_0, W_0) \leq a(D_0[W_0, \theta]) < a(\delta) < \delta_1,$$

the relations (3.8.2), (3.8.3), (3.8.4) and (3.8.5) yield

$$b(\varepsilon) \leq V_0(t_1, U(t_1)) \leq r_0(t_1, t_0, w_0) < b(\varepsilon),$$

where $r_0(t, t_0, w_0) = \sum_{i=1}^{N} r_i(t, t_0, w_0)$. This contradiction proves that the trivial solution of (3.7.1) is equi-stable.

Suppose next that the trivial solution of (3.7.6) is quasi-equi-asymptotically stable . Set $\varepsilon = \rho$ and $\hat{\delta_0} = \delta(t_0, \rho)$. Let $0 < \eta < \rho$. Then given $b(\eta)$ and $t_0 \in \mathbb{R}_+$, there exist $\delta_1^* = \delta_1(t_0) > 0$ and $T = T(t_0, \eta) > 0$ satisfying

$$\sum_{i=1}^{N} w_{i0} < \delta_1^* \text{ implies } \sum_{i=1}^{N} w_i(t, t_0, w_0) < b(\eta), \quad t \geq t_0 + T. \tag{3.8.6}$$

Choosing $w_0 = V(t_0, W_0)$ as before, we find $\delta_0^* = \delta_0(t_0) > 0$ such that $a(\delta_0^*) < \delta_1^*$. Let $\delta_0 = \min(\delta_1^*, \delta_0^*)$ and $D_0[W_0, \theta] < \delta_0$. This implies $D_0[U(t), \theta] < \rho$, $t \geq t_0$ and therefore the estimate (3.8.5) holds for all $t \geq t_0$.

Suppose now that there is a sequence $\{t_k\},\ t_k \geq t_0 + T,\ t_k \to \infty$ as $k \to \infty$, and $\eta \leq D_0[U(t_k), \theta]$, where $U(t)$ is any solution of (3.7.1) with $D_0[W_0, \theta] < \delta_0$.

In view of (3.8.6), this leads to the contradiction

$$b(\eta) \leq V_0(t_k, U(t_k)) \leq r_0(t_k, t_0, w_0) < b(\eta).$$

Hence the trivial solution of (3.7.1) is equi-asymptotically stable and the proof is complete.

In order to apply the method of vector Lyapunov functions to concrete problems, it is necessary to know the properties of solutions of the comparison system (3.7.6), which is difficult in general, except when $G(t, w) = Aw$, where A is a quasi-monotone $N \times N$ stability matrix. Hence we shall present some simple and useful techniques to deal with this problem.

We shall first prove a result which reduces the study of the properties of solutions of (3.7.6) to that of a scalar differential equation

$$v' = G_0(t, v), \quad v(t_0) = v_0 \geq 0 \tag{3.8.7}$$

where $G_0 \in C[\mathbb{R}_+^2, \mathbb{R}]$. Specifically, we have the following result.

Lemma 3.8.1 *Assume that* $L \in C^1[\mathbb{R}_+, \mathbb{R}_+^N]$, $G \in C[\mathbb{R}_+ \times \mathbb{R}_+^N, \mathbb{R}^N], G_0 \in C[\mathbb{R}_+^2, \mathbb{R}]$ *and* G, G_0 *are smooth enough to assure the existence and uniqueness of solutions for* $t \geq t_0$ *of (3.7.6) and (3.8.7) respectively. Suppose further that for* $(t, v) \in \mathbb{R}_+^2$,

$$G(t, L(v)) \leq \frac{dL(v)}{dv} G_0(t, v).$$

Then $w_0 \leq L(v_0)$ *implies*

$$w(t, t_0, w_0) \leq L(v(t, t_0, v_0)), \quad t \geq t_0, \tag{3.8.8}$$

where $w(t, t_0, w_0), v(t, t_0, w_0)$ *are the solutions of (3.7.6) and (3.8.7) respectively.*

Proof Set $m(t) = L(v(t, t_0, v_0)$, so that $m(t_0) = L(v_0)) \geq w_0$ and

$$\begin{aligned} m'(t) &= \frac{dL(v(t, t_0, v_0))}{dv} G_0(t, v(t, t_0, v_0)) \\ &\geq G(t, L(v(t, t_0, v_0))) = G(t, m(t)). \end{aligned}$$

hence by the comparison Theorem 1.4.1 in Lakshmikantham and Leela[1] , we get the stated result in view of uniqueness of solutions.

Let us give an example to illustrate Lemma 3.8.1.

Suppose that $G_1 = -2w_1^2$, $G_2 = -2w_2^3 + 2w_1 w_2^{\frac{3}{2}}$, so that

$$\begin{aligned} w_1' &= -2w_1^2, \\ w_2' &= -2w_2^3 + 2w_1 w_2^{\frac{3}{2}}. \end{aligned} \tag{3.8.9}$$

Choosing $L_1(v) = \frac{3}{5} v^{\frac{3}{2}}, L_2(v) = v$ and

$$G_0(t, v) = \begin{cases} -\frac{2}{5} v^3, & 0 \leq v < 1, \\ -\frac{2}{5} v^{\frac{5}{2}}, & 1 \leq v, \end{cases}$$

the assumptions of Lemma 3.8.1 are satisfied. Clearly, the trivial solution of (3.8.7) is uniformly asymptotically stable and therefore the trivial solution of (3.8.9) is also uniformly asymptotically stable.

Lemma 3.8.2. Assume that $Q \in C^1[\mathbb{R}_+^N, \mathbb{R}_+]$, $G \in C[\mathbb{R}_+ \times \mathbb{R}_+^N, \mathbb{R}^N]$, $G_0 \in C[\mathbb{R}_+^2, \mathbb{R}]$ and for $(t, w) \in \mathbb{R}_+ \times \mathbb{R}_+^N$,

$$\frac{dQ(w)}{dw} G(t, w) \leq G_0(t, Q(w)). \tag{3.8.10}$$

Then, any solution $w(t) = w(t, t_0, w_0)$ of (3.7.6) existing for $t \geq t_0$, satisfies

$$Q(w(t)) \leq v(t), \quad t \geq t_0,$$

where $v(t) = v(t, t_0, v_0)$ is the maximal solution of (3.8.7) existing for $t \geq t_0$, provided $Q(w_0) \leq v_0$.

Proof Let $w(t) = w(t, t_0, w_0)$ be any solution of (3.7.6) existing for $t \geq t_0$.

Set $p(t) = Q(w(t))$, Then, we have

$$p'(t) = \frac{dQ(w(t))}{dw} G(t, w(t)) \leq G_0(t, Q(w(t))) = G_0(t, p(t))),$$

and $p(t_0) \leq v_0$. Hence by the Theorem 1.4.1 in Lakshmikantham and Leela [1], it follows that $p(t) \leq v(t), t \geq t_0$, where $v(t)$ is the maximal solution of (3.8.7). Hence the proof is complete.

As an example, consider the case $G(t, w) = Aw$ where A is an $N \times N$ matrix with $a_{ij} \geq 0, i \neq j$, and A is quasi-diagonally dominant, that is, for some $d_i > 0$,

$$d_i |a_{ii}| > \sum_{j=1, i \neq j}^{n} d_j |a_{ij}|. \tag{3.8.11}$$

Choosing $Q(w) = \sum_{i=1}^{N} d_i w_i$ for some $d_i > 0$, we see that (3.8.10) is satisfied by $G_0(t, v) = -\gamma v$ for some $\gamma > 0$ in view of (3.8.11). Consequently, the trivial solution of (3.8.7) is exponentially asymptotically stable which implies that the trivial solution of (3.7.6) does have the same property.

3.9 Nonsmooth Analysis

In the previous sections of this chapter, we developed several results in stability theory by utilizing continuous Lyapunov-like functions and investigated analogous results parallel to standard Lyapunov stability theory for set differential equations.

In this and the next section, we concern ourselves with Lyapunov stability theory employing Lyapunov-like functions, which are only lower semi continuous (lsc) and this requires introducing the concepts and results of nonsmooth analysis extending suitably to the present set up. We have already sketched in Sections 2.7 and 2.8, some results related to the existence of Euler solutions and flow invariance in terms of proximal normals. There is an intimate connection between proximal normal theory and subdifferentials of lower semicontinuous functions that we develop before proceeding to build Lyapunov theory in the framework of lsc functions and set differential equations. We shall embark on this aspect in this section and consider Lyapunov's theory in the following section.

Let us start with proximal normals again, since we did not provide in Section 2.8, all the necessary tools needed for our purpose.

Recall that $D[A, \theta] = \|A\| = sup_{a \in A} \|a\|$, and

$$\|A + B\|^2 \leq \|A\|^2 + \|B\|^2 + 2 < A, B >, \tag{3.9.1}$$

where for $A, B \in K_c(\mathbb{R}^n)$,

$$< A, B > = \sup\{(a \cdot b) : a \in A, b \in B\}. \tag{3.9.2}$$

Let $f : K_c(\mathbb{R}^n) \to \mathbb{R}$. Let $U, V \in K_c(\mathbb{R}^n)$ be such that there exists a $Z \in K_c(\mathbb{R}^n)$ satisfying $V = U + Z$. So $V - U$ the Hukuhara difference of V, U, exists.

If there exists an element $A(U) \in K_c(\mathbb{R}^n)$ such that

$$|f(V) - f(U) - < A(U), Z > | \leq \varepsilon \|Z\|, \ \varepsilon > 0,$$

where $< A(U), Z >$ is defined as in (3.9.2), then we say that $f_U(U) = A(U)$ is the derivative of f at U. We note that $f_U : K_c(\mathbb{R}^n) \to K_c(\mathbb{R}^n)$.

Consider next, $F(U) = f_U(U)$. For $A, B \in K_c(\mathbb{R}^n)$, we first define (A, B) as an element of $\mathbb{R}^n$ whose i^{th} component is given by

$$(A, B)_i = sup\{(a_i b_i) : a \in A, b \in B\}, \ 1 \leq i \leq n.$$

Suppose there exists a map $\tilde{f}$ defined for each $U \in K_c(\mathbb{R}^n)$, mapping $K_c(\mathbb{R}^n)$ into $\tilde{K}_c(\mathbb{R}^n) = \{$the set of compact, convex subsets in $K_c(\mathbb{R}^n)\}$.

Then, we define,

$$D[F(V), F(U) + \ << \tilde{f}(U), Z >>] \ \leq \ \varepsilon \|Z\|, \ \ \varepsilon > 0,$$

where

$$<< \tilde{f}(U), Z >> = \ \{(B, Z) : B \in \tilde{f}(U) \text{ and } \tilde{f} : K_c(\mathbb{R}^n) \to \tilde{K}_c(\mathbb{R}^n)\}.$$

Then, we say that $F_U(U) = \tilde{f}(U)$ is the derivative of F at U.

With these preliminaries, we consider, as in Section 2.8, the set differential equation

$$D_H U = F(t, U), \ U(t_0) = U_0 \in K_c(\mathbb{R}^n), \tag{3.9.3}$$

where $F : [t_0, \infty) \times K_c(\mathbb{R}^n) \to K_c(\mathbb{R}^n)$ is any function, and the scalar differential equation

$$u' = g(t, u), \ u(t_0) = u_0 \geq 0, \tag{3.9.4}$$

where $g \in C[\mathbb{R}^2_+, \mathbb{R}_+]$, as in (2.8.4)

Let $\Omega \subset K_c(\mathbb{R}^n)$ be a nonempty, closed set. Let $U \in K_c(\mathbb{R}^n)$ be not lying in Ω. Suppose that the Hukuhara difference $U - S$ exists for every $S \in \Omega$. That is, for each $S \in \Omega$, there exists a $Z \in K_c(\mathbb{R}^n)$ such that $U = S + Z$. Suppose further that, there exists an element $S \in \Omega$ whose distance to U is minimal, that is,

$$D_0[U, \Omega] = \|U - S\| = \inf_{S_0 \in \Omega} \|U - S_0\|. \tag{3.9.5}$$

We call such an $S \in \Omega$ a projection of U onto Ω and denote the set of all such elements by $\text{proj}_\Omega(U)$. The element $U - S$ will be called the proximal normal direction to Ω at S. Any nonnegative multiple $\xi = t(U - S), \ t \geq 0$, we call a proximal normal to Ω at S. The set of all ξ obtained in this manner is said to be a proximal normal cone to Ω at S and is denoted by $N^P_\Omega(S)$. Suppose that $S \in \Omega$ is such that $S \notin proj_\Omega(U)$, for any $U \in K_c(\mathbb{R}^n)$, not in Ω, then we set $N^P_\Omega(S) = \{\theta\}$. When $S \notin \Omega$, $N^P_\Omega(S)$ is not defined.

Suppose that $S \in proj_{\Omega}(U)$. Then, $\|U - \tilde{S}\| \geq \|U - S\|$ for all $\tilde{S} \in \Omega$. As a result, we have using (3.9.1),

$$\begin{aligned} \|U - \tilde{S}\|^2 &= \|U - S + S - \tilde{S}\|^2 \\ &\leq \|U - S\|^2 + \|S - \tilde{S}\|^2 + 2 < U - S, S - \tilde{S} >, \end{aligned}$$

which implies

$$< U - S, \tilde{S} - S > \leq \frac{1}{2} \|\tilde{S} - S\|^2 \text{ for all } \quad \tilde{S} \in \Omega. \tag{3.9.6}$$

However, any element $C = U - S$, satisfying (3.9.6) need not be such that $S \in proj_{\Omega}(U)$. Consequently, we set $N_{\Omega}^{P}(S) = \{\theta\}$, for any $\xi = t(U - S)$, satisfying

$$< \xi, \tilde{S} - S > \quad \leq \quad \sigma \, \|\tilde{S} - S\|^2 \quad \text{for all} \quad \tilde{S} \in \Omega, \tag{3.9.7}$$

where $\sigma = \sigma(\xi, S) > 0$, but $S \notin proj_{\Omega}(U)$. Thus, we assume as an axiom, the following proposition.

Proposition 3.9.1 *For any given $\delta > 0$, $\quad \xi \in N_{\Omega}^{P}(S)$, if and only if there exists a $\sigma = \sigma(\xi, S) > 0$, such that*

$$< \xi, \tilde{S} - S > \quad \leq \quad \sigma \|\tilde{S} - S\|^2, \quad \text{for all} \quad \tilde{S} \in \Omega \cap B(S, \delta). \tag{3.9.8.}$$

It can be verified that $S \in proj_{\Omega}(U)$ is equivalent to $S \in proj_{\Omega}(S + \delta(U - S))$.

We shall next consider the subgradients of lower semicontinuous(*lsc*) functions.

Let $f : K_c(\mathbb{R}^n) \to (-\infty, \infty]$ be a lsc function with $dom(f) = \{X \in K_c(\mathbb{R}^n) : f(X) < \infty\}$. Suppose that $(\zeta, -\lambda) \in K_c(\mathbb{R}^n) \times \mathbb{R}$ belongs to $N_{epi(f)}^{P}(X, r)$. It can be verified that $\;(i)\;\; \lambda \geq 0$. $\;(ii)\;\; r = f(X) \; if \lambda > 0$, and $(iii)\; \lambda = 0 \; if \; r > f(X)$.

An element $\zeta \in K_c(\mathbb{R}^n)$ is said to be the proximal subgradient of f at $x \in$ dom (f) provided that $(\zeta, -1) \in N_{epi(f)}(X, f(X))$, where $epi(f) = \{(X, r) \in K_c(\mathbb{R}^n) \times \mathbb{R} : f(X) \leq r\}$.

The set of all such ζ is denoted by $\partial_P f(X)$ and is referred to as proximal subdifferential or P-subdifferential. Let us note that because a cone is involved, if $\lambda > 0$, and $(\zeta, -\lambda) \in N_{epi(f)}^{P}(X, f(X))$, then $\frac{\zeta}{\lambda} \in \partial_P f(X)$.

Also, the following result concerns the approximation of horizontal proximal normals to epigraphs, by nonhorizontal proximal normals, and is needed for our later use.

Theorem 3.9.1 *Let $f : K_c(\mathbb{R}^n) \to \mathbb{R}$ be a* lsc *function and let $(\theta, 0) \in N_{epi(f)}(X, f(X))$. Then, for every $\epsilon > 0$ there exists $X' \in X + \epsilon B$ and $(\zeta, -\lambda) \in N_{epi(f)} \; (X', f(X'))$ such that*

$$\lambda > 0, \quad | \; f(X') - f(X) \; | < \epsilon, \quad \|(\theta, 0) - (\zeta, -\lambda)\| < \epsilon \tag{3.9.9}$$

We are now in a position to prove the following proximal subgradient inequality.

Theorem 3.9.2 *Let $f : K_c(\mathbb{R}^n) \to (-\infty, \infty]$ be a* lsc *function and let $X \in dom(f)$. Then, $\zeta \in \partial_P f(X)$ if and only if there exists positive numbers σ, and η such that*

$$f(Y) \geq f(X) + < \zeta, Y - X > -\sigma \|Y - X\|^2 \tag{3.9.10}$$

for all $Y \in B(X, \eta)$.

Proof Suppose that (3.9.10) holds. we then have by adding $\sigma(\alpha - f(X))^2$,

$$\alpha - f(X) + \sigma[\|Y - X\|^2 + (\alpha - f(X))^2] \ \geq \ < \zeta, Y - X >$$

for all $\alpha \geq f(Y)$. This implies,

$$< (\zeta, -1), [(Y, \alpha) - (X, f(X))] > \ \leq \ \sigma \|(Y, \alpha) - (X, f(X))\|^2$$

for all points $(Y, \alpha) \in \text{epi}(f)$ near $(X, f(X))$.

In view of Proposition 3.9.1, this implies,

$$(\zeta, -1) \in N^P_{epi(f)}(X, f(X)).$$

Conversely, suppose that $(\zeta, -1) \in N^P_{epi(f)}(X, f(X))$. Then there exists a $\delta > 0$ such that

$$(X, f(X)) \in proj_{epi(f)}((X, f(X)) + \delta(\zeta, -1)).$$

This implies,

$$\|\delta(\zeta, -1)\|^2 \leq \|[(X, f(X)) + \delta(\zeta, -1)] - (Y, \alpha)\|^2,$$

for all $(Y, \alpha) \in \text{epi}(f)$.

Upon taking $\alpha = f(Y)$, we get from the last inequality

$$\delta^2 + \delta^2 \|\zeta\|^2 \leq \|X - Y + \delta\zeta\|^2 + (f(X) - f(Y) - \delta)^2,$$

which can be rewritten as

$$(f(Y) - f(X) + \delta)^2 \geq \delta^2 + 2\delta < \zeta, Y - X > -\|X - Y\|^2. \tag{3.9.11}$$

Clearly, the left hand side of the expression (3.9.11) is positive for all Y, sufficiently near X, that is $Y \in B(X, \eta)$ for some $\eta > 0$. Since f is *lsc*, by shrinking η if necessary, we can make sure that

$$f(Y) - f(X) + \delta > 0, \quad \text{for all } \ Y \in B(X, \eta).$$

Consequently, taking square roots of (3.9.11) we get,

$$f(Y) \geq h(Y) = f(X) - \delta + [\delta^2 + 2\delta < \zeta, Y - X > -\|Y - X\|^2]^{\frac{1}{2}} \tag{3.9.12}$$

for all $Y \in B(X, \eta)$.

Considering the function $h(Y)$, we can calculate and show that $H(X) = h_X(X) = \zeta$, and $H_X(X)$ is bounded by say $2\sigma > 0$ in a neighbourhood of X.

Hence the function h satisfies the inequality, for some $\eta > 0$,

$$h(Y) \geq h(X) + < \zeta, Y - X > -\sigma \|Y - X\|^2, \quad \text{for all} \quad Y \in B(X, \eta). \tag{3.9.13}$$

But, then noting that $f(X) = h(X)$, the relations (3.9.12) and (3.9.13) yield

$$f(Y) \geq f(X) + < \zeta, Y - X > -\sigma \|Y - X\|^2 \tag{3.9.14}$$

for all $Y \in B(X, \eta)$ which is (3.9.10) as desired. The proof is complete.

3.10 Lyapunov Stability Criteria

In this section, we plan to provide the Lyapunov stability criteria for Euler solutions of set differential equation (3.9.3). We begin with the following definition of the Lyapunov function.

Let $V : \mathbb{R}_+ \times K_c(\mathbb{R}^n) \to (-\infty, \infty]$ be an lsc function with domain $\text{dom}(V) = [(t,X) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n) : V(t,X) < \infty]$.

Definition 3.10.1 *The pair (V,F) is said to be weakly decreasing if for any U_0 there exists an Euler solution $U(t) = U(t,t_0,U_0)$ of (3.9.3) satisfying*

$$V(t,U(t)) \leq V(t_0,U_0),\ t \geq t_0 \geq 0.$$

One can easily verify that, (V,F) is weakly decreasing if and only if $(\text{epi}V, \{1\} \times F(t,U) \times \{0\})$ is weakly invariant.

In order to deduce the Lyapunov theory of stability from the present set up, we need a sufficient condition, which assures the weakly decreasing nature of the system (V,F).

Theorem 3.10.1 *(V,F) is weakly decreasing if for all $(\theta,\zeta) \in \partial_P V(t,Z), (t,Z) \in dom(V)$, we have*

$$\theta + \ < F(t,Z),\ \zeta > \leq\ 0. \tag{3.10.1}$$

Proof Suppose that for $(t,Z) \in \text{dom}(V)$, $(\theta,\zeta) \in \partial_P V(t,Z)$, and (3.10.1) holds. (V,F) is weakly decreasing if and only if $(\text{epi}V, \{1\} \times F \times \{0\})$ is weakly invariant.

To show that $(\text{epi}V,\ \{1\} \times F \times \{0\})$ is weakly invariant, it suffices to show that for any $(\theta,\zeta,\lambda) \in N^P_{\text{epi}V}(t,Z,r)$

$$< (\{1\}, F(t,Z), \{0\}),\ (\theta,\zeta,\lambda) > \leq\ 0.$$

Let $(\theta,\zeta,\lambda) \in N^P_{\text{epi}V}(t,Z,r)$. Then, $\lambda \leq 0$.

If $\lambda < 0$, then $(\theta,\zeta,\lambda) \in N^P_{\text{epi}V}(t,Z,r)$.

This implies $(\frac{\theta}{-\lambda}, \frac{\zeta}{-\lambda}, -1) \in N^P_{\text{epi}V}(t,Z,r)$, which in turn leads to $(\frac{\theta}{-\lambda}, \frac{\zeta}{-\lambda}) \in \partial_P V(t,Z)$.

By hypothesis, we get $\frac{\theta}{-\lambda} + \ < F(t,Z), \frac{\zeta}{-\lambda} > \leq\ 0$. This implies, $\theta + \ < F(t,Z), \zeta > \leq\ 0,\quad (t,Z) \in \text{dom}(V)$, which is the required condition.

In case, $\lambda = 0$, then we have $(\theta,\zeta,0) \in N^P_{\text{epi}V}(t,Z,V(t,Z))$. We invoke Theorem 3.9.1 to deduce the existence of sequences $(\theta_i,\zeta_i,-\epsilon_i) \in N^P_{\text{epi}V}(t_i,Z_i,V(t_i,Z_i))$ with $\epsilon_i > 0$, and $(\theta_i, Z_i, V(t_i,Z_i))$ such that $(\theta_i,\zeta_i,-\epsilon_i) \to (\theta,\zeta,0)$,

$$(t_i, Z_i, V(t_i,Z_i)) \to (t,Z,V(t,Z)).$$

Then, as in case $\lambda < 0$ above, $\theta_i + \ < F(t_i,Z_i),\ \zeta_i > \leq\ 0$. Since F is locally bounded, the sequence $F(t_i,Z_i)$ is bounded. Passing to a subsequence, we may suppose that $F(t_i,Z_i)$ converges to $F(t,Z)$.

This in turn implies $\theta + < F(t,Z), \zeta > \leq 0$ and hence the proof is complete.

We shall now consider the Lyapunov theory of stability, employing the *lsc* functions $V(t,X)$ so that we can utilize the set of proximal subdifferentials of V, namely $\partial_P V(t,X)$, for providing sufficient conditions for Lyapunov stability in the weak sense. We shall derive the weak stability results from Theorem 3.10.1. We assume that $F(t,\theta) \equiv \theta$ so that we can discuss the weak stability of $X = \theta$. As in section 3.4, we assume that, given $U_0, V_0 \in K_c(\mathbb{R}^n)$, the Hukuhara difference $U_0 - V_0 \equiv X_0$ exists, and we consider solutions

$U(t) = U(t,t_0,X_0)$ or $X(t) = X(t,t_0,X_0)$ for stability purposes.

We list the following conditions concerning V:

(i) $V : \mathbb{R}_+ \times K_c(\mathbb{R}^n) \to [0,\infty]$ is *lsc* with dom $V = \{(t,X) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n) : V(t,X) < \infty\}$ and $V(t,X)$ is positive definite.

(ii) the sets $[V]^q = \{(t,X) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n) : V(t,X) \leq q\}$ are compact for every $q > 0$.

(iii) $\theta + < F(t,X), \zeta > -G(t, V(t,X)) \leq 0$, for $(t,X) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n), (\theta,\zeta) \in \partial_P V(t,X))$ where $G \in C[\mathbb{R}_+^2, \mathbb{R}], G(t,0) \equiv 0$ and satisfies a nonlinear growth condition similar to F. (See Theorem 2.8.1).

(iv) $\theta + < F(t,X), \zeta > +W(t,X) \leq 0, \quad (t,X) \in K_c(\mathbb{R}^n)$, where $W \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$, and F is bounded on bounded sets. Here $W(t,X)$ is positive definite and satisfies a nonlinear growth condition similar to F.

We can now prove the following result on weak stability.

Theorem 3.10.2 *Assume that (i), (ii) and (iii) hold. Then the weak stability properties of the scalar differential equation*

$$w' - G(t,w) = 0, \;\; w(0) = w_0 = V(t_0, X_0), \tag{3.10.2}$$

imply that the corresponding weak stability properties of $X = \theta$ of (3.9.3).

Proof Let us define $Q : \mathbb{R}_+ \times K_c(\mathbb{R}^n) \times \mathbb{R} \to (-\infty,\infty]$ by $Q(t,X,w) = V(t,X) - w$ and the function $\tilde{F}(t,X,w) = F(t,X) \times \{-G(t,w)\}$. We observe that $\tilde{F}$ is also an *usc* function satisfying the nonlinear growth condition similar to the one satisfied by F.

Let us claim that the system $(Q,\tilde{F})$ is weakly decreasing. We wish to apply Theorem 3.10.1 and so let $(\theta,\zeta,\eta) \in \partial_P Q(t,X,w)$.

Then, $(\theta,\zeta) \in \partial_P V(t,X)$ and $\eta = -1$.

The condition $\theta + < F(t,X), \zeta > -G(t,V(t,X)) \leq 0$, provides the condition that

$$\theta + < (F(t,X), -G(t,V(t,X)), (\zeta,-1) > \; \leq \; 0,$$

which verifies the inequality of the Theorem 3.10.1.

Hence, we deduce the existence of a solution (t,X,w) of $\tilde{F}$ with (t_0,X_0,w_0) satisfying

$$Q(t,X(t),w(t)) \leq Q(t_0,X_0,w_0) = V(t_0,X_0) - w_0 = 0, \quad t \geq t_0,$$

which reduces to the comparison principle

$$V(t, X(t)) \leq w(t), \quad t \geq t_0, \tag{3.10.3}$$

where $X(t) = X(t.t_0, X_0)$ is a solution of (3.9.3), and $w(t)$ is a solution of (3.10.2).

Once we have the comparison principle given by (3.10.3), if we suppose that $w = 0$, of (3.10.2) possesses any weak stability property, it follows employing the standard arguments, see Lakshmikantham and Leela [1] and Lakshmikantham, Leela and Martynyuk [1] that the corresponding weak stability property of $X = \theta$ of (3.9.3) holds. Hence the proof is complete.

In a similar manner, one can prove the following result relative to the condition (iv).

Theorem 3.10.3 *Assume that (i), (ii) and (iv) hold. then, $X = \theta$ of (3.9.3) is weakly asymptotically stable.*

Proof In this case we set, $Q(t, X, y) = V(t, X) + y$ and $\tilde{F}(t, X, y) = F(t, X) \times \{W(t, X)\}$ and, similar to the proof of Theorem 3.10.2, we can deduce from Theorem 2.8.1, the existence of a solution (X, y) of $\tilde{F}$ at $(X_0, 0)$ such that

$$Q(t, X(t), y(t)) \leq Q(t_0, X_0, 0) = V(t_0, X_0), \quad t \geq 0.$$

This in turn reduces to

$$V(t, X(t)) + \int_0^t W(s, X(s))\, ds \leq V(t_0, X_0)), \quad t \geq 0, \tag{3.10.4}$$

where $X(t)$ is a solution of (3.9.3).

The weak stability of $X = \theta$ of (3.9.3) follows immediately from (3.10.4) using (i) and (ii) in a straightforward way.

To prove $\|X(t)\| \to 0$ as $t \to \infty$, we observe that (3.10.4) implies that $V(t, X(t))$ and $\int_0^t W(s, X(s))\, ds$ are bounded on $\mathbb{R}_+$, as well as $X(t)$. Since F is bounded on bounded sets, we also have $D_H X(t)$ bounded, and as a consequence $X(t)$ is globally Lipschitz. With this information, it is now standard to show that $\|X(t)\| \to 0$ as $t \to \infty$, for otherwise the divergence of $\int_0^\infty W(s, X(s))\, ds$ results. Hence the proof is complete.

3.11 Notes and Comments

Lyapunov-like functions and needed comparison theorems including global existence described in Sections 3.2 and 3.3 are from Lakshmikantham, Leela and Vatsala [2]. The stability criteria provided in Section 3.4, parallel to the original Lyapunov results for ODE, is from Gnana Bhaskar and Vasundhara Devi [1]. The idea of utilizing the concept of Hukuhara difference in choosing the initial values suitably to delete any undesirable part of the solutions that may be present in certain cases, is from Lakshmikantham, Leela and Vasundhara

Devi [1]. The example worked to demonstrate this idea is also from the above mentioned paper. The example uses interval analysis, for which, see Moore [1] and Markov [1].

The contents of Sections 3.5 and 3.6, where the method of perturbing Lyapunov functions is employed to prove nonuniform stability and boundedness results, are from Gnana Bhaskar and Vasundhara Devi [2].For the idea of perturbing Lyapunov functions pertaining to ODE, refer to Lakshmikantham and Leela [4]. See also Lakshmikantham, Leela and Martynyuk [1].The introduction of set differential systems and extension of the method of vector Lyapunov functions for such systems reported in Sections 3.7 and 3.8, is adapted from Gnana Bhaskar and Vasundhara Devi [3]. For more details on the method of vector Lyapunov functions, see Lakshmikantham, Matrosov and Sivasundaram [1]. The criteria described in Sections 3.9 and 3.10 is taken from Gnana Bhaskar and Lakshmikantham [3], where the ideas of nonsmooth analysis and weaker lower semicontinuous functions are employed.

Chapter 4

Connection to Fuzzy Differential Equations

4.1 Introduction

When a real world problem is transferred into a deterministic IVP of ordinary differential equations, namely,

$$x' = f(t, x), \quad x(t_0) = x_0,$$

we cannot usually be sure that the model is perfect. If the underlying structure of the model depends upon subjective choices, one way to incorporate these into the model, is to utilize the aspect of fuzziness, which leads to the consideration of fuzzy differential equations(FDE). There exists sufficient literature on the theory of FDEs. The intricacies involved in incorporating fuzziness into the theory of ordinary differential equations pose a certain disadvantage and other possibilities are being explored to address this problem. One of the approaches is to transform FDEs into multivalued differential inclusions so as to employ the existing theory of differential inclusions. Another approach is to connect FDEs to SDEs and examine the interconnection between them. In this chapter we shall be concerned with the latter approach, since the former framework is already known, see Lakshmikantham and Mohapatra [1].

In Section 4.2, we provide a short account of the necessary preliminary material on fuzzy set theory, formulate FDEs and list the required known results concerning local and global existence, uniqueness, continuous dependence of solutions and comparison results. Section 4.3 is devoted to the investigation of Lyapunov stability theory through Lyapunov-like functions. For this purpose, we develop a comparison result in terms of Lyapunov-like functions and then give some simple stability results. We sketch an example to expose the difficulties involved in general and suggest a way out in those cases where there is a problem, by suitably choosing the initial value in order to weed out the undesirable part of the solution. This may be considered parallel to partial stability or conditional stability in ODEs. The interconnection of FDEs with SDEs is explored in Section 4.4. We indicate also the alternative formulation of fuzzy IVPs into a sequence of multivalued differential inclusions and the advantage of the approach.

In Section 4.5, we continue to study the interconnection for the case when the function involved in the SDE is only upper semicontinuous and prove some results parallel to the continuous case considered in Section 4.4. Section 4.6 introduces impulses into FDEs and shows how the impulses help to overcome the disadvantage that exists in the study of FDEs. Some important results relative to impulsive fuzzy differential equations are proved in this section and a familiar example is discussed to point out the advantage gained by adding impulses. Hybrid fuzzy differential equations are considered in Section 4.7 and the required structure is developed for the investigation of the stability of such systems. Section 4.8 introduces another concept of differential equations in a metric space which can be applied to study FDEs. Notes and comments are given in Section 4.9.

4.2 Preliminaries

Let $K_c(\mathbb{R}^n)$ denote the family of all nonempty, compact, convex subsets of $\mathbb{R}$. If $\alpha, \beta \in \mathbb{R}$ and $A, B \in K_c(\mathbb{R}^n)$, then

$$\alpha(A+B) = \alpha A + \alpha B, \quad \alpha(\beta A) = (\alpha\beta)A, \quad 1A = A$$

and if $\alpha, \beta \geq 0$, then $(\alpha+\beta)A = \alpha A + \beta A$.

Let $I = [t_0, t_0 + a]$, $t_0 \geq 0$ and $a > 0$ and E^n be the set of all functions $u : \mathbb{R}^n \to [0, 1]$ such that u satisfies (i)-(iv) mentioned below :

(i) u is normal, that is , there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$;

(ii) u is fuzzy convex , that is , for $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$,

$$u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\};$$

(iii) u is upper semicontinuous;

(iv) $[u]^0 =$ closure of $\{x \in \mathbb{R}^n : u(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$, we denote $[u]^\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\}$. Then from (i)-(iv), it follows that the $\alpha-$ level sets $[u]^\alpha \in K_c(\mathbb{R}^n)$, for $0 \leq \alpha \leq 1$.

Let $D[A, B]$ be the Hausdorff distance between the sets $A, B \in K_c(\mathbb{R}^n)$. We define,

$$D_0[u, v] = \sup_{0 \leq \alpha \leq 1} D[[u]^\alpha, [v]^\alpha], \tag{4.2.1}$$

which is a metric in E^n, and (E^n, D_0) is a complete metric space.

We list the following properties of $D_0[u, v]$, which are similar to $D[A, B]$ where $A, B \in K_c(\mathbb{R}^n)$:

$$D_0[u + w, v + w] = D_0[u, v], \tag{4.2.2}$$

$$D_0[\lambda u, \lambda v] = |\lambda| D_0[u, v], \tag{4.2.3}$$

$$D_0[u, v] \leq D_0[u, w] + D_0[w, v], \tag{4.2.4}$$

for all $u, v, w \in E^n$ and $\lambda \in \mathbb{R}$.

For $x, y \in E^n$ if there exists a $z \in E^n$ such that $x = y + z$, then z is called the Hukuhara difference of x and y and is denoted by $x - y$. A mapping $F : I \to E^n$ is differentiable at $t \in I$ if there exists a $D_H F(t) \in E^n$ such that the limits

$$\lim_{h \to 0+} \frac{F(t+h) - F(t)}{h}, \quad \text{and} \quad \lim_{h \to 0+} \frac{F(t) - F(t-h)}{h} \tag{4.2.5}$$

exist and are equal to $D_H F(t)$. Here the limits are taken in the metric space (E^n, D_0).

Moreover, if $F : I \to E^n$ is continuous , then it is integrable and

$$\int_{t_0}^{t_2} F = \int_{t_0}^{t_1} F + \int_{t_1}^{t_2} F, \quad t_0 \leq t_1 \leq t_2 \leq t_0 + a.$$

Further, If $F, G : I \to E^n$ are integrable, $\lambda \in \mathbb{R}$, then the following properties of the integral hold:

$$\int (F + G) = \int F + \int G;$$

$$\int \lambda F = \lambda \int F, \quad \lambda \in \mathbb{R};$$

$$D_0[F, G] \ \text{is integrable;}$$

$$D_0\left[\int F, \int G\right] \leq \int D_0[F, G].$$

Finally, let $F : I \to E^n$ be continuous. Then the integral $G(t) = \int_{t_0}^{t} F(s)\, ds$ is differentiable and $D_H G(t) = F(t)$. Furthermore,

$$F(t) - F(t_0) = \int_{t_0}^{t} D_H F(s)\, ds.$$

See for details Lakshmikantham and Mohapatra[1].

Consider the fuzzy differential system

$$D_H u = f(t, u), \quad u(t_0) = u_0, \tag{4.2.6}$$

where $f \in C[I \times E^n, E^n]$ and $I = [t_0, t_0 + a],\ t_0 \geq 0,\ a > 0$.

Before proceeding further, we note that a mapping $u : I \to E^n$ is a solution of the initial value problem (4.2.6) if and only if it is continuous and satisfies the integral equation

$$u(t) = u_0 + \int_{t_0}^{t} f(s, u(s)),\ ds \quad \text{for} \ t \in I.$$

Using the properties of $D_0[u,v]$ and of the integral listed above, and the theory of differential and integral inequalities, one can establish the following results concerning comparison principles, existence and uniqueness, continuous dependence and global existence of solutions of (4.2.6). We merely state such results whose proofs are almost similar to the corresponding results for set differential equations discussed in Chapter 2. One can also see the proofs in Lakshmikantham and Mohapatra [1].

Theorem 4.2.1 *Assume that $f \in C[I \times E^n, E^n]$ and for $t \in I,\ u, v \in E^n$,*

$$D_0[f(t,u), f(t,v)] \quad \leq \quad g(t, D_0[u,v]),$$

where $g \in C[I \times \mathbb{R}_+, \mathbb{R}_+]$ and $g(t,w)$ is nondecreasing in w for each t. Suppose further that the maximal solution $r(t) = r(t, t_0, w_0)$ of the scalar differential equation

$$w' \quad = \quad g(t,w), \quad w(t_0) = w_0 \quad \geq 0, \tag{4.2.7}$$

exists on I. Then, if $u(t), v(t)$ are any two solutions of (4.2.6) through (t_0, u_0), (t_0, v_0) respectively on I, and if $D_0[u_0, v_0] \leq w_0$, we have

$$D_0[u(t), v(t)] \leq r(t, t_0, w_0), \quad t \in I. \tag{4.2.8}$$

Remark 4.2.1 *If we employ the theory of differential inequalities instead of integral inequalities, we can dispense with the monotone character of $g(t,w)$ assumed in Theorem 4.2.1. This is the next comparison principle.*

Theorem 4.2.2 *Let the assumptions of Theorem 4.2.1 hold except for the nondecreasing property of $g(t,w)$ in w. Then the conclusion (4.2.8) is valid.*

The next comparison result provides an estimate under weaker assumptions.

Theorem 4.2.3 *Assume that $f \in C[I \times E^n, E^n]$ and*

$$\limsup_{h \to 0+} \frac{D_0[u + hf(t,u), v + hf(t,v)] - D_0[u,v]}{h} \leq g(t, D_0[u,v]),\ t \in I,$$

where $g \in C[I \times \mathbb{R}_+, \mathbb{R}],\ u, v \in E^n$. The maximal solution $r(t, t_0, w_0)$ of (4.2.7) exists on I. Then, the conclusion of the Theorem 4.2.1 is valid.

We wish to remark that in Theorem 4.2.3, $g(t,w)$ need not be non-negative, and therefore the estimate (4.2.8) would be finer than the estimates in Theorems 4.2.1 and 4.2.2.

As a special case of Theorems 4.2.1, 4.2.2, 4.2.3 we have the following important corollary.

Corollary 4.2.1 *Assume that $f \in C[I \times E^n, E^n]$ and either*

(a) $D_0[f(t,u), \hat{0}] \leq g(t, D_0[u, \hat{0}])$ *or*

(b) $\displaystyle\limsup_{h \to 0+} \frac{D_0[u + hf(t,u), \hat{0}] - D_0[u, \hat{0}]}{h} \leq g(t, D_0[u, \hat{0}]),\ t \in I,$

where $g \in C[I \times \mathbb{R}_+, \mathbb{R}]$. *Then, if* $D_0[u_0, \hat{0}\,] \le w_0$, *we have*

$$D_0[u(t), \hat{0}\,] \le r(t, t_0, w_0), \quad t \in I,$$

where $r(t, t_0, w_0)$ *is the maximal solution of (4.2.7) on* I *and* $\hat{0} \in E^n$ *is defined as*

$$\hat{0}\,(x) = \begin{cases} 1 & if\, x = 0, \\ 0 & if\, x \neq 0. \end{cases}$$

Theorem 4.2.4 *Assume that*

(a) $f \in C[R_0, E^n]$, $D_0[f(t,x), \hat{0}\,] \le M_0$, *on* R_0; *where* $R_0 = I \times B(u_0, b)$, $B(u_0, b) = \{x \in E^n : D_0[u, u_0] \le b\}$ *and*

(b) $g \in C[I \times [0, 2b], \mathbb{R}_+], 0 \le g(t,w) \le M_1$ *on* $I \times [0, 2b]$, $g(t, 0) = 0$, $g(t,w)$ *is nondecreasing in* w *for each* $t \in I$ *and* $w(t) \equiv 0$ *is the unique solution of (4.2.7) on* I;

(c) $D_0[f(t,u), f(t,v)] \le g(t, D_0[u,v])$ *on* R_0.

Then the successive approximations defined by

$$u_{n+1}(t) = u_0 + \int_{t_0}^{t} f(s, u_n(s))\, ds, \quad n = 0, 1, 2, \cdots,$$

exist on $[t_0, t_0 + \eta]$, *where* $\eta = min\{a, \frac{b}{M}\}$, $M = max\{M_0, M_1\}$, *as continuous functions and converge uniformly to the unique solution* $u(t)$ *of the IVP (4.2.6) on* $[t_0, t_0 + \eta]$.

Theorem 4.2.5 *Suppose that the assumptions of Theorem 4.2.4. hold. Also, further that the solutions* $w(t, t_0, w_0)$ *of (4.2.7) through every point* (t_0, w_0) *are continuous with respect to* (t_0, w_0). *Then the solutions* $u(t, t_0, u_0)$ *of (4.2.6) are continuous relative to* (t_0, u_0).

Theorem 4.2.6 *Assume that* $f \in C[\mathbb{R}_+ \times E^n, E^n]$ *and*

$$D_0[f(t,u), \hat{0}\,] \le g(t, D_0[\,u, \hat{0}\,]\,), \quad (t,u) \in \mathbb{R}_+ \times E^n,$$

where $g \in C[\mathbb{R}_+^2, \mathbb{R}_+]$, $g(t,w)$ *is nondecreasing in* w *for each* $t \in \mathbb{R}_+$ *and the maximal solution* $r(t, t_0, w_0)$ *of (4.2.7) exists on* $[t_0, \infty)$. *Suppose further that* f *is smooth enough to guarantee local existence of solutions of (4.2.6) for any* $(t_0, u_0) \in \mathbb{R}_+ \times E^n$. *Then the largest interval of existence of any solution* $u(t, t_0, u_0)$ *of (4.2.6) such that* $D_0[u_0, \hat{0}\,] \le w_0$ *is* $[t_0, \infty)$.

4.3 Lyapunov-like functions

Consider the fuzzy differential equation

$$D_H u = f(t,u), \quad u(t_0) = u_0, \tag{4.3.1}$$

where $f \in C[\mathbb{R}_+ \times S(\rho), E^n]$ and $S(\rho) = \{u \in E^n : D_0[u, \hat{0}\} < \rho]$.

We assume that $f(t, \hat{0}) = \hat{0}$, so that we have the trivial solution for (4.3.1).

To investigate the stability criteria, the following comparison result in terms of the Lyapunov-like function is very important and can be proved via the theory of ordinary differential inequalities. Here the Lyapunov-like function serves as a vehicle to transform the fuzzy differential equation into scalar comparison differential equation, and therefore it is enough to consider the qualitative properties of the simpler comparison equation.

Theorem 4.3.1 *Assume that*

(i) $V \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ *and* $|\ V(t, u_1) - V(t, u_2)\ | \leq\ L\ D_0[u_1, u_2]$*, where* $L > 0$,

(ii) $D^+V(t, u) \equiv \limsup_{h \to 0^+} \frac{1}{h}[V(t+h, u+hf(t, u)) - V(t, u)] \leq g(t, V(t, u))$*, where* $g \in C[\mathbb{R}_+^2, \mathbb{R}]$.

Then, if $u(t)$ *is any solution of (4.3.1) existing on* (t_0, ∞) *such that* $V(t_0, u_0) \leq w_0$*, we have*

$$V(t, u(t)) \leq r(t, t_0, w_0), \quad t \geq t_0,$$

where $r(t, t_0, w_0)$ *is the maximal solution of the scalar differential equation*

$$w' = g(t, w), \quad w(t_0) = w_0 \geq 0, \tag{4.3.2}$$

existing on $[t_0, \infty)$.

Proof Let $u(t)$ be any solution of (4.3.1) existing on $[t_0, \infty)$. Define $m(t) = V(t, u(t))$ so that $m(t_0) = V(t_0, u_0) \leq w_0$. For small $h > 0$,

$$\begin{aligned} m(t+h) - m(t) &= V(t+h, u(t+h)) - V(t, u(t)) \\ &= V(t+h, u(t+h)) - V(t+h, u(t) + hf(t, u(t))) \\ &\quad + V(t+h, u(t) + hf(t, u(t))) - V(t, u(t)) \\ &\leq LD_0[u(t+h), u(t) + hf(t, u(t))] \\ &\quad + V(t+h, u(t) + hf(t, u(t))) - V(t, u(t)), \end{aligned}$$

using the Lipschitz condition given in (i). Thus

$$\begin{aligned} D^+m(t) &= \limsup_{h \to 0^+} \frac{1}{h}[m(t+h) - m(t)] \\ &\leq D^+V(t, u(t)) + L \limsup_{h \to 0^+} \frac{1}{h}[D_0[u(t+h), u(t) + hf(t, u(t))]]. \end{aligned}$$

Let $u(t+h) = u(t) + z(t, h)$, where $z(t, h)$ is the Hukuhara difference for small $h > 0$, which is assumed to exist. Hence employing the properties of $D_0[u, v]$, we see that

$$\begin{aligned} D_0[u(t+h), u(t) + hf(t, u(t))] &= D_0[u(t) + z(t, h), u(t) + hf(t, u(t))] \\ &= D_0[z(t, h), hf(t, u(t))] \\ &= D_0[u(t+h) - u(t), hf(t, u(t))]. \end{aligned}$$

Consequently,

$$\frac{1}{h}D_0[u(t+h), u(t)+hf(t,u(t))] = D_0[\frac{u(t+h)-u(t)}{h}, f(t,u(t))],$$

and therefore,

$$\begin{aligned}\limsup_{h\to 0^+} &\frac{1}{h}[D_0[u(t+h), u(t)+hf(t,u(t))]]\\ &= \limsup_{h\to 0^+}\frac{1}{h}D_0\left[\frac{u(t+h)-u(t)}{h}, f(t,u(t))\right],\\ &= D_0[D_H u(t), f(t,u(t))] \equiv 0,\end{aligned}$$

since $u(t)$ is a solution of (4.3.1). We therefore have the scalar differential inequality

$$D^+m(t) \le g(t,m(t)), \quad m(t_0) \le w_0, \ \ t \ge t_0,$$

which yields by the theory of differential inequalities (see Lakshmikantham and Leela [1])

$$m(t) \le r(t,t_0,w_0), \quad t \ge t_0.$$

This proves the claimed estimate of the theorem.

The following corollaries are useful.

Corollary 4.3.1 *The function $g(t,w) \equiv 0$ is admissible in Theorem 4.3.1 to yield the estimate*

$$V(t,u(t)) \le V(t_0,u_0), \quad t \ge t_0.$$

Corollary 4.3.2 *If, in Theorem 4.3.1, we strengthen the assumption on $D^+V(t,u)$ to*

$$D^+V(t,u) \le -c[w(t,u)] + g(t,V(t,u)),$$

where $w \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$, $c \in \mathcal{K} = \{a \in C[[0,\rho),\mathbb{R}_+] : a(w)$ is increasing in w and $a(0)=0\}$, and $g(t,w)$ is nondecreasing in w for each $t \in \mathbb{R}_+$, then we get the estimate

$$V(t,u(t)) + \int_{t_0}^{t} c[w(s,u(s))]\,ds \ \le \ r(t,t_0,w_0), \quad t \ge t_0,$$

whenever $V(t_0,u_0) \le w_0$.

Proof Set $L(t,u(t)) = V(t,u(t)) + \int_{t_0}^{t} c[w(s,u(s))]\,ds$, and note that

$$\begin{aligned} D^+L(t,u(t)) &\le D^+V(t,u(t)) + c[w(t,u(t))]\\ &\le g(t,V(t,u(t))) \le g(t,L(t,u(t))),\end{aligned}$$

using the monotone character of $g(t,w)$. We then get immediately by Theorem 4.3.1, the estimate

$$L(t,u(t)) \ \le r(t,t_0,w_0), \ \ t \ge t_0,$$

where $u(t)$ is any solution of (4.3.1). This implies the stated estimate.

A simple example of $V(t,u)$ is $D_0[u,\hat{0}]$ so that

$$D^+V(t,u) = \limsup_{h\to 0^+} \frac{1}{h}[D_0[u+hf(t,u),\hat{0}\,] - D_0[u,\hat{0}\,]].$$

Having the necessary comparison results in terms of Lyapunov-like functions, it is easy to establish stability results analogous to original Lyapunov results for fuzzy differential equations.

We shall assume that (4.3.1) possesses the trivial solution, and the solutions exist and are unique for $t \geq 0$.

Recall that the approach in the formulation of fuzzy differential equation (FDE) (4.3.1) is based on the fuzzification of the differential operator, whose values are in E^n and therefore suffers from the disadvantage (in view of Corollary 2.5.1 in Lakshmikantham and Mohapatra[1]) that the solution $u(t)$ of (4.3.1) has the property that $\text{diam}[u(t)]^\alpha$ is nondecreasing as t increases. Consequently, it is concluded that this formulation of FDE is not suitable for reflecting the rich behaviour of the solutions of the corresponding ordinary differential equation (ODE). As a result, following the suggestion of Hüllermeier[1], an alternative formulation leading to multivalued differential inclusions has been investigated, which we shall discuss in the next section. Here we shall utilize the Hukuhara difference in the initial values in such a way that a subset of the solutions of (4.3.1) matches the behaviour of the solutions of the corresponding ODE.

For this purpose, we suppose that for any $u_0, v_0 \in E^n$, there exists a $z_0 \in E^n$ such that Hukuhara difference $u_0 - v_0 = z_0$ exists. Then, we consider the stability of the solutions $u(t,t_0,u_0 - v_0) = u(t,t_0,z_0)$ of (4.3.1) with respect to the trivial solution of (4.3.1). This approach helps to delete an undesirable part of the solution generated.

On the other hand, if the FDE is given by

$$D_H u = \lambda(t)u, \;\; \lambda \in C[\mathbb{R}_+,\mathbb{R}_+], \;\; \lambda \in L^1(\mathbb{R}_+),$$

the foregoing situation does not exist, and therefore it matches in its behaviour with corresponding ODE. Hence there is no need to use the Hukuhara difference in the initial values to obtain the desirable part of the solution.

Let us consider the following standard example. Let $a \in E^1$ have level sets $[a]^\alpha = [a_1^\alpha, a_2^\alpha]$ for $\alpha \in I = [0,1]$ and suppose that a solution $u : [0,T] \to E^1$ of the fuzzy differential equation

$$D_H u = au, \;\; u(0) = u_0 \in E^1, \tag{4.3.3}$$

on E^1 has level sets $[u(t)]^\alpha = [u_1^\alpha(t), u_2^\alpha(t)]$, for $\alpha \in I$ and $t \in [0,T]$.

The Hukuhara derivative $D_H u$ also has level sets

$$\left[\frac{du}{dt}(t)\right]^\alpha = \left[\frac{du_1^\alpha}{dt}(t), \frac{du_2^\alpha}{dt}(t)\right],$$

for $\alpha \in I$ and $t \in [0,T]$.

By the extension principle, the fuzzy set $f(t,u(t)) = au(t)$ has level sets,

$$[au(t)]^\alpha = [min\{a_1^\alpha u_1^\alpha(t), a_2^\alpha u_1^\alpha(t), a_1^\alpha u_2^\alpha(t), a_2^\alpha u_2^\alpha(t)\},$$
$$max\{a_1^\alpha u_1^\alpha(t), a_2^\alpha u_1^\alpha(t), a_1^\alpha u_2^\alpha(t), a_2^\alpha u_2^\alpha(t)\}],$$

for all $\alpha \in I$ and $t \in [0,T]$.

Thus the fuzzy differential equation (4.3.3) is equivalent to the coupled system of ordinary differential equations

$$\begin{aligned} \frac{du_1^\alpha}{dt} &= min\{a_1^\alpha u_1^\alpha, a_2^\alpha u_1^\alpha, a_1^\alpha u_2^\alpha, a_2^\alpha u_2^\alpha\}, \\ \frac{du_2^\alpha}{dt} &= max\{a_1^\alpha u_1^\alpha, a_2^\alpha u_1^\alpha, a_1^\alpha u_2^\alpha, a_2^\alpha u_2^\alpha\}, \end{aligned} \tag{4.3.4}$$

for $\alpha \in I$.

For $a = \chi_{\{-1\}} \in E^1$, the fuzzy differential equation (4.3.3) becomes

$$D_H u = -u, \quad u(0) = u_0, \tag{4.3.5}$$

and the system of ordinary differential equations (4.3.4) reduces to

$$\frac{du_1^\alpha}{dt} = -u_2^\alpha, \quad \frac{du_2^\alpha}{dt} = -u_1^\alpha,$$

for $\alpha \in I$.

If the level sets of the initial value $u_0 \in E^1$ are $[u_0]^\alpha = [u_{01}^\alpha, u_{02}^\alpha]$ for $\alpha \in I$, then the level sets of the solution u of (4.3.5)are given by, $[u(t)]^\alpha = [u_1^\alpha(t), u_2^\alpha(t)]$ where

$$u_1^\alpha(t) = \frac{1}{2}(u_{01}^\alpha - u_{02}^\alpha)e^t + \frac{1}{2}(u_{01}^\alpha + u_{02}^\alpha)e^{-t}, \tag{4.3.6}$$

$$u_2^\alpha(t) = \frac{1}{2}(u_{02}^\alpha - u_{01}^\alpha)e^t + \frac{1}{2}(u_{01}^\alpha + u_{02}^\alpha)e^{-t}, \tag{4.3.7}$$

for $0 \le \alpha \le 1$ and $t \ge 0$.

Let us suppose that for $v_0, z_0 \in E^1$, such that the Hukuhara difference $u_0 - v_0 = z_0$ exists, we have

$$[u_0]^\alpha = [v_0 + z_0]^\alpha = [v_0]^\alpha + [z_0]^\alpha.$$

Let us suppose that for $v_0, z_0 \in E^1$, such that the Hukuhara difference $u_0 - v_0 = z_0$ exists, we have

$$[u_0]^\alpha = [v_0 + z_0]^\alpha = [v_0]^\alpha + [z_0]^\alpha.$$

Since $[u_0]^\alpha = [u_{01}^\alpha, u_{02}^\alpha]$, let us choose $[v_0]^\alpha = \left[\frac{u_{01}^\alpha - u_{02}^\alpha}{2}, \frac{u_{02}^\alpha - u_{01}^\alpha}{2}\right]$, so that $[z_0]^\alpha = \left[\frac{u_{01}^\alpha + u_{02}^\alpha}{2}, \frac{u_{02}^\alpha + u_{01}^\alpha}{2}\right]$.

Then it follows, assuming $u_{01}^\alpha \neq -u_{02}^\alpha$, that

$$[u(t,u_0)]^\alpha = \left[-\frac{u_{02}^\alpha - u_{01}^\alpha}{2}, \frac{u_{02}^\alpha - u_{01}^\alpha}{2}\right]e^t + \left[\frac{u_{01}^\alpha + u_{02}^\alpha}{2}, \frac{u_{02}^\alpha + u_{01}^\alpha}{2}\right]e^{-t},$$

$$[u(t,v_0)]^\alpha = \left[\frac{1}{2}(u_{01}^\alpha - u_{02}^\alpha), \frac{1}{2}(u_{02}^\alpha - u_{01}^\alpha)\right] e^t,$$

and

$$[u(t,z_0)]^\alpha = \left[\frac{1}{2}(u_{01}^\alpha + u_{02}^\alpha), \frac{1}{2}(u_{02}^\alpha + u_{01}^\alpha)\right] e^{-t},\ t \geq 0.$$

If on the other hand, $u_{01}^\alpha = -u_{02}^\alpha$, that is $[u_0]^\alpha = [-d^\alpha, d^\alpha]$ with $d^\alpha = u_{02}^\alpha$. Then the choice of $[v_0]^\alpha = [c^\alpha - d^\alpha, c^\alpha + d^\alpha]$ for some c^α so that $[z_0]^\alpha = [c^\alpha, c^\alpha]$ and we find $[v_0]^\alpha = [u_0]^\alpha + [z_0]^\alpha$, by changing the roles of u_0, v_0.

We note that, in general, for any initial value $[u_0]^\alpha$ for which $u_{01}^\alpha \neq u_{02}^\alpha$, the solution of (4.3.5) contains both desired and undesired parts of solution compared with the solution of the corresponding ODE. In order to isolate the desired part of the solution $u(t, u_0)$ of (4.3.5) that matches the solution of ODE, we need to utilize the initial values satisfying the desired Hukuhara difference.

We are now ready to prove the following stability results by means of Lyapunov-like functions utilizing the solutions $u(t, t_0, z_0) = u(t, t_0, u_0 - v_0)$ of (4.3.1). For this purpose, we list a typical definition of stability so that others can be formulated on the basis of this and standard definitions in stability theory.

Definition 4.3.1 *The trivial solution of (4.3.1) is said to be equi-stable if for each $\varepsilon > 0$ and $t_0 \geq 0$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $D_0[z_0, \hat{0}] < \delta$ implies $D_0[u(t, t_0, z_0), \hat{0}\,] < \varepsilon,\ t \geq t_0$.*

Theorem 4.3.2 *Assume that the following hold:*

(i) $V \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$, $|V(t, u_1) - V(t, u_2)| \leq LD_0[u_1, u_2],\ L > 0$ and for $(t, u) \in \mathbb{R}_+ \times S(\rho)$, where $S(\rho) = \{u \in E^n : D_0[u, \hat{0} < \rho\}$,

$$D^+V(t,u) \equiv \limsup_{h \to 0^+} \frac{1}{h}[V(t+h, u + hf(t,u)) - V(t,u)] \leq 0; \qquad (4.3.8)$$

(ii) $b(D_0[u, \hat{0}\,]) \leq V(t, u) \leq a(t, D_0[u, \hat{0}\,])$, for $(t, u) \in \mathbb{R}_+ \times S(\rho)$ where $b, a(t, .) \in \mathcal{K} = \{\sigma \in C[[0, \rho), \mathbb{R}_+] : \sigma(0) = 0$ and $\sigma(\omega)$ is increasing in $\omega\}$.

Then, the trivial solution of (4.3.1) is equi-stable.

Proof Let $0 < \varepsilon < \rho$ and $t_0 \in \mathbb{R}_+$ be given. Choose a $\delta = \delta(t_0, \varepsilon)$ such that

$$a(t_0, \delta) < b(\varepsilon). \qquad (4.3.9)$$

We claim that with this δ, equi-stability holds. If not, there would exist a solution $u(t) = u(t, t_0, z_0)$ of (4.3.1) with $D_0[z_0, \hat{0}] < \delta$ and $t_1 > t_0$ such that

$$D_0[u(t_1), \hat{0}\,] = \varepsilon \quad \text{and} \quad D_0[u(t), \hat{0}\,] \leq \varepsilon < \rho, \quad t_0 \leq t \leq t_1. \qquad (4.3.10)$$

By Corollary 4.3.1, we then have

$$V(t, u(t)) \leq V(t_0, z_0), \quad t_0 \leq t \leq t_1.$$

Consequently, using (ii), (4.3.9) and (4.3.10), we arrive at the following contradiction:

$$b(\varepsilon) = b(D_0[u(t_1), \hat{0}\,]) \le V(t_1, u(t_1)) \le V(t_0, z_0) \le a(t_0, D_0[z_0, \hat{0}\,]) < b(\varepsilon).$$

Hence equi-stability holds, completing the proof.

The next result provides sufficient conditions for equi-asymptotic stability. In fact, it gives exponential asymptotic stability.

Theorem 4.3.3 *Let the assumptions of Theorem 4.3.2 hold except that the estimate (4.3.8) be strengthened to*

$$D^+V(t,u) \le -\beta V(t,u), \quad (t,u) \in \mathbb{R}_+ \times S(\rho). \tag{4.3.11}$$

Then the trivial solution of (4.3.1) is equi-asymptotically stable.

Proof Clearly, the trivial solution of (4.3.1) is equi-stable. Hence taking $\varepsilon = \rho$ and designating $\delta_0 = \delta(t_0, \rho)$, we have by Theorem 4.3.2.

$$D_0[z_0, \hat{0}\,] < \delta_0 \text{ implies } D_0[u(t), \hat{0}\,] < \rho, \quad t \ge t_0.$$

Consequently, we get from the assumption (4.3.11), the estimate

$$V(t, u(t)) \le V(t_0, z_0)\; exp[-\beta(t - t_0)], \quad t \ge t_0.$$

Given $\varepsilon > 0$, we choose $T = T(t_0, \varepsilon) = \dfrac{1}{\beta}\; ln \dfrac{a(t_0, \delta_0)}{b(\varepsilon)} + 1$. Then, it is easy to see that,

$$b(D_0[u(t), \hat{0}\,]) \le V(t, u(t)) \le a(t_0, \delta)\; e^{-\beta(t - t_0)} \; < \; b(\varepsilon), \quad t \ge t_0 + T.$$

The proof is complete.

We shall next consider the uniform stability criteria.

Theorem 4.3.4 *Assume that, for* $(t,u) \in \mathbb{R}_+ \times S(\rho) \cap S^c(\eta)$ *for each* $0 < \eta < \rho$, $V \in C[\mathbb{R}_+ \times S(\rho) \cap S^c(\eta), \mathbb{R}_+]$, $|V(t,u_1) - V(t,u_2)| \le LD_0[u_1, u_2]$, $L > 0$,

$$D^+V(t,u) \le 0, \tag{4.3.12}$$

and

$$b(D_0[u, \hat{0}\,]) \le V(t,u) \le a(D_0[u, \hat{0}\,]), \quad a, b \in \mathcal{K}. \tag{4.3.13}$$

Then the trivial solution of (4.3.1) is uniformly stable.

Proof Let $0 < \varepsilon < \rho$ and $t_0 \in \mathbb{R}_+$ be given. Choose $\delta = \delta(\varepsilon) > 0$ such that $a(\delta) < b(\varepsilon)$. Then we claim that with this δ, uniform stability follows. If not, there would exist a solution $u(t) = u(t, t_0, z_0)$ of (4.3.1), and a $t_2 > t_1 > t_0$ satisfying

$$D_0[u(t_1), \hat{0}\,] = \delta, \;\; D_0[u(t_2), \hat{0}\,] = \varepsilon \quad \text{and} \;\; \delta \le D_0[u(t), \hat{0}\,] \le \epsilon < \rho. \tag{4.3.14}$$

Taking $\eta = \delta$, we get from (4.3.12), the estimate

$$V(t_2, u(t_2)) \leq V(t_1, u(t_1)).$$

This, together with (4.3.13), (4.3.14), and the definition of δ, yield

$$\begin{aligned} b(\varepsilon) &= b(D_0[u(t_2), \hat{0}\,]) \\ &\leq V(t_2, u(t_2)) \leq V(t_1, u(t_1)) \\ &\leq a(D_0[u(t_1), \hat{0}\,]) = a(\delta) < b(\varepsilon). \end{aligned}$$

This contradiction proves uniform stability, completing the proof.

Finally, we shall prove uniform asymptotic stability.

Theorem 4.3.5 *Let the assumptions of Theorem 4.3.4 hold except that (4.3.12) is strengthened to*

$$D^+V(t, u) \leq -c(D_0[u, \hat{0}\,]), \quad c \in \mathcal{K}. \tag{4.3.15}$$

Then the trivial solution of (4.3.1) is uniformly asymptotically stable.

Proof By Theorem 4.3.4, uniform stability follows. And, for $\varepsilon = \rho$, we designate $\delta_0 = \delta_0(\rho)$. This means,

$$D_0[z_0, \hat{0}\,] < \delta_0 \text{ implies } D_0[u(t), \hat{0}\,] < \rho, \quad t \geq t_0,$$

where $u(t) = u(t, t_0, z_0)$ is the solution of (4.3.1).

In view of the uniform stability, it is enough to show that there exists a t^* such that for $t_0 \leq t^* \leq t_0 + T$, where $T = 1 + \dfrac{a(\delta_0)}{c(\delta)}$,

$$D_0[u(t^*), \hat{0}\,] < \delta. \tag{4.3.16}$$

If this is not true, $\delta \leq D_0[u(t), \hat{0}\,]$, for $t_0 \leq t \leq t_0 + T$. Then, (4.3.15) gives,

$$V(t, u(t)) \leq V(t_0, z_0) - \int_{t_0}^{t} c(D_0[u(s), \hat{0}\,])\, ds, \quad t_0 \leq t \leq t_0 + T.$$

As a result, we have, in view of the choice of T,

$$0 \leq V(t_0 + T, u(t_0 + T)) \leq a(\delta_0) - c(\delta)T < 0,$$

a contradiction. Hence there exists a t^* satisfying (4.3.16) and from uniform stability we conclude that

$$D_0[z_0, \hat{0}\,] < \delta_0 \text{ implies } D_0[u(t), \hat{0}\,] < \varepsilon, \quad t \geq t_0 + T,$$

and the proof is complete.

4.4 Connection with Set Differential Equations

Recall that the IVP for fuzzy differential equations proposed in Section 4.2, is of the type

$$D_H u = f(t, u), \quad u(t_0) = u_0 \in E^n, \tag{4.4.1}$$

where $f \in C[\mathbb{R}_+ \times E^n, E^n]$, for which basic results are listed there. As noted in Section 4.3., this approach is based on the fuzzification of the differential operator, whose values are in E^n and therefore suffers from the disadvantage that the solution $u(t)$ of (4.4.1) has the property that $diam[u(t)]^\alpha$ is nondecreasing as t increases. Consequently, this formulation cannot fully reflect the rich behaviour of solutions of corresponding ordinary differential equations.

Recently, Hüllermeier [1] has suggested an alternative formulation of fuzzy IVPs by replacing the RHS of a system of ordinary differential equations by a fuzzy function

$$f : \mathbb{R}_+ \times \mathbb{R}^n \to E^n,$$

with the initial fuzzy set $x_0 \in E^n$, so that one can consider the fuzzy multivalued differential inclusion

$$x' \in f(t, x), \quad x(t_0) = x_0 \in E^n, \tag{4.4.2}$$

on $J = [t_0, T]$, where now f is defined from $\mathbb{R}_+ \times \mathbb{R}^n \to E^n$, rather than $\mathbb{R}_+ \times E^n \to E^n$, as in (4.4.1). However, instead of (4.4.2), a family of multivalued differential inclusions,

$$x'_\beta \in F(t, x_\beta; \beta), \;\; x_\beta(t_0) \in [x_0]^\beta, \quad 0 \le \beta \le 1, \tag{4.4.3}$$

is investigated on J where $F(t, x, \beta) \equiv [f(t, x)]^\beta$ and $F(t, x, 0) = \overline{supp}(f(t, x))$. The idea is that the set of all solutions $S_\beta(x_0, T),\ t_0 \le t \le T$, would be $\beta-$ level of a fuzzy set $S(x_0, T)$, in the sense that all attainable sets $A_\beta(x_0, t),\ t_0 < t \le T$, are levels of a fuzzy set on $\mathbb{R}^n$. Considering $S(x_0, T)$ to be the solution of (4.4.1) thus captures both uncertainty and the rich behaviour of differential inclusion in one and the same technique.

For this purpose, the standard results of multivalued differential inclusions, under the usual conditions on F in (4.4.3) yield that the attainable set $A_\beta(x_0, t)$ is compact subset of $\mathbb{R}^n$. If F is assumed to be quasiconcave in addition, one can conclude, under reasonable assumptions, utilizing the representation theorem, the existence of a fuzzy set $u(t)$ such that $[u(t)]^\beta = A_\beta(x_0, t)$ with a similar relation for the solution set $S_\beta(x_0, T)$. See for details Lakshmikantham and Mohapatra [1].

In the literature, the following example is often quoted to demonstrate the advantage gained by the alternative approach when compared to the original one.

Consider the crisp initial value problem of ODE with unknown initial value x_0, that is ,

$$x' = -x,\ x(0) = x_0 \in [-1, 1]. \tag{4.4.4}$$

The solution of (4.4.4) when restricted to the interval $[-1, 1]$ is

$$x(t) = [-e^{-t}, e^{-t}],\ t \geq 0.$$

The fuzzy differential equation corresponding to (4.4.4) in E^1 is

$$D_H x = -x,\ x(0) = x_0 = [-1, 1],\ x_0 \in E^1. \tag{4.4.5}$$

Suppose that $[x]^\beta = [x_1^\beta, x_2^\beta],\ [D_H x]^\beta = \left[\frac{dx_1^\beta}{dt}, \frac{dx_2^\beta}{dt}\right]$ are the $\beta-$level sets for $0 \leq \beta \leq 1$. By extension principle, (4.4.5) becomes

$$\frac{dx_1^\beta}{dt} = -x_2^\beta, \quad \frac{dx_2^\beta}{dt} = -x_1^\beta, \quad 0 \leq \beta \leq 1. \tag{4.4.6}$$

The solution of (4.4.6) is given by $x_1^\beta(t) = -e^t,\ x_2^\beta(t) = e^t$ and therefore the fuzzy function $x(t)$ solving (4.4.5) is $[x(t)]^\beta = [-e^t, e^t],\ t \geq 0$, which shows that the $diam[x(t)]^\beta \to \infty$ as $t \to \infty$.

In the framework of Hüllermeier, on the other hand, fuzzy differential equation (4.4.5) is replaced by the family of inclusions

$$x'_\beta \in F(t, x_\beta; \beta) = [-x_2^\beta, -x_1^\beta],\ \ x_\beta(0) = [-1, 1]; \tag{4.4.7}$$

which has a fuzzy solution set $S([-1, 1], T)$, and fuzzy attainable set $A_\beta([-1, 1], t)$, $0 \leq t \leq T$ respectively which are defined by $\beta-$ level sets

$$S_\beta([-1, 1], t) = \{x(.) : x(t) \in [-e^{-t}, e^{-t}]\}, \quad 0 \leq t \leq T, \tag{4.4.8}$$

$$A_\beta[[-1, 1], t) = [-e^{-t}, e^{-t}], \tag{4.4.9}$$

which matches the kind of behaviour a fuzzification of the ordinary differential equation (4.4.4) should have.

The new approach shows that a fuzzification of ODE has no effect on the behaviour of solutions. Then, a question arises, why should we bother to introduce fuzziness in the originally modelled ODE without fuzziness? It is natural, in fact, when one incorporates in the ODE other phenomena such as randomness, delay, uncertainties such as fuzziness or even transform ODE into a difference equation or generate a set differential equation (SDE), to name a few, the corresponding dynamic system should exhibit the effect of such phenomena. It is not natural to expect always to have the same behaviour as that of the solutions of ODE from which the new dynamic systems are generated. Generally speaking, the theory of the corresponding dynamic systems is a lot richer than the theory of ODEs and therefore it would be interesting to investigate it as an independent discipline.

Seen from this point of view, the original formulation of FDEs, does meet the criteria. Let us give some examples, including the often quoted one described earlier, to show other possibilities. Let us start with the ODEs, that is, crisp IVPs:

(1a) $u' = -u$, or (1b) $u' + u = 0$, $u(0) = u_0$,

(2a) $u' = u$, or (2b) $u' - u = 0$, $u(0) = u_0$.

Clearly the solutions of (1) and (2) are $u(t) = u_0 e^{-t}$ and $u(t) = u_0 e^t$, $t \geq 0$, respectively.

The corresponding IVPs for FDEs are, respectively ,

$$
\begin{array}{lll}
(i) & D_H u = (-1)u, & u(0) = u_0 \in E^1, \\
(ii) & D_H u + u = 0, & u(0) = u_0 \in E^1, \\
(iii) & D_H u = u, & u(0) = u_0 \in E^1, \\
(iv) & D_H u + (-1)u = 0, & u(0) = u_0 \in E^1.
\end{array}
$$

Since (ii) and (iv) are not essentially FDE's, we introduce as a forcing term a $\sigma \in E^1$ and consider the following FDE's:

$$
\begin{array}{lll}
(ii^*) & D_H u + u = \sigma(t), & u(0) = u_0 \in E^1, \\
(iv^*) & D_H u + (-1)u = \sigma(t), & u(0) = u_0 \in E^1.
\end{array}
$$

Suppose that the solutions of FDEs have level sets

$$[u(t)]^\alpha = [u_1^\alpha(t), u_2^\alpha(t)], \quad [u_0]^\alpha = [\alpha - 1, 1 - \alpha],$$

and $[D_H u]^\alpha = [\frac{du_1^\alpha}{dt}, \frac{du_2^\alpha}{dt}]$ and $\sigma^\alpha(t) = [(\alpha - 1), (1 - \alpha)]e^{-t}$, for $0 \leq \alpha \leq 1$.

The FDE's $(i), (ii^*), (iii)$ and (iv^*) reduce to the following systems of ODEs

$$(I) \ [\frac{du_1^\alpha}{dt}, \frac{du_2^\alpha}{dt}] = [-u_2^\alpha, -u_1^\alpha],$$

$$(II^*) \ [\frac{du_1^\alpha}{dt}, \frac{du_2^\alpha}{dt}] + [u_1^\alpha, u_2^\alpha] = [(\alpha - 1), (1 - \alpha)]e^{-t},$$

$$(III) \ [\frac{du_1^\alpha}{dt}, \frac{du_2^\alpha}{dt}] = [u_1^\alpha, u_2^\alpha],$$

$$(IV^*) \ [\frac{du_1^\alpha}{dt}, \frac{du_2^\alpha}{dt}] + [-u_2^\alpha, -u_1^\alpha] = [(\alpha - 1), (1 - \alpha)]e^{-t}.$$

with the same initial condition $[u_0]^\alpha = [\alpha - 1, 1 - \alpha]$. The solutions of these equations, using the methods of interval analysis, are

$$
\begin{array}{lll}
(Ia) & [u(t)]^\alpha & = [\alpha - 1, 1 - \alpha]e^t, \\
(II^*a) & [u(t)]^\alpha & = [\alpha - 1, 1 - \alpha](e^{-t}(1 + t)), \\
(IIIa) & [u(t)]^\alpha & = [\alpha - 1, 1 - \alpha]e^t, \\
(IV^*a) & [u(t)]^\alpha & = [\alpha - 1, 1 - \alpha](e^{-t}(1 + t)).
\end{array}
$$

The solutions (Ia) and (II^*a) represent typical behaviours corresponding to ODE from which they are generated. They show that introducing fuzziness into

ODEs, some times destroys the good behaviour of solutions and helps at some other times. For a discussion that solution behavior depends on the choice of the forcing term, see Kaleva [3].

On the other hand, if we consider the family of multivalued differential inclusions (MDI), for (IV^*), we obtain,

$$0 \in u'_\alpha + [-u_2^\alpha, -u_1^\alpha] + [(\alpha - 1), (1 - \alpha)]e^{-t}, \;\; u(0) = [\alpha - 1, 1 - \alpha],$$

which has as its attainable set

$$A_\alpha[[\alpha - 1, 1 - \alpha], t] = [\alpha - 1, 1 - \alpha] \; (\frac{3}{2}e^t - e^{-t}), \; t \geq 0.$$

The foregoing analysis of examples indicates a variety of behaviors of solutions of FDEs compared to that of ODEs, from which they are generated if the initial values are chosen appropriately. Thus, it appears that study of FDEs as originally formulated is much richer than expected and needs further investigation.

We next state a known result that relates the solution of set differential equation to the attainable set of multivalued differential inclusion is the following (see Tolstonogov [1]).

Theorem 4.4.1 *Assume that* $F \in C[\mathbb{R}_+ \times \mathbb{R}^n, K_c(\mathbb{R}^n)]$;

$$D[F(t,x), F(t,y)] \leq g(t, \|x - y\|), \quad (t,x)\;(t,y) \;\in \mathbb{R}_+ \times \mathbb{R}^n,$$

and $D[F(t,x), \theta] \leq q(t, \|x\|), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, *where* g *and* q *satisfy the following assumptions. The functions* $g, q \in C[\mathbb{R}_+^2, \mathbb{R}_+]$, $g(t,w)$, $q(t,w)$ *are nondecreasing in* w *for each* $t \in \mathbb{R}_+$, $w(t) \equiv 0$ *is the only solution of the scalar differential equation*

$$w' = g(t,w), \;\; w(t_0) = 0,$$

on $[t_0, \infty)$ *and* $r(t, t_0, w_0)$ *is the maximal solution of the scalar differential equation*

$$w' = q(t,w), \quad w(t_0) = w_0 > 0,$$

existing on $[t_0, \infty)$. *Then, there exists a unique solution* $U(t) = U(t, t_0, U_0)$ *on* $[t_0, \infty)$ *of IVP (3.2.1) and the attainable set* $A(U_0, t)$ *of differential inclusion*

$$x' \in F(t,x), \quad x(t_0) \in U_0,$$

satisfying $A(U_0, t) \subset U(t), \quad t_0 \leq t < \infty$.

Finally, we need the following representation result (see Lakshmikantham and Mohapatra [1]).

Theorem 4.4.2 *Let* $Y_\beta \subset \mathbb{R}^n$, $0 \leq \beta \leq 1$ *be a family of compact subsets satisfying*

(i) $Y_\beta \in K(\mathbb{R}^n)$ *for all* $0 \leq \beta \leq 1$;

(ii) $Y_\beta \subseteq Y_\alpha$ *whenever* $\alpha \leq \beta$, $\alpha, \beta \in [0,1]$;

(iii) $Y_\beta = \prod_{i=1}^{\infty} Y_{\beta_i}$, *for any nondecreasing sequence* $\beta_i \to \beta$ *in* $[0,1]$.

Then, there is a fuzzy set $u \in D^n$*, such that* $[u]^\beta = Y_\beta$. *If* Y_β *are also convex, then* $u \in E^n$. *(Here* D^n *denotes the set of usc normal fuzzy sets with compact support and thus* $E^n \subset D^n$*). Conversely, the level sets* $[u]^\beta$*, of any* $u \in E^n$*, are convex and satisfy these conditions.*

It should be noted that Theorem 4.4.2 can be easily generalized from $\mathbb{R}^n$ to a Banach space.

We propose here another formulation of fuzzy differential equation (4.4.1) by a set differential equation which is generated by $\beta-$ level set of the R.H.S. of (4.4.1) where $f : \mathbb{R}_+ \times \mathbb{R}^n \to E^n$, as before.

For this purpose, consider the level set for each $\beta,\ 0 \le \beta \le 1$, and write

$$F(t,x;\beta) = [f(t,x)]^\beta \in K_c(\mathbb{R}^n).$$

Next generate the mapping $H : \mathbb{R}_+ \times K_c(\mathbb{R}^n) \times I \to K_c(\mathbb{R}^n),\ I = [0,1]$ by defining

$$H(t,A;\beta) = \overline{co} F(t,A;\beta) \tag{4.4.10}$$

for each $A \in K_c(\mathbb{R}^n)$. Then consider the family of set differential equations given by

$$D_H U_\beta = H(t,U_\beta;\beta) \quad U_\beta(t_0) = U_{0\beta} \in K_c(\mathbb{R}^n), \tag{4.4.11}$$

on $[t_0,T],\ T > t_0$, where $D_H U_\beta$ is the Hukuhara derivative for each β.

Let us list the following conditions.

(i) $F(t,x;\beta)$ is quasi-concave, that is,

1. for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,\ \alpha,\beta \in I,\ F(t,x;\alpha) \supseteq F(t,x;\beta)$, whenever $\alpha \le \beta$;
2. if β_n is nondecreasing sequence in I, converging to β, then for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,\ \cap_{n=1}^{\infty} F(t,x;\beta_n) = F(t,x;\beta)$;

(ii) $D[H(t,A;\beta),\ H(t,B;\beta)] \le g(t, D[A,B])$, for $t \in \mathbb{R}_+,\ A,B \in K_c(\mathbb{R}^n)$, $\beta \in I$;

(iii) $g \in C[\mathbb{R}_+^2, \mathbb{R}_+],\ g(t,0) \equiv 0,\ g(t,w)$ is nondecreasing in w for each $t \in \mathbb{R}_+$ and $w(t) \equiv 0$ is the only solution of

$$w' = g(t,w), \quad w(t_0) = 0, \ \text{for}\ \ t \ge t_0;$$

(iv) $D[H(t,A;\alpha),\ H(t,A;\beta)] \le L|\alpha - \beta|,\ \alpha,\beta \in I$, $(t,A) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n),\ L > 0$.

We are now in a position to prove the following result.

Theorem 4.4.3 *Suppose that the conditions (i) to (iv) are satisfied. Then there exists a unique solution* $U_\beta(t) = U_\beta(t,t_0,U_{0\beta}) \in K_c(\mathbb{R}^n),\ \beta \in I$ *of (4.4.11) and* $U_\beta(t)$ *is quasiconcave in* β *for* $t \ge t_0$*. Moreover, there exists a fuzzy set* $u(t) \in E^n$ *such that*

$$[u(t)]^\beta = U_\beta(t), \ \ t \ge t_0.$$

Proof Since f is continuous on $\mathbb{R}_+ \times \mathbb{R}^n$, $F(t,x;\beta)$ is also continuous for $(t,x,\beta) \in \mathbb{R}_+ \times \mathbb{R}^n \times I$. This implies that $H(t,A;\beta)$ is continuous map for $(t,A;\beta) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n) \times I$. Consequently, by Theorems 2.3.1 and 2.6.1 it follows that there exists a unique solution $U_\beta(t) = U(t,t_0,U_{0\beta}) \in K_c(\mathbb{R}^n)$, for $t \geq t_0$ of (4.4.11).

We first show that if $\alpha \leq \beta$, then $U_\beta(t) \subseteq U_\alpha(t)$ for $t \geq t_0$. From the definition of quasi-concavity of $F(t,x;\beta)$, it follows that $H(t,A;\beta)$ is also quasi-concave in β. Let $U_\alpha(t), U_\beta(t)$ be the solution of

$$D_H U_\alpha = H(t,U_\alpha;\alpha), \quad U_\alpha(t_0) = U_0 \in K_c(\mathbb{R}^n),$$

$$D_H U_\beta = H(t,U_\beta;\beta) \quad U_\beta(t_0) = U_0 \in K_c(\mathbb{R}^n),\ \alpha \leq \beta.$$

Then we find, using quasi-concavity

$$D_H U_\alpha = H(t,U_\alpha;\alpha) \supseteq H(t,U_\alpha;\beta), \quad \alpha \leq \beta.$$

But $H(t,U_\alpha;\beta) = H(t,U_\beta;\beta)$ because of $U_\alpha(t_0) = U_0 = U_\beta(t_0)$ and therefore $U_\alpha(t) \equiv U_\beta(t)$ by uniqueness of solutions of (4.4.11).

Thus it is clear that $U_\beta(t) \subseteq U_\alpha(t),\ \alpha \leq \beta,\ t \geq t_0$.

We shall next prove that if β_n is a nondecreasing sequence, $\beta_n \in I$, converging to β, then $U_{\beta_n}(t) \to U_\beta(t)$, uniformly on compact subsets of $[t_0,\infty)$. For this purpose, set $m(t) = D[U_{\beta_n}(t), U_\beta(t)]$ and note that $D[U_{0\beta_n}, U_{0\beta}] = m(t_0)$. We shall assume that $U_{0\beta_n} \to U_{0\beta}$, as $n \to \infty$. Then employing the properties of the metric D, the definition of Hukuhara derivative and the conditions (ii) and (iv), we arrive at the scalar differential inequality,

$$D^+ m(t) \leq g(t,m(t)) + L|\beta_n - \beta|,\ m(t_0) = D[U_{0\beta_n}, U_{0\beta}], t \geq t_0.$$

Hence by Lemma 1.3.1 in (Lakshmikantham and Leela [1]), we obtain

$$m(t) \leq r_n(t,t_0,\eta_n),$$

on any compact set $J \subset [t_0,\infty)$, where $\eta_n = D[U_{0\beta_n}, U_{0\beta}]$ and $r_n(t,t_0,\eta_n)$ is the maximal solution of

$$w' = g(t,w) + L|\beta_n - \beta|, \quad w(t_0) = \eta_n, \text{ on } J.$$

By assumption (iii), $r_n(t,t_0,\eta_n) \to r(t,t_0,0) \equiv 0$, uniformly on J as $n \to \infty$. Since $\beta_n \to \beta$ as $n \to \infty$, it follows that $m(t) \equiv 0$ on J, which in turn implies that $D[U_{\beta_n}(t), U_\beta(t)] \to 0$ as $n \to \infty$. Thus, $U_\beta(t)$ is quasi-concave in $\beta \in I$, for $t \geq t_0$. Consequently, by Theorem 4.4.2, there exists a fuzzy set $u(t) \in E^n$ such that $[u(t)]^\beta = U_\beta(t), t \geq t_0$, and this completes the proof.

To find the connection between the solution $U_\beta(t)$ of (4.4.11) and the attainability set $A_\beta(U_0,t)$ of (4.4.3) we have the following result.

Theorem 4.4.4 *Let $F \in C[\mathbb{R}_+ \times \mathbb{R}^n \times I, K_c(\mathbb{R}^n)]$ satisfy the assumptions of Theorem 4.4.1 for each $\beta \in I = [0,1]$ and assume that it is also quasiconcave in β as well. Then there exists a unique solution $U_\beta(t) = U_\beta(t,t_0,U_0)$ of (4.4.1) for $t \geq t_0$ and the attainable set $A_\beta(U_0,t)$ of the inclusion (4.4.3) such that*

$$A_\beta(U_0,t) \subset U_\beta(t,t_0,U_0), \ \ t \geq t_0. \tag{4.4.12}$$

Proof It is easy to verify that when $F(t,x;\beta)$ satisfies the assumptions required in Theorem 4.4.1, the desired conditions in Theorems 2.3.1 and 2.6.1 are also satisfied, in view of the monotone nondecreasing nature of the functions $g(t,w),\ q(t,w)$, the definition of D and the fact $H(t,A;\beta)$ is generated by $F(t,x;\beta)$. We have assumed conditions in terms of $H(t,A;\beta)$ since the set differential equations are treated as an independent subject. Thus for each $\beta \in I$, Theorem 4.4.1 yields the relation (4.4.12). Also, both $U_\beta(t)$ and $A_\beta(U_0,t)$ satisfy the assumptions of Theorem 4.4.2., because one can prove similarly the quasi-concavity of $A_\beta(U_0,t)$. Therefore, there exist fuzzy sets $u(t), v(t) \in E^n$ such that

$$[v(t)]^\beta \ = \ A_\beta(U_0,t) \text{ and } [u(t)]^\beta = U_\beta(t), \quad t \geq t_0.$$

The proof is complete.

We note that, in general, since $A_\beta(U_0,t)$ is only compact and not convex, only (4.4.12) holds. Equality in (4.4.12) is valid only in some special cases.

Recalling (4.4.7) in the example, let us generate the set differential equation from F in (4.4.7), that is

$$H(t,U,\beta) = \overline{co}F(t,A;\beta), \text{ for } A \in K_c(\mathbb{R}),$$

and thus,

$$D_H U_\beta = -U_\beta, \ \ U_\beta(0) = U_{0\beta} \in K_c(\mathbb{R}). \tag{4.4.13}$$

Since the values of the solution (4.4.13) are intervals, the equation (4.4.13) can be written as,

$$[u'_{1\beta},\ u'_{2\beta}] = [-u_{2\beta}, -u_{1\beta}], \tag{4.4.14}$$

where $U = [u_{1\beta}, u_{2\beta}]$. The relation (4.4.14) is equivalent to, taking $U_{0\beta} = [u_{10\beta}, u_{20\beta}]$, the system of equations,

$$\begin{aligned} u'_{1\beta} &= -u_{2\beta}, \qquad u_{1\beta}(0) = u_{10\beta}, \\ u'_{2\beta} &= -u_{1\beta}, \qquad u_{2\beta}(0) = u_{20\beta}, \end{aligned}$$

whose solution corresponds to (4.3.6) and (4.3.7), duly altered to the present framework, for $0 \leq \beta \leq 1$ and $t \geq 0$,

$$\begin{aligned} u_{1\beta}(t) &= \frac{1}{2}[u_{10\beta} + u_{20\beta}]e^{-t} + \frac{1}{2}[u_{10\beta} - u_{20\beta}]e^{t}, \\ u_{2\beta}(t) &= \frac{1}{2}[u_{20\beta} + u_{10\beta}]e^{-t} + \frac{1}{2}[u_{20\beta} - u_{10\beta}]e^{t}. \end{aligned}$$

Given $U_{0\beta} \in K_c(\mathbb{R})$, there exists $V_{0\beta}, W_{0\beta} \in K_c(\mathbb{R})$ such that $U_{0\beta} = V_{0\beta} + W_{0\beta}$, and hence the Hukuhara difference $U_{0\beta} - V_{0\beta} = W_{0\beta}$ exists.

Choose

$$V_{0\beta} = [\frac{1}{2}[(u_{10\beta} - u_{20\beta}), (u_{20\beta} - u_{10\beta})],$$

so that $$W_{0\beta} = \frac{1}{2}[(u_{10\beta} + u_{20\beta}), (u_{20\beta} + u_{10\beta})].$$

It then follows, assuming that $u_{10\beta} \neq -u_{20\beta}$, that

$$\begin{aligned}
U_\beta(t, U_{0\beta}) &= \frac{1}{2}[-(u_{20\beta} - u_{10\beta}),\ (u_{20\beta} - u_{10\beta})]e^t \\
&\qquad + \frac{1}{2}[(u_{10\beta} + u_{20\beta}),\ (u_{10\beta} + u_{20\beta})]e^{-t} \\
U_\beta(t, V_{0\beta}) &= \frac{1}{2}[(u_{10\beta} - u_{20\beta}),\ (u_{20\beta} - u_{10\beta})]e^t, \text{ and} \\
U_\beta(t, W_{0\beta}) &= \frac{1}{2}[(u_{10\beta} + u_{20\beta}),\ (u_{10\beta} + u_{20\beta})]e^{-t},\ t \geq 0.
\end{aligned}$$

It therefore follows that

$$A_\beta(W_{0\beta}, t) = U_\beta(t, W_{0\beta}) \subset U_\beta(t, U_{0\beta}),\ t \geq 0. \tag{4.4.16}$$

4.5 Upper Semicontinuous Case Continued

Recall that from fuzzy differential equation (4.4.1), we did generate the sequence of set differential equations given by

$$D_H U_\beta = H(t, U_\beta; \beta),\ \ U_\beta(t_0) = U_{0\beta} \in K_c(\mathbb{R}^n), \tag{4.5.1}$$

where $H(t, A; \beta) = \overline{co}F(t, A; \beta)$ and $F(t, X; \beta) = [f(t, X)]^\beta,\ 0 \leq \beta \leq 1$.(see equations (4.4.10) and (4.4.11).)

In this section, we continue to investigate the upper semicontinuous (usc) case discussed in Section 2.9 and prove some results parallel to the continuous case considered in Section 4.4.

Let us list the following hypotheses, H(F).

$F : J \times \mathbb{R}^n \times I \to \mathbb{R}^n,\ \ I = [0, 1], J = [t_0, b],\ t_0 \geq 0$ and $b \in [t_0, \infty)$, is a multifunction with compact convex values such that

(a) $(t, X) \to F(t, X; \beta)$ is $\mathcal{L} \bigoplus \mathcal{B}(\mathbb{R}^n)$ measurable for every $\beta \in I$;

(b) for every $(x, \beta) \in \mathbb{R}^n \times I$ and $\varepsilon > 0$ for almost every $t \in J$ there exists $\delta > 0$ such that for all $\alpha,\ \beta - \delta < \alpha \leq \beta,\ y \in x + \varepsilon.B$, the inclusion

$$F(t, y; \alpha) \subset F(t, x; \beta) + \varepsilon.B \tag{4.5.2}$$

holds;

(c) $F(t, x; \beta)$ is quasiconcave, that is for $x \in \mathbb{R}^n, \alpha, \beta \in I, \alpha \leq \beta$,

$$F(t, x; \alpha) \supset F(t, x; \beta) \text{ a.e.} \tag{4.5.3}$$

(d) the inequality (2.9.10) holds for every $\beta \in I$.

Lemma 4.5.1 *Assume that hypotheses H(F) hold. Then for every $\beta \in I$ the multifunction $H(t, A; \beta)$ has all properties listed in Lemma 2.9.1. and* *(i)* $H(t, A; \beta)$ *is quasiconcave a.e.;*

(ii) if $U_k \in K_c(\mathbb{R}^n)$, $k \geq 1$, *is a nonincreasing sequence with respect to inclusion, and* $\beta_k \in I$, $k \geq 1$, *is a nondecreasing sequence converging to* β.

Then the sequence $H(t, U_k; \beta_k)$, $k \geq 1$, *converges a.e. to* $H(t, \cap_{k=1}^{\infty} U_k;\ \beta)$ *in space* $K_c(\mathbb{R}^n)$.

Proof By (4.5.2) for fixed $\beta \in I$ the multifunction $x \to F(t, x; \beta)$ is usc for almost every $t \in J$. Hence for fixed $\beta \in I$ the multifunction $(t, x) \to H(t, x; \beta)$ for almost every $t \in J$ has compact convex values and possesses all properties listed in Lemma 2.9.1.

According to (4.5.3) for every $A \in K_c(\mathbb{R}^n)$,

$$F(t, A; \alpha) \supset F(t, A; \beta), \ \ \alpha \leq \beta, \text{ a.e.} \tag{4.5.4}$$

Hence (i) is true.

Let a sequence $U_k \in K_c(\mathbb{R}^n)$, $k \geq 1$ be nonincreasing with respect to inclusion, and $\beta_k \in I$, $k \geq 1$, be a nondecreasing sequence converging to β. Then $V = \cap_{k=1}^{\infty} U_k$ is a nonempty compact convex set and sequence U_k, $k \geq 1$, converges to V in $K_c(\mathbb{R}^n)$. By using (4.5.4) and the monotonicity of $F(t, U; \beta)$ with respect to U we obtain

$$F(t, V; \beta) \subset \bigcap_{k=1}^{\infty} F(t, V; \beta_k) \subset \bigcap_{k=1}^{\infty} F(t, U_k; \beta_k). \tag{4.5.5}$$

Let $y \in \cap_{k=1}^{\infty} F(t, U_k; \beta_k)$. Then there exists a sequence $x_k \in U_k \subset U_1$ such that $y \in F(t, x_k; \beta_k)$, $k \geq 1$. Since $x_k \in U_1, k \geq 1$, without loss of generality we can assume that x_k, $k \geq 1$, converges to x. It is clear that $x \in V$. Take any $\varepsilon > 0$. According to (4.5.2) there exists a number $N \geq 1$ such that

$$y \in F(t, x_k; \beta_k) \subset F(t, x; \beta) + \varepsilon.B \tag{4.5.6}$$

for all $k \geq N$. Since $\varepsilon > 0$ is arbitrary then $y \in F(t, x; \beta) \subset F(t, V; \beta)$. Hence by (4.5.5)

$$F(t, V; \beta) \subset \cap_{k=1}^{\infty} F(t, U_k; \beta_k) \subset \ F(t, V; \beta). \tag{4.5.7}$$

The inclusion (4.5.7) tells us that the sequence $F(t, U_k; \beta_k)$, $k \geq 1$ converges to $F(t, V; \beta)$ in $K_c(\mathbb{R}^n)$. Hence the sequence $H(t, U_k; \beta_k) = \overline{co} F(t, U_k; \beta_k), k \geq 1$, converges to $H(t, \cap_{k=1}^{\infty} U_k; \beta)$ in $K_c(\mathbb{R}^n)$. Thus the Lemma is proved.

We can now prove the following result which provides the connection between the fuzzy differential equation (4.4.1) and the sequence of set differential equations (4.5.1).

Theorem 4.5.1 *Assume that the multifunction* $F(t, x; \beta)$ *satisfies hypotheses* $H(F)$. *Then there exists a solution* $U_\beta(t) = U_\beta(t, t_0, U_{0\beta}) \in K_c(\mathbb{R}^n), \beta \in I, t \in J$ *of the equation (4.5.1). If the solution* $U_\beta(t), \beta \in I$, *is unique then* $U_\beta(t)$ *is quasiconcave for* $t \in J$. *Moreover, there exists a fuzzy set* $u(t) \in E^n$ *such that* $[u(t)]^\beta = U_\beta(t), \beta \in I, t \in J$ *and fuzzy set* $t \to u(t)$ *is continuous from* J *to* E^n.

Proof The existence of solution $U_\beta(t),\ \beta \in I$ follows from Lemma 4.5.1. and Corollary 2.9.1. Let us show that $U_\beta(t),\ t \in J$, is quasiconcave if the solution $U_\beta(t), \beta \in I$, is unique.

Let $\alpha < \beta$ and $U_\alpha(t)$ be a solution of equation (4.5.1). Then

$$U_\alpha(t) = U_{0\alpha} + \int_{t_0}^{t} H(s, U_\alpha(s); \alpha)\, ds,\ t \in J. \tag{4.5.8}$$

Denote by V_α the collection of all functions $x \to \mathbb{R}^n$ representable as

$$x(t) = x_0 + \int_{t_0}^{t} v(s)\, ds,\ t \in J,\ x_0 \in U_{0\alpha}, \tag{4.5.9}$$

where $v(s)$ is a Bochner integrable selector of $H(s, U_\alpha(s); \alpha)$. Then V_α is compact convex set of $C(J, \mathbb{R}^n)$ and $V_\alpha(t) = U_\alpha(t),\ t \in J$.

Using (1.8.2) , (4.5.8), (4.5.9) and the definition of the operator $T(V_0, F, V)$ we obtain

$$V_\alpha = T(U_{0\alpha}, H_\alpha, V_\alpha). \tag{4.5.10}$$

Let V_β^0 be a collection of all functions $x : J \to \mathbb{R}^n$ representable as (4.5.9) with $x_0 \in U_{0\beta}$ and $v(s)$ being a Bochner integrable selector of $H(s, U_\alpha(s); \alpha)$.

As has been shown in the proof of Theorem 2.9.1, V_β^0 is a compact convex set of $C(J, \mathbb{R}^n)$,

$$V_\beta^0 = T(U_{0\beta}, H_\alpha, V_\alpha), \tag{4.5.11}$$

and

$$V_\beta^0 \subset V_\alpha, \tag{4.5.12}$$

because $U_{0\beta} \subset U_{0\alpha}$.

From quasiconcavity and monotonicity of $H(t, A; \alpha)$ and (4.5.11) , (4.5.12) it follows

$$V_\beta^1 = T(U_{0\beta}, H_\beta, V_\beta^0) \subset T(U_{0\beta}, H_\alpha, V_\beta^0) \subset T(U_{0\beta)}, H_\alpha, V_\alpha) = V_\beta^0, \tag{4.5.13}$$

and V_β^1 is compact convex subset of $C(J, \mathbb{R}^n)$.

We now define $V_\beta^2 = T(U_{0\beta}, H_\beta, V_\beta^1)$. Because $V_\beta^1 \subset V_\beta^0$ we have

$$V_\beta^2 = T(U_{0\beta}, H_\beta, V_\beta^1) \subset T(U_{0\beta}, H_\beta, V_\beta^0) = V_\beta^1.$$

Continuing this process we obtain a sequence $\{V_\beta^k\}, k \geq 1$, of compact convex subsets of $C(J, \mathbb{R}^n)$ decreasing relative to the inclusion. Repeating the proof of Theorem 2.9.1 we obtain that $U_\beta = \cap_{k=1}^{\infty} V_\beta^k = T(U_{0\beta}, H_\beta, U_\beta)$ and $U_\beta(t) = \{x(t) : x(.) \in U_\beta\}$ is a solution of equation (4.5.1). Since $U_\beta \subset V_\beta^k \subset V_\alpha$ then $U_\beta(t) \subset V_\alpha(t) = U_\alpha(t),\ t \in J$.

For the construction of $U_\beta(t)$ we use the solution $U_\alpha(t)$ of equation (4.5.1). Since the equation (4.5.1) has a unique solution, the solution $U_\beta(t)$ does not depend on $\alpha < \beta$. Hence $U_\beta(t) \subset U_\alpha(t),\ t \in J$, for any $\alpha, \beta \in I, \alpha < \beta$.

Let $r_0 = \max\{\|x\|; x \in U_{00}\}$ and $r(t) = r(t, t_0, r_0)$ be a maximal solution of (2.9.3) on J. By hypothesis $H(F)(d)$ we have

$$\|U_\beta(t)\| \leq \|U_{0\beta}\| + \int_0^t \|H(s, U_\beta(s); \beta)\| \, ds \leq \int_0^t g(s, \|U_\beta(s)\|) \, ds, \ t \in J.$$

Hence by Lemma 1.3.1. in Lakshmikantham and Leela[1] we obtain

$$\|H(t, U_\beta(t); \beta)\| \leq g(t, r(t)) = r'(t). \tag{4.5.14}$$

If $t_* \leq t$, then

$$U_\beta(t) = U_\beta(t_*) + \int_{t_*}^t H(s, U_\beta(s); \beta) \, ds. \tag{4.5.15}$$

If $t \leq t_*$, then

$$U_\beta(t_*) = U_\beta(t) + \int_t^{t_*} H(s, U_\beta(s); \beta) \, ds. \tag{4.5.16}$$

Taking into consideration (1.3.9),(4.5.14),(4.5.15),(4.5.16) we obtain

$$D(U_\beta(t), U_\beta(t_*)) \ \leq \ |\int_{t_*}^t r'(s) \, ds|, \ \beta \in J.$$

Hence the family of functions $U_\beta(.), \beta \in I$ is equicontinuous from J to $K_c(\mathbb{R}^n)$. Let $\beta_k \in I, k \geq 1$, be any nondecreasing sequence converging to β. We claim that $U_\beta(t) = \cap_{k=1}^\infty U_{\beta_k}(t), t \in J$. Since for every $t \in J$ the sequence $U_{\beta_k}(t)$ is nondecreasing with respect to inclusion, the sequence $U_{\beta_k}(t)$ is converging pointwise to a function $V(t) = \cap_{k=1}^\infty U_{\beta_k}(t)$ in $K_c(\mathbb{R}^n)$. Moreover, the sequence $U_{\beta_k}(t), \ k \geq 1$, converges uniformly to $V(t)$ because the sequence $U_{\beta_k}(t)$ is equicontinuous. Hence $V : J \to K_c(\mathbb{R}^n)$ is continuous.

Taking into consideration the statement (ii) of Lemma 4.5.1 we obtain

$$H(t, U_{\beta_k}(t); \beta_k) \to H(t, V(t); \beta) \text{ in } K_c(\mathbb{R}^n). \tag{4.5.17}$$

From (4.5.14) it follows

$$D(H(t, V(t); \beta), H(t, U_{\beta_k}(t); \beta_k)) \leq 2r'(t). \tag{4.5.18}$$

Let

$$U(t) = U_{0\beta} + \int_0^t H(s, V(s); \beta) \, ds.$$

Then the inequality

$$D(U(t), U_{\beta_k}(t)) \leq D(U_{0\beta}, U_{0\beta_k}) + \int_0^t D(H(s, V(s); \beta), H(t, U_{\beta_k}(t), U_{\beta_k})) \, ds, \ \ t \in J, \tag{4.5.19}$$

holds. Since $U_{0\beta} = [x_0]^\beta$, then

$$\lim_{k \to \infty} D(U_{0\beta}, U_{0\beta_k}) = 0. \tag{4.5.20}$$

Now from (4.5.18) to (4.5.20) and Lebesgue bounded convergence theorem we obtain $\lim_{k\to\infty} D(U(t), U_{\beta_k}(t)) = 0$.

Since $U_{\beta_k}(t) \to V(t)$ then $U(t) = V(t),\ t \in J$ and the equality

$$V(t) = U_{0\beta} + \int_0^t H(s, V(s))\, ds,\ t \in J$$

is true.

Hence $V(t)$ is a solution of equation (4.5.1). Due to the uniqueness of the solution of equation (4.5.1)

$$U_\beta(t) = V(t) = \bigcap_{k=1}^{\infty} U_{\beta_k}(t),\ t \in J.$$

Consequently, by Theorem 4.4.2, there exists a fuzzy set $u(t) \in E^n$ such that $[u(t)]^\beta = U_\beta(t),\ t \in J$. Since the family $U_\beta(t),\ \beta \in J$, is equicontinuous , then the fuzzy set $u(t)$ is continuous from J to E^n and this completes the proof.

Remark 4.5.1 *In the original formulation of fuzzy differential equations (FDEs), the function f in (4.4.1)is assumed to be continuous to prove several basic results. This implies that the function $F(t, x; \beta)$ is continuous for each β. In Section 4.4, under the assumption of continuity several results are investigated. The function f is assumed to be usc, $F(t, x; \beta)$ is usc for each β, and, consequently , the standard results of multivalued inclusions can be utilized to capture the rich behaviour of solutions of inclusions. See Lakshmikantham and Mohapatra [1] for further details. In this section, we took a step further to study set differential equations (SDEs) which are generated by FDEs, since SDEs have several advantages.*

Remark 4.5.2 *If $u \in D^n$, that is u is not assumed fuzzy convex , then the level set $[u]^\beta$ need not be convex. Hence, when the fuzzy convexity is discarded, $[f(t, x)]^\beta = F(t, x; \beta)$ need not be convex. Nonetheless, the generated function $H(t, A; \beta)$ is convex and therefore, we can still apply our results.*

4.6 Impulsive Fuzzy Differential Equations

Let PC denote the class of piecewise continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$ with discontinuities of first kind only at $t = t_k,\ k = 1, 2, \ldots$. We need the following known result (see Lakshmikantham, Bainov and Simeonov [1]).

Theorem 4.6.1 *Assume that*

(A_0) The sequence $\{t_k\}$ satisfies $0 \le t_0 < t_1 < t_2, \ldots$, with $\lim_{k\to\infty} t_k = \infty$;

(A_1) $m \in PC^1[\mathbb{R}_+, \mathbb{R}]$ and $m(t)$ is left continuous at $t_k,\ k = 1, 2, \ldots$;

(A_2) for $k = 1, 2, ...,$ $t \geq t_0$, and $m'(t) \leq g(t, m(t)), \quad t \neq t_k, \quad m(t_0) \leq w_0,$

$$m(t_k^+) \leq \psi_k(m(t_k)), \tag{4.6.1}$$

where $g \in C[\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}], \quad \psi_k : \mathbb{R} \to \mathbb{R},$ $\psi_k(w)$ *is nondecreasing in* w*;*

(A_3) $r(t) = r(t, t_0, w_0)$ *is the maximal solution of*

$$w' = g(t, w), \quad t \neq t_k, \quad w(t_0) = w_0,$$

$$w(t_k^+) = \psi_k(w(t_k)), \quad t_k > t_0 \geq 0, \tag{4.6.2}$$

existing on $[t_0, \infty)$*. Then* $m(t) \leq r(t), \quad t \geq t_0$.

Proof For $t \in [t_0, t_1]$, we have by the classical comparison theorem $m(t) \leq r(t)$. Hence using the facts that $\psi_1(w)$ is nondecreasing in w and $m(t_1) \leq r(t_1)$ we obtain

$$m(t_1^+) \leq \psi_1(m(t_1)) \leq \psi_1(r(t_1)) = w_1^+.$$

Now, for $t_1 < t \leq t_2$, it follows, using again the classical comparison theorem $m(t) \leq r(t)$, where $r(t) = r(t, t_1, w_1^+)$ is the maximal solution of (4.6.2) on the interval $t_1 \leq t \leq t_2$. Moreover, as before, we get

$$m(t_2^+) \leq \psi_2(m(t_2)) \leq \psi_2(r(t_2)) = w_2^+.$$

Repeating the arguments, we finally arrive at the desired result, and the proof is complete. Repeating the arguments, we finally arrive at the desired result, and the proof is complete.

Let us consider now the impulsive fuzzy differential equation

$$\begin{aligned} u' &= f(t, u), \quad t \neq t_k, \\ u(t_k^+) &= u(t_k) + I_k(u(t_k)), \quad u(t_0) = u_0, \end{aligned} \tag{4.6.3}$$

where (A_0) holds and $f : \mathbb{R}_+ \times E^n \to E^n$, $I_k : E^n \to E^n$, f is continuous in $(t_{k-1}, t_k] \times E^n$ and for each $u \in E^n$, $\lim f(t, v) = f(t_k^+, u)$ exists as $(t, v) \to (t_k^+, u)$.

We assume that, for each $(t_{k-1}, t_k] \times E^n$, there exists a unique solution $u(t)$ of (4.6.3) in each interval $[t_{k-1}, t_k]$. As a result, employing impulsive condition in (4.6.3) at each t_k, we can define the solution $u(t)$ on the entire interval $[t_0, \infty)$.

Theorem 4.6.2 *Assume that* $f \in C[\mathbb{R}_+ \times E^n, E^n]$ *and*

$$\limsup_{h \to 0^+} \frac{1}{h} \left[D_0[u + hf(t, u), v + hf(t, v)] - D_0[u, v] \right]$$

$$\leq g(t, D_0[u, v]), \quad t \in \mathbb{R}_+, \quad u, v \in E^n, \quad t \neq t_k,$$

where $g \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}]$*. Suppose that*

$$D_0[u + I_k(u), v + I_k(v)] \leq \psi_k(D_0[u, v])$$

where $\psi_k : \mathbb{R}_+ \to \mathbb{R}_+,\ \psi_k(w)$ *is nondecreasing in* w. *The maximal solution* $r(t) = r(t, t_0, w_0)$ *of (4.6.2) exists for* $t \geq t_0$. *Then*

$$D_0[u(t), v(t)] \leq r(t), \quad t \geq t_0,$$

where $u(t), v(t)$ *are the solutions of (4.6.3) existing on* $[t_0, \infty)$.

Proof Proceeding as in the proof of Theorem 4.3.1, we find that for $t \neq t_k$,

$$m(t+h) - m(t) = D_0[u(t+h), v(t+h)] - D_0[u(t), v(t)]$$

$$\leq D_0[u(t+h), u(t) + hf(t, u(t)] + D_0[v(t) + hf(t, v(t), v(t+h)]$$

$$+ D_0[u(t) + hf(t, u(t)), v(t) + hf(t, v(t))] - D_0[u(t), v(t)].$$

Hence

$$\begin{aligned} D^+m(t) \quad &= \quad \limsup_{h\to 0^+} \frac{1}{h}[m(t+h) - m(t)] \\ &\quad + \ \leq \limsup_{h\to 0^+} \frac{1}{h}[D_0(u(t) + hf(t, u(t)), v(t) + hf(t, v(t))] \\ &\quad -D_0[u(t), v(t)] \\ &\quad + \ \limsup_{h\to 0^+} D_0[\frac{u(t+h) - u(t)}{h}, f(t, u(t))] \\ &\quad + \limsup_{h\to 0^+} D_0[f(t, v(t)), \frac{v(t+h) - v(t)}{h}], \quad t \neq t_k, \\ &\leq \quad g(t, D_0[u(t), v(t)]) = g(t, m(t)), \quad t \neq t_k. \end{aligned}$$

Also, for $t = t_k$,

$$\begin{aligned} m(t_k^+) \quad &= \quad D_0[u(t_k^+), v(t_k^+)] \\ &= \quad D_0[u(t_k) + I_k(u(t_k)), v(t_k) + I_k(v(t_k))] \\ &\leq \quad \psi_k(D_0[u(t_k), v(t_k)] = \psi_k(m(t_k)). \end{aligned}$$

We therefore obtain from Theorem 4.6.1, the stated result, namely,

$$D_0[u(t), v(t)] \leq r(t), \quad t \geq t_0,$$

where $r(t) = r(t, t_0, w_0)$ is the maximal solution of (4.6.2) provided $D_0[u_0, v_0] \leq w_0$, completing the proof.

Let $V : \mathbb{R}_+ \times E^n \to \mathbb{R}_+$. Then V is said to belong to class V_0 if

(i) V is continuous in $(t_{k-1}, t_k] \times E^n$ and for each $u \in E^n,\ k = 1, 2, ...,$ $\lim_{(t,v)\to(t_k^+,u)} V(t, v) = V(t_k^+, u)$ exists;

(ii) V satisfies $|V(t, u) - V(t, v)| \leq LD_0[u, v],\ L \geq 0$.

For $(t, u) \in (t_{k-1}, t_k] \times E^n$, we define

$$D^+V(t,u) = \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, u+hf(t,u)) - V(t,u)],$$

then we can prove the following comparison theorem.

Theorem 4.6.3 *Let $V : \mathbb{R}_+ \times E^n \to \mathbb{R}_+$ and $V \in V_0$. Suppose that*

$$D^+V(t,u) \leq g(t, V(t,u)), \quad t \neq t_k, \tag{4.6.4}$$

$$V(t, u + I_k(u)) \leq \psi_k(V(t,u)), \quad t = t_k, \tag{4.6.5}$$

where $g : \mathbb{R}_+^2 \to \mathbb{R}$ is continuous in $(t_{k-1}, t_k] \times \mathbb{R}_+$ and for each $w \in \mathbb{R}_+$, $\lim_{(t,z)\to(t_k^+,w)} g(t,z) = g(t_k^+, w)$ exists, $\psi_k : \mathbb{R}_+ \to \mathbb{R}$ is nondecreasing. Let $r(t)$ be the maximal solution of the scalar impulsive differential equation (4.6.2) existing for $t \geq t_0$. Then $V(t_0^+, u_0) \leq w_0$ implies

$$V(t, u(t)) \leq r(t), t \geq t_0.$$

Proof Let $u(t) = u(t, t_0, u_0)$ be any solution of (4.6.3) existing on $t \geq t_0$, such that $V(t_0^+, u_0) \leq w_0$. Define $m(t) = V(t, u(t))$ for $t \neq t_k$. Then using standard arguments, we arrive at the differential inequality

$$D^+m(t) \leq g(t, m(t)), \quad t \neq t_k.$$

From (4.6.5), we get for $t = t_k$,

$$m(t_k^+) = V(t_k^+, u(t_k^+)) = V(t_k^+, u(t_k) + I_k(u(t_k))) \leq \psi_k(V(t_k, u(t_k))) = \psi_k(m(t_k)).$$

Hence by Theorem 4.6.1, $m(t) \leq r(t), t \geq t_0$, which proves the claim of Theorem 4.6.3.

Some special cases of $g(t, w)$ and $\psi_k(w)$ which are instructive and useful are given below as a corollary.

Corollary 4.6.1 *In Theorem 4.6.3, suppose that*

(i) $g(t,w) = 0, \quad \psi_k(w) = w$ *for all* k, *then* $V(t, u(t))$ *is nondecreasing in* t *and* $V(t, u(t)) \leq V(t_0^+, u_0), \quad t \geq t_0;$

(ii) $g(t,w) \equiv 0, \quad \psi_k(w) = d_k w, d_k \geq 0$ *for all* k, *then*

$$V(t, u(t)) \leq V(t_0^+, u_0) \prod_{t_0 < t_k < t} d_k, \quad t \geq t_0;$$

(iii) $g(t,w) = -\alpha w, \alpha > 0, \quad \psi_k(w) = d_k w, d_k \geq 0$ *for all* k, *then*

$$V(t, u(t)) \leq \; V(t_0^+, u_0) \left(\prod_{t_0 < t_k < t} d_k \right) e^{-\alpha(t-t_0)}, \quad t \geq t_0;$$

(iv) $g(t,w) = \lambda'(t)w, \quad \psi_k(w) = d_k w, \ d_k \geq 0$ *for all* $k, \lambda \in C^1[\mathbb{R}_+, \mathbb{R}_+]$.

Then

$$V(t, u(t)) \leq \ V(t_0^+, u_0) \left(\prod_{t_0 < t_k < t} d_k \right) \exp[\lambda(t) - \lambda(t_0)], \quad t \geq t_0;$$

Recall the example considered in Section 4.3 and note that when we choose $[x_0]^\alpha = [\alpha - 1, 1 - \alpha], \ 0 \leq \alpha \leq 1$, we get

$$[x(t)]^\alpha = [(\alpha - 1), (1 - \alpha)]e^t = [-1, 1](1 - \alpha)e^t, \ \ t \geq 0.$$

In particular, diam $[x(t)]^\alpha = 2(1 - \alpha)e^t, \ t \geq 0$.

In order to show the effect of impulses, we now introduce the impulsive condition as in (4.6.3), that is, $[x_k]^{+\alpha} = [x_{1k}^{+\alpha,}, x_{2k}^{+\alpha}]$ and $[x_k]^\alpha = [x_{1k}^\alpha, x_{2k}^\alpha]$ for each k , where $x_k^+ = x(t_k^+)$ and $x_k = x(t_k)$. Because of impulse condition $[x(t)]^\alpha$ reduces to

$$[x(t)]^\alpha = \left[(\alpha - 1) \prod_{0 < t_k < t} d_k, (1 - \alpha) \prod_{0 < t_k < t} d_k \right] e^t, \ t \geq 0.$$

It follows that, if $d_k, \ t_k$ satisfy the condition

$$t_{k+1} + ln \ d_k \leq t_k,$$

then $x = 0$ is stable and if $t_{k+1} + \beta \ ln \ d_k \leq t_k, \beta > 0$, then $x = 0$ is asymptotically stable.

This demonstrates that the impulsive action helps to obtain stability of FDE without utilizing Hukuhara difference for the initial values, as we have proposed in Section 4.3.

4.7 Hybrid Fuzzy Differential Equations

The problem of stabilizing a continuous plant governed by differential equation through the interaction with a discrete time controller has recently been investigated. This study leads to the consideration of hybrid systems. In this section, we shall extend this approach to fuzzy differential equations.

Consider the hybrid fuzzy differential system

$$u'(t) = f(t, u(t), \lambda_k(z)), \quad u(t_k) = z, \tag{4.7.1}$$

on $[t_k, t_{k+1}]$ for any fixed $z \in E^n, k = 0, 1, 2, ...,$ where $f \in C[\mathbb{R}_+ \times E^n \times E^n, E^n]$, and $\lambda_k \in C[E^n, E^n]$. Here we assume that $0 \leq t_0 < t_1 < t_2 < ...,$ are such that $t_k \to \infty$ as $k \to \infty$ and the existence and uniqueness of solutions of the hybrid system hold on each $[t_k, t_{k+1}]$. To be specific, the system is of the form

$$u'(t) = \begin{cases} u_0'(t) = f(t, u_0(t), \lambda_0(u_0)), & u_0(t_0) = u_0, & t_0 \le t \le t_1, \\ u_1'(t) = f(t, u_1(t), \lambda_1(u_1)), & u_1(t_1) = u_1, & t_1 \le t \le t_2, \\ \vdots & \vdots & \vdots \\ u_k'(t) = f(t, u_k(t), \lambda_k(u_k)), & u_k(t_k) = u_k, & t_k \le t \le t_{k+1}, \\ \vdots & \vdots & \vdots \end{cases}$$

where $u_k = u_{k-1}(t_k)$ for each k. By the solution of (4.7.1), we therefore mean the following function

$$u(t) = u(t, t_0, u_0) = \begin{cases} u_0(t), & t_0 \le t \le t_1, \\ u_1(t), & t_1 \le t \le t_2, \\ \vdots & \vdots \\ u_k(t), & t_k \le t \le t_{k+1}, \\ \vdots & \vdots \end{cases}$$

We note that the solutions of (4.7.1) are differentiable in each interval for $t \in (t_k, t_{k+1})$ for any fixed $u_k \in E^n$ and $k = 0, 1, 2, ...$

Let $V \in C[E^n, \mathbb{R}_+]$. For $t \in [t_k, t_{k+1}], u, z \in E^n$, we define

$$D^+V(u; z) = \limsup_{h \to 0^+} \frac{1}{h}[V(u + hf(t, u, \lambda_k(z))) - V(u)].$$

We can then prove the following comparison theorem in terms of Lyapunov-like function V.

Theorem 4.7.1 *Assume that*

(i) $V \in C[E^n, \mathbb{R}_+], V(u)$ *satisfies* $|V(u) - V(v)| \le L\, D_0[u, v], L > 0$ *for* $u, v \in E^n$;

(ii) $D^+V(u; z) \le g(t, V(u), \sigma_k(V(z))), t \in (t_k, t_{k+1}]$, *where* $g \in C[\mathbb{R}_+^3, R]$, $\sigma_k \in C[\mathbb{R}_+, \mathbb{R}_+],\ u, z \in E^n, k = 0, 1, 2, ...;$

(iii) the maximal solution $r(t) = r(t, t_0, w_0)$ *of the hybrid scalar differential equation.*

$$\begin{aligned} w' &= g(t, w(t),\ \sigma_k(w_k)),\ t \in (t_k, t_{k+1}], \\ w(t_k) &= w_k, \quad k = 0, 1, 2, ..., \end{aligned} \tag{4.7.2}$$

exists on $[t_0, \infty)$.

Then any solution $u(t) = u(t, t_0, u_0)$ *of (4.7.1) such that* $V(u_0) \le w_0$ *satisfies the estimate*

$$V(u(t)) \le r(t), \quad t \ge t_0.$$

Proof Let $u(t)$ be any solution of (4.7.1) existing on $[t_0, \infty)$ and set $m(t) = V(u(t))$. Then using (i) and (ii), and proceeding as in the proof of Theorem 4.3.1 and Theorem 4.6.2, we get the differential inequality

$$D^+ m(t) \leq g(t, m(t), \sigma_k(m_k)) \text{ for } t_k < t \leq t_{k+1},$$

where $m_k = V(u(t_k))$. For $t \in [t_0, t_1]$, since $m(t_0) = V(u_0) \leq w_0$, we obtain

$$V(u_0(t)) \leq r_0(t, t_0, w_0), \quad t_0 \leq t \leq t_1,$$

where $r_0(t) = r_0(t, t_0, w_0)$ is the maximal solution of

$$w_0' = g(t, w_0, \sigma_0(w_0)), \quad w_0(t_0) = w_0 \geq 0, \quad t_0 \leq t \leq t_1,$$

and $u_0(t)$ is the solution of

$$u_0' = f(t, u_0(t), \lambda_0(u_0)), \quad u(t_0) = u_0 \geq 0, \quad t_0 \leq t \leq t_1.$$

Similarly, for $t \in [t_1, t_2]$, it follows that

$$V(u_1(t)) \leq r_1(t, t_1, w_1), \quad t_1 \leq t \leq t_2,$$

where $w_1 = r_0(t_1, t_0, w_0), r_1(t, t_1, w_1)$ is the maximal solution of

$$w_1' = g(t, w_1, \sigma_1(w_1)), \quad w_1(t_1) = w_1 \geq 0, \quad t_1 \leq t \leq t_2,$$

and $u_1(t)$ is the solution of

$$u_1' = f(t, u_1(t), \lambda_1(u_1)), \quad u_1(t_1) = u_1, \quad t_1 \leq t \leq t_2.$$

Proceeding similarly, we can obtain

$$V(u_k(t)) \leq r_k(t, t_k, w_k), \quad t_k \leq t \leq t_{k+1},$$

where $u_k(t)$ is the solution of

$$u_k'(t) = f(t, u_k(t), \lambda_k(u_k)), \quad u_k(t_k) = u_k, \quad t_k \leq t \leq t_{k+1},$$

and $r_k(t, t_k, w_k)$ is the maximal solution of

$$w_k' = g(t, w_k(t), \sigma_k(w_k)), \quad w_k(t_k) = w_k, \quad t_k \leq t \leq t_{k+1},$$

where $w_k = r_{k-1}(t_k, t_{k-1}, r_{k-2}(t_{k-1}, t_{k-2}, w_{k-1}))$.

Thus defining $r(t, t_0, w_0)$as the maximal solution of the comparison hybrid system (4.7.2) as

$$r(t, t_0, w_0) = \begin{cases} r_0(t, t_0, w_0), & t_0 \leq t \leq t_1, \\ r_1(t, t_1, w_1), & t_1 \leq t \leq t_2, \\ \vdots & \vdots \\ r_k(t, t_k, w_k), & t_k \leq t \leq t_{k+1}, \\ \vdots & \vdots \end{cases}$$

and taking $w_0 = V(u_0)$, we obtain the desired estimate

$$V(u(t)) \leq r(t), \quad t \geq t_0.$$

The proof is therefore complete.

Consider now the hybrid impulsive fuzzy differential system given by

$$\begin{bmatrix} u' = f(t, u, \lambda(t_k, u_k)), & t \in [t_k, t_{k+1}], \\ u(t_k^+) = u(t_k) + I_k(u(t_k)), & t = t_k, \\ u(t_0^+) = u_0, & \end{bmatrix} \tag{4.7.3}$$

where $f \in C[\mathbb{R}_+ \times E^n \times E^n, E^n]$, $I_k : E^n \to E^n$, $\lambda_k \in C[\mathbb{R}_+, \times E^n, E^n]$, and $k = 0, 1, 2, \ldots$.

We assume that $I_0(u_0) = 0$, and the existence of solution of the system

$$\begin{bmatrix} u' = f(t, u, \lambda(t_k, z)), & t \in (t_k, t_{k+1}], \\ u(t_k^+) = z + I_k(z), & t \neq t_k, \\ u(t_0^+) = u_0 & \end{bmatrix} \tag{4.7.4}$$

on $(t_k, t_{k+1}]$ for any fixed $z \in E^n$ and all $k = 0, 1, 2, \ldots$.

Note that the solution of (4.7.4) is a piecewise continuous function with points of discontinuity of the first type at $t = t_k$ at which they are assumed to be left continuous.

Let $V : \mathbb{R}_+ \times E^n \to \mathbb{R}_+$. Then V is said to belong to class V_0, if

(i) V is continuous in $(t_k, t_{k+1}] \times E^n$ and for each $u \in E^n, k = 1, 2, \ldots.$ $\lim_{(t,v) \to (t_k^+, u)} V(t, v) = V(t_k^+, u)$ exists;

(ii) V is locally Lipschitzian in u. Then we define, as before,

$$D^+V(t, u, z) \equiv \limsup_{h \to 0^+} \frac{1}{h}[V(t + h, u + hf(t, u, \lambda_k(t_k, z)) - V(t, u)].$$

We need the following comparison result.

Theorem 4.7.2 *Assume that*

(i) $V \in C[\mathbb{R}_+ \times E^n, \mathbb{R}_+], V(t, u)$ *is locally Lipschitzian in* u *that is* $|V(t, u) - V(t, v)| \leq L\ D_0[u, v], L > 0,$ *and*

$$D^+V(t, u, z) \leq g(t, V(t, u),\ \sigma_k(t_k, z)), \quad t \in (t_k, t_{k+1}],$$

$u, z \in E^n$, *where* $\sigma_k \in C[\mathbb{R}_+ \times E^n, \mathbb{R}]$, $g \in C[\mathbb{R}_+^3, \mathbb{R}]$;

(ii) there exist a $\psi_k \in C[\mathbb{R}_+, \mathbb{R}_+]$, $\psi_k(w)$ *is nondecreasing in* w *and*

$$V(t, u + I_k(u)) \leq \psi_k(V(t, u)),\ k = 1, 2, \cdots;\ u \in E^n;$$

(iii) the maximal solution $r(t) = r(t, t_0, w_0)$ of the scalar hybrid impulsive differential equation

$$\begin{cases} w' = g(t, w, \eta(t_k, w_k)), & t \in (t_k, t_{k+1}], \\ w(t_k^+) = \psi(w(t_k)), & t = t_k, \\ w(t_0) = w_0 \geq 0, \end{cases} \tag{4.7.5}$$

existing on $[t_0, \infty)$, where $\eta \in C[\mathbb{R}_+^2, \mathbb{R}]$ and $w_k = V(t_k, u_k)$. Then any solution $u(t) = u(t, t_0, u_0)$ of (4.7.3) satisfies

$$V(t, u(t)) \leq r(t, t_0, w_0), \quad t \geq t_0,$$

provided $w_0 \geq V(t_0, u_0)$.

The proof of this comparison theorem follows on similar lines to Theorem 4.7.2 defining $u(t)$ and $r(t)$, piece by piece suitably. We omit the proof to avoid monotony. Having the foregoing comparison result at our disposal, we can formulate stability criteria of the solutions of (4.7.3) relative to any kind of stability such as Lyapunov stability , practical stability, stability in terms of two different measures, which includes several known stability concepts or the new concept of stability, which includes Lyapunov and orbital stability as special cases. We simply state a typical result whose proof can be constructed based on stability criteria of impulsive differential equations.

Theorem 4.7.3 *Assume that*

(i) $V \in V_0$ and $V(t, u)$ is positive definite and decrescent;

(ii) $D^+V(t, u, z) \leq g(t, V(t, u), \sigma_k(t_k, z))$, $t \in (t_k, t_{k+1}]$, $u, z \in E^n, \sigma_k, g$ are as defined in Theorem 4.7.2 ;

(iii) $V(t, u + I_k(u)) \leq \psi_k(V(t, u)), t = t_k, u \in E^n$, where $\psi_k(w)$ is nondecreasing in w, as in Theorem 4.7.2.

Then stability properties of the trivial solution $w = 0$ of (4.7.5) imply the corresponding stability properties of (4.7.3) respectively.

All that is needed to get any kind of stability properties of (4.7.3) is to require positive definiteness and decresence of $V(t, u)$ suitably relative to that particular stability demands. For example if we want stability criteria in terms of two different measures say, (h_0, h), then $V(t, u)$ need to satisfy positive definiteness relative to h and decrescence with respect to h_0, where h_0 is finer than h. See for details Lakshmikantham and Liu [1].

4.8 Another Formulation

Consider a differential equation in a given space X

$$u'(t) = f(t, u(t)),\ t \in [0, T], \tag{4.8.1}$$

with a specific initial condition

$$u(0) \ = \ u_0, \tag{4.8.2}$$

where $T > 0$, and $f : [0, T] \times X \to X$.

Now, suppose that X is a Banach space with norm $\|.\|$ inducing a distance d. If f is continuous, then equation (4.8.1) is equivalent to

$$\lim_{h \to 0} \frac{\|u(t+h) - u(t) - f(t, u(t))h\|}{h} = 0,\ t \in [0, T).$$

Therefore, any solution of (4.8.1) satisfies

$$\lim_{h \to 0^+} \frac{d(u(t+h), F(t, h, u(t)))}{h} = 0, \tag{4.8.3}$$

where

$$F : [0, T] \times \mathbb{R}_+ \times X \to X, \quad F(t, h, u) = u + hf(t, u). \tag{4.8.4}$$

With the notation of Kloeden, Sadovsky and Vasiyeva [1], (4.8.1) is equivalent to

$$u(t + dt) - u(t) - D_{t,u(t)}(dt) = o(dt), \tag{4.8.5}$$

with

$$D_{t,u}(dt) = f(t, u)dt = F(t, dt, u) - u. \tag{4.8.6}$$

There are three possible definitions for continuous processes in a Banach space.

Formulation (4.8.5) allows the study of nonsmooth systems such as "stop nonlinearities" and is called an *equation with a nonlinear differential*. With an adequate choice of nonlinear differential D it is possible to obtain existence results for classical ordinary differential equations, Caratheodory differential equations, and differential inclusions with maximal monotone operators.

However, (4.8.1) and (4.8.5) have both the same shortcoming: One needs an algebraic structure in the underlying space. On the other hand, (4.8.3) seems adequate to study the evolution of a process in a metric space making it possible to obtain results of calculus and differential equations without employing any concept of derivative or requiring that the underlying metric space be linear.

This motivates the following definition.

Let (X, d) be a complete metric space and $F : [0, T] \times \mathbb{R}_+ \times X \to X$. We shall consider (X, F) as a *metric differential equation* in the following sense: A function $u : [0, T] \to X$ is a solution of the metric differential equation (X, F) with initial condition (4.8.2) if u satisfies (4.8.3) and $u(0) = u_0$.

This conception is related to the concept of quasi-differential equations and to mutations in a metric space. See Melnik [1], Panasyuk [1], Plotnikov [1], Aubin [1,2].

Of course, if $X = \mathbb{R}^n$ and F is given by (4.8.4) with f continuous, then we have a classical ordinary differential equation, i.e., a continuous dynamical system.

If $X = E^n$, note that for a function $f : [0,T] \times E^n \to E^n$, (4.8.4) makes sense, and we can reconsider a fuzzy differential equation as a metric differential equation in the metric space E^n. As we know that in E^n the difference of two elements is not always well defined which precludes us from using (4.8.5) to study fuzzy differential equations.

We recall that a fuzzy subset of $\mathbb{R}^n$ is just a map

$$u : \mathbb{R}^n \to [0,1]$$

where $u(x)$ is the grade of membership of $x \in \mathbb{R}^n$ to the fuzzy set. For each $\alpha \in (0,1]$ the $\alpha-$ level set $[u]^\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\}$. The support of u , denoted by $[u]^0$ is the closure of the union of all its level sets. Of course, any classical subset $A \subset \mathbb{R}^n$ is identified with its characteristic function χ_A. The set of normal, fuzzy convex, upper semicontinuous functions, with compact support is denoted by E^n.

It is possible to define addition in E^n levelwise:

$$u, v \in E^n, [u+v]^\alpha = [u]^\alpha + [v]^\alpha, \alpha \in [0,1].$$

For $c \neq 0$, scalar multiplication is defined also levelwise:

$$u \in E^n,\ [cu]^\alpha = c[u]^\alpha.$$

Note that it is not possible to define $0u$ levelwise since $0u = \chi_\phi \notin E^n$. We define $u - v = u + (-1)v$. Observe that $u + v = \chi_{\{0\}}$ implies $u = -v$, but $u = -v$ does not imply, in general, that $u + v = \chi_{\{0\}}$. Indeed, for example, for $u = \chi_{[0,1]}$, $-u = \chi_{[-1,0]}$, and $u - u = \chi_{[-1,1]}$.

The distance between elements of E^n is given by the supremum of the Hausdorff distance between the level sets:

$$u, v \in E^n,\ D_0[u,v] = \sup_{\alpha \in [0,1]} D[[u]^\alpha, [v]^\alpha].$$

Thus, (E^n, D_0) is a complete metric space (see Lakshmikantham and Mohapatra [1]). Moreover, for $u, v, w, z \in E^n$ and $c, c' \neq 0$, we have

$$\begin{aligned} D_0[cu, cv] &= |c| D_0[u,v], \\ D_0[u+w, v+w] &= D_0[u,v], \\ D_0[u+w, v+z] &= D_0[u,v] + D_0[w,z], \\ D_0[cu, c'u] &= |c - c'| D_0[u, \chi_{\{0\}}]. \end{aligned}$$

If $a \in \mathbb{R}^n$, then $\chi_{\{a\}} \in E^n$, and for $a, b \in \mathbb{R}^n$. $D_0[\chi_{\{a\}}, \chi_{\{b\}}] = |a - b|$.

Now, consider a map $f : E^n \to E^n$.

Definition 4.8.1 *For fixed $u, v \in E^n$ we say that f is differentiable at u in the direction v if there exists $w \in E^n$ such that*

$$\lim_{h\to 0^+} \frac{D_0[f(u+hv), f(u)+hw]}{h} = 0.$$

We say that w is the derivative of f at u in the direction v and write $w = f'(u)v$.

Definition 4.8.2 *For a function $u : [0,T] \to E^n$ and $t \in [0,T)$ we say that u is differentiable at t if there exists $w \in E^n$ such that*

$$\lim_{h\to 0^+} \frac{D_0[u(t+h), u(t)+hw]}{h} = 0,$$

and we write $D_H u(t) = w$.

Example 4.8.1 *Let $u_0 \in E^n$ and take $u : [0,T] \to E^n$, $u(t) = u_0$. Then, $u'(t) = \chi_{\{0\}}$ for every $t \in [0,T)$. Indeed,*

$$\begin{aligned} \lim_{h\to 0^+} \frac{D_0[u(t+h), u(t)+h\chi_{\{0\}}]}{h} &= \lim_{h\to 0^+} \frac{D_0[u_0, u_0+h\chi_{\{0\}}]}{h} \\ &= \lim_{h\to 0^+} \frac{D_0[u_0, u_0]}{h} = 0. \end{aligned}$$

Example 4.8.2 *For $u_0, u_1 \in E^n$ let $u(t) = u_0 + tu_1$. Then, $D_H u(t) = u_1$ since*

$$\lim_{h\to 0^+} \frac{D_0[u(t+h), u(t)+hu_1]}{h} = \lim_{h\to 0^+} \frac{D_0[u_0+(t+h)u_1, u_0+tu_1+hu_1]}{h} = 0.$$

Example 4.8.3 *Let $f : [0,T] \to \mathbb{R}^n$ be differentiable. Define*

$$\hat{f} : [0,T] \to E^n,\ \hat{f}(t) = \chi_{\{f(t)\}}.$$

Then $\hat{f}$ is differentiable and $\hat{f}'(t) = \chi_{\{f'(t)\}}$.

Example 4.8.4 *For any $\lambda > 0$, $u_0 \in E^n$, the function $u(t) = e^{\lambda t}u_0$ satisfies $D_H u(t) = \lambda u(t)$. Indeed,*

$$\begin{aligned} &\lim_{h\to 0^+} \frac{1}{h}(D_0[u(t+h), u(t)+h\lambda e^{\lambda t}u_0]) \\ &= \lim_{h\to 0^+} \frac{1}{h}(D_0[e^{\lambda(t+h)}u_0, e^{\lambda t}u_0 + h\lambda e^{\lambda t}u_0]) \\ &= \lim_{h\to 0^+} \frac{e^{\lambda t}}{h} D_0[e^{\lambda h}u_0, (1+\lambda h)u_0] \\ &= \lim_{h\to 0^+} \frac{e^{\lambda t}}{h}(e^{\lambda h} - (1+\lambda h))D_0[u_0, \chi_{\{0\}}] = 0. \end{aligned}$$

Definition 4.8.3 *Given $v : [0,T] \to E^n$, a primitive of v is a function $u : [0,T] \to E^n$ such that $D_H u(t) = v(t)$ a.e. $[0,T]$, i.e. for almost all $t \in [0,T]$ we have*

$$\lim_{h \to 0^+} \frac{D_0[u(t+h), u(t) + hv(t)]}{h} = 0.$$

If u is a primitive of v satisfying the initial condition 4.8.2., we say that u is a primitive starting at u_0.

For example, a primitive of $v(t) = e^t u_0$ is itself.

Lemma 4.8.1 *If $u : [0,T] \to E^n$ is differentiable at t, then u is right continuous at t.*

Proof Let $D_H u(t) = v(t)$. For every $\varepsilon > 0$, there exists $\delta > 0$ such that for $h \in (0, \delta)$, we have

$$D_0[u(t+h), u(t)] \leq D_0[u(t+h), u(t) + hv(t)] + D_0[u(t) + hv(t), u(t)]$$

$$\leq \varepsilon h + D_0[hv(t), \chi_{\{0\}}] = \varepsilon h + h D_0[v(t), \chi_{\{0\}}].$$

This shows the right continuity of u at t.

Lemma 4.8.2 *Suppose that $v \in [0,T] \to E^n$ is piecewise constant, then v has a primitive starting at u_0. Moreover, if $t_0 = 0 < t_1 < t_2 < \cdots < t_k = T$, and $v(t) = v_i \in E^n$ for $t \in (t_i, t_{i+1}), i = 0, 1, \ldots\ldots, k-1$, it is possible to construct a Lipschitz continuous primitive with Lipschitz constant equal to $max_{0 \leq i \leq k-1} D_0[v_i, \chi_{\{0\}}]$.*

Proof Let $u(0) = u_0$ and for $t \in (0, t_1)$, define $u(t) = u_0 + tv_0$. Thus, $D_H u(t) = v_0 = v(t)$ for every $t \in [0, t_1)$. For $i \geq 1$, set $u(t_i) = u(t_{i-1}) + (t_i - t_{i-1})v_{i-1}$, and for $t \in (t_i, t_{i+1})$, $u(t) = u(t_i) + (t - t_i)v_i$. It is clear that $D_H u(t) = v(t)$ for every $t \in (t_i, t_{i+1}), i = 0, 1\ldots\ldots, k-1$. Now, if $t, \tau \in [t_i, t_{i+1}]$, then

$$\begin{aligned} D_0[u(t), u(\tau)] &= D_0[u(t_i) + (t - t_i)v_i, u(t_i) + (\tau - t_i)v_i] \\ &= D_0[(t - t_i)v_i, (\tau - t_i)v_i] \\ &= |t - \tau| D_0[v_i, \chi_{\{0\}}]. \end{aligned}$$

For arbitrary $t, \tau \in [0,T]$, suppose that $t \in [t_i, t_{i+1}]$, and $\tau \in [t_j, t_{j+1}], i < j$. Hence,

$$\begin{aligned} D_0[u(t), u(\tau)] &\leq D_0[u(t), u(t_{i+1})] + D_0[u(t_{i+1}), u(t_{i+2})] + \cdots \\ &\quad \cdots + D_0[u(t_{j-1}), u(t_j)] + D_0[u(t_j), u(\tau)] \\ &\leq |t - t_{i+1}| D_0[v_i, \chi_{\{0\}}] + |t_{i+2} - t_{i+1}| D_0[v_{i+1}, \chi_{\{0\}}] + \cdots \\ &\quad \cdots + |t_j - t_{j-1}| D_0[v_{j-1}, \chi_{\{0\}}] + |\tau - t_j| D_0[v_j, \chi_{\{0\}}] \\ &\leq \max_{0 \leq i \leq (k-1)} D_0[v_i, \chi_{\{0\}}] |\tau - t|. \end{aligned}$$

Lemma 4.8.3 *Let $v_1, v_2 : [0,T] \to E^n$ with primitives u_1, u_2. Suppose that the function $s \in [0,T] \to D_0[v_1(s), v_2(s)]$ is integrable on $[0,T]$ (for example if v_1 and v_2 are piecewise continuous). Then,*

$$D_0[u_1(t), u_2(t)] \;\le\; D_0[u_1(0), u_2(0)] + \int_0^t D_0[v_1(s), v_2(s)]\, ds. \tag{4.8.7}$$

Proof Define $\xi(t) = D_0[u_1(t), u_2(t)], t \in [0,T]$. We have

$$\begin{aligned}
&\xi(t+h) - \xi(t)\\
&\quad \le \; D_0[u_1(t+h), u_1(t) + hv_1(t)] + D_0[u_1(t) + hv_1(t), u_2(t) + hv_1(t)]\\
&\qquad + D_0[u_2(t) + hv_1(t), u_2(t) + hv_2(t)] + D_0[u_2(t) + hv_2(t), u_2(t+h)]\\
&\quad - \; D_0[u_1(t), u_2(t)]\\
&\quad = \; D_0[u_1(t+h), u_1(t) + hv_1(t)] + D_0[u_1(t), u_2(t)] + hD_0[v_1(t), v_2(t)]\\
&\qquad + D_0[u_2(t) + hv_2(t), u_2(t+h)] - D_0[u_1(t), u_2(t)].
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{\xi(t+h) - \xi(t)}{h} &\le \frac{1}{h} D_0[u_1(t+h), u_1(t) + hv_1(t)]\\
&\quad + D_0[v_1(t), v_2(t)] + \frac{1}{h} D_0[u_2(t) + hv_2(t), u_2(t+h)].
\end{aligned}$$

Therefore, $D^+\xi(t) \;\le\; D_0[v_1(t), v_2(t)],\; t \in [0,T]$, and

$$\xi(t) \;\le\; \xi(0) + \int_0^t D_0[v_1(s), v_2(s)]\, ds.$$

Corollary 4.8.1 *(Uniqueness) : For a given initial state we have at most one continuous primitive.*

Lemma 4.8.4 *Let u be a continuous primitive of v. Then it satisfies the following inequality:*

$$\frac{1}{h} D_0[u(t+h), u(t) + hv(t)] \;\le\; \frac{1}{h} \int_0^h D_0[v(t+s), v(t)]\, ds. \tag{4.8.8}$$

Moreover, assume that $\sup_{\tau \in [0,T]} D_0[v(\tau), \chi_{\{0\}}] = k < +\infty$, then u is Lipschitz continuous with Lipschitz constant k.

Proof Fix $t_0 \in [0,T]$, and note that

$$\mu : [0,T] \to E^n,\; \mu(t) = u(t_0) + tv(t_0)$$

is a primitive of the constant function $v(t_0)$ with $\mu(0) = u(t_0)$. Also the function

$$u_{t_0} : [0, T - t_0] \to E^n,\; u_{t_0}(t) = u(t + t_0)$$

is a primitive of $v_{t_0} : [0, T - t_0] \to E^n, v_{t_0}(t) = v(t + t_0)$ with $u_{t_0}(0) = u(t_0)$. Using (4.8.7) we get

$$D_0[\mu(h), u_{t_0}(h)] \leq D_0[\mu(0), u_{t_0}(0)] + \int_0^h D_0[v(t_0), v(s + t_0)]\, ds$$

and

$$D_0[u(t_0) + hv(t_0), u(h + t_0)] \leq \int_0^h D_0[v(t_0 + s), v(t_0)]\, ds$$

Dividing by h we obtain (4.8.8).

Now, the constant function $u(t)$ is a primitive of the constant function $\chi_{\{0\}}$ starting at $u(t)$. Let $t' > t$. hence,

$$\begin{aligned} D_0[u(t'), u(t)] &= D_0[u_t(t' - t), u(t)] \leq \int_0^{t'-t} D_0[v_t(s), \chi_{\{0\}}]\, ds \\ &= \int_0^{t'-t} D_0[v(t + s), \chi_{\{0\}}]\, ds \leq k.(t' - t). \end{aligned}$$

We now prove the main result of the Section: Any continuous function has a primitive.

Theorem 4.8.1 *Let $v : [0, T] \to E^n$ be continuous. Then there exists a unique primitive of v starting at a given u_0.*

Proof For $\varepsilon > 0$ there exists $\delta > 0$ such that $D_0[v(t), v(s)] < \varepsilon$ whenever $|t - s| < \delta$ since v is uniformly continuous on $[0, T]$. Take $m > \frac{T}{\delta}$, $h = \frac{T}{m}$, and for $i = 0, 1, 2...., m - 1$ define the piecewise continuous function

$$v_m : [0, T] \to E^n, \ v_m(t) = v(ih), \ t \in (ih, (i + 1)h).$$

For any $t \in [0, T]$, let $t \in (ih, (i + 1)h)$, then

$$D_0[v_m(t), v(t)] = D_0[v(ih), v(t)] < \varepsilon,$$

since $|t - ih| < h = \frac{T}{m} < \delta$. Also, for every $m = 1, 2, ...$ and $t \in [0, T]$ we have

$$D_0[v_m(t), \chi_{\{0\}}] \leq \sup_{\tau \in [0,T]} D_0[v(\tau), \chi_{\{0\}}] = k < \infty.$$

In view of Lemma 4.8.2., let u_m be the primitive of v_m starting at u_0. By Lemma 4.8.4, u_m is Lipschitz continuous with Lipschitz constant k. Now, using Lemma 4.8.3., for $l, m > \frac{T}{\delta}$ and $t \in [0, T]$ we have

$$D_0[u_m(t), u_l(t)] \leq \int_0^t D_0[v_m(s), v_l(s)]\, ds \ \leq 2\varepsilon t \ \leq \ 2T\varepsilon. \tag{4.8.9}$$

In consequence, for every $t \in [0, T]$ we see that the sequence $\{u_m(t)\}_{m=1}^{\infty}$ is a Cauchy sequence in E^n. Therefore, there exists $\lim_{m\to\infty} u_m(t) = u(t)$. Passing to the limit when $l \to \infty$ in (4.8.9) we see that $\{u_m\}_{m=1}^{\infty}$ converges uniformly to u.

Now, for $m = 1, 2.....$ consider the functions

$$\mu_m : [0, T] \to E^n,\ \mu_m(t) = u_m(t) + hv_m(t),$$

and

$$\mu : [0, T] \to E^n,\ \mu(t) = u(t) + hv(t).$$

Using (4.8.8) we can write

$$D_0[u_m(t + h), u_m(t) + hv_m(t)] \leq \int_0^h D_0[v_m(t + s), v_m(t)]\ ds.$$

On the other hand,

$$\lim_{m\to\infty} D_0[u_m(t + h), u_m(t) + hv_m(t)] = D_0[u(t + h), u(t) + hv(t)],$$

and

$$\lim_{m\to\infty} \int_0^h D_0[v_m(t + s), v_m(t)]\ ds = \int_0^h D_0[v(t + s), v(t)]\ ds.$$

Hence,

$$\frac{1}{h} D_0[u(t + h), u(t) + hv(t)] \leq \frac{1}{h} \int_0^h D_0[v(t + s), v(t)]\ ds.$$

Now, v is uniformly continuous on $[0, T]$ and hence

$$\lim_{h\to 0^+} \frac{1}{h} \int_0^h D_0[v(t + s), v(t)]\ ds = 0,$$

uniformly on $t \in [0, T]$. Therefore,

$$\lim_{h\to 0^+} \frac{1}{h} D_0[u(t + h), u(t) + hv(t)] = 0,$$

uniformly on $t \in [0, T]$, and u is a primitive of v. For $f : E^n \to E^n$, consider the fuzzy differential equation

$$D_H u(t) = f(u(t)),\ t \in [0, T], \tag{4.8.10}$$

with the fuzzy initial condition

$$u(0) = u_0 \in E^n. \tag{4.8.11}$$

Definition 4.8.4 *We say that $u : [0, T] \to E^n$ is a solution of (4.8.10)-(4.8.11) if u is a primitive of $f(u)$ starting at u_0.*

For $u \in C([0, T], E^n)$, define the function

$$Fu : [0, T] \to E^n,\ [Fu](t) = f(u(t)),$$

and denote by Gu the unique continuous primitive of Fu starting at u_0. Hence, a function $u \in C([0, T], E^n)$ is a solution of the initial value problem (4.8.10)-(4.8.11) if and only if u coincides with Gu.

Theorem 4.8.2 *Suppose that $f : E^n \to E^n$ is such that there exists $k \geq 0$ with*

$$D_0[f(x), f(y)] \leq kD_0[x, y],\ x, y \in E^n. \tag{4.8.12}$$

Then the initial fuzzy problem (4.8.10),(4.8.11) has a unique solution.

Proof . In the space $C([0, T], E^n)$, consider the metric

$$\tilde{D}[u_1, u_2] = \sup_{t\in[0,T]} D_0[u_1(t), u_2(t)]e^{-kt}.$$

Thus, using Lemma 4.8.3, we have for any $t \in [0, T]$,

$$\begin{aligned} D_0[[Gu_1](t), [Gu_2](t)] &\leq \int_0^t D_0[[Fu_1])(s), [Fu_2](s)]\, ds \\ &= \int_0^t D_0[f(u_1(s)), f(u_2(s))]\, ds \end{aligned}$$

$$\begin{aligned} &\leq k \int_0^t D_0[u_1(s), u_2(s)]\, ds \\ &= k \int_0^t e^{ks} e^{-ks} D_0[u_1(s), u_2(s)]\, ds \\ &\leq k \int_0^t e^{ks} \tilde{D}[u_1, u_2]\, ds = (e^{kt} - 1)\tilde{D}[u_1, u_2]. \end{aligned}$$

Hence

$$\tilde{D}[Gu_1, Gu_2] \ \leq\ [1 - e^{-kT}]\, \tilde{D}[u_1, u_2],$$

and G is a contraction and has a unique fixed point.

4.9 Notes and Comments

The preliminaries introduced for the formulation of fuzzy differential equations and the basic results reported for such equations in Section 4.2, are adapted from Lakshmikantham and Mohapatra [1]. For further results on basic fuzzy set theory see Kaleva [1, 2], Seikkala [1], Vorobiev and Seikkala [1], and O'Regan, Lakshmikantham and Nieto [1]. For the results of Section 4.3 concerning stability criteria in terms of Lyapunov-like functions with necessary comparison principles see Gnana Bhaskar, Lakshmikantham, and Vasundhara Devi [1]. To eliminate the possible undesirable part of the solutions, the Hukuhara difference in initial conditions is employed suitably extending the ideas given Lakshmikantham, Leela and Vasundhara Devi [1]. For the suggestion to reduce FDEs to a sequence of multivalued differential equations, see Hüllermeier [1]. See Diamond[1], Diamond and Watson [1] and Lakshmikantham and Mohapatra [1] where Hülermeier's approach is exploited fruitfully. For recent results in this connection see Agarwal, O'Regan, and Lakshmikantham [1]. For results in multivalued differential equations see Deimling [1]. The interconnection between

FDEs and SDEs sketched in Section 4.4, dealing with the continuous case, is taken from Lakshmikantham, Leela and Vatsala [1]. See Lakshmikantham and Tolstonogov [1] for similar results in USC case described in Section 4.5 and also Tolstonogov [1]. The introduction to the theory of impulsive FDEs in Section 4.6 and hybrid FDEs in Section 4.7 and the corresponding results reported are adapted from Lakshmikantham and Vatsala [2]. See Lakshmikantham and Nieto [1] for the formulation of differential equations in metric space and related results for FDEs studied in Section 4.8. For other kinds of formulation of differential equations in metric spaces, see also Aubin [1], Kloeden, Sadovsky and Vasiyeva [1], Melnik [1], Panasyuk [1] and Plotnikov [1].

Chapter 5

Miscellaneous Topics

5.1 Introduction

This chapter is devoted to the introduction of several topics in the framework of set differential equations, that need further investigation.

We begin Section 5.2 with set differential equations involving impulsive effects and prove certain basic results. In Section 5.3, we consider impulsive set differential equations and develop the fruitful monotone iterative technique in a general set up so as to include several special important results.

Section 5.4 is dedicated to the investigation of set differential equations with delay. Here we provide some fundamental results for such equations. In Section 5.5, we introduce impulses into the study of SDEs with delay and consider suitable interesting results. The discussion of set difference equations forms the content of Section 5.6. Employing Causal or nonanticipative maps of Volterra type, we discuss, in Section 5.7, set differential equations involving such maps and extend appropriate basic results to such equations. Lyapunov-like functions whose values are in the metric space $(K_c(\mathbb{R}^d), D)$ are introduced in Section 5.8. A necessary comparison theorem in terms of such Lyapunov-like functions is proved using suitable partial order, to discuss qualitative properties of solutions of set differential systems. The general set up considered for Lyapunov-like functions covers not only existing single, vector, matrix and cone-valued Lyapunov function theory, but also provides a very general framework for further progress.

Since all through this book, we did employ the metric space $(K_c(\mathbb{R}^n), D)$ for the investigation of several situations, in Section 5.9 we indicate how one can extend most of the results discussed to the metric space $(K_c(E), D)$, where E is any real Banach space with suitable modifications demanded by the infinite dimensional framework. Finally, we provide notes and contents in Section 5.10.

5.2 Impulsive Set Differential Equations

Many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. These processes are subject to short term perturbations whose duration is negligible in comparison to the duration of the process. Consequently, it is natural to suppose that these perturbations act instantaneously, that is, in the form of impulses. Thus impulsive differential equations have become a natural description of observed evolution phenomena of several real world problems. The study of impulsive differential equations has been growing as a well deserved discipline, and a systematic treatment of the theory is available. There has been much progress in the investigation of impulsive dynamic systems of other kinds.

In this section, we shall extend the ideas of impulsive ordinary differential equations, to set differential equations and investigate some basic properties.

We first introduce the following notation.

(i) Let $\{t_k\}$ be a sequence such that $0 \leq t_1 < t_2 < \ldots < t_k < \ldots$ with $\lim_{k\to\infty} t_k = \infty$.

(ii) $F \in PC[\mathbb{R}_+ \times K_c(\mathbb{R}^n),\ K_c(\mathbb{R}^n)]$, implies $F : \mathbb{R}_+ \times K_c(\mathbb{R}^n) \to K_c(\mathbb{R}^n)$ is continuous in $(t_{k-1}, t_k] \times K_c(\mathbb{R}^n)$, for each $k = 1, 2, ...$, and for each $U \in K_c(\mathbb{R}^n)$, $k = 1, 2, \ldots$, $\lim_{(t,Y)\to(t_k^+,U)} F(t, Y) = F(t_k^+, U)$ exists.

(iii) $g \in PC[\mathbb{R}_+ \times \mathbb{R}_+ ,\ \mathbb{R}\ ]$ if $g : (t_{k-1}, t_k] \times \mathbb{R}_+ \to \mathbb{R}$ is continuous and for each $w \in \mathbb{R}_+$, $\lim_{(t,z)\to(t_k^+,w)} g(t,z) = g(t_k^+, w)$ exists.

(iv) $h \in PC^1[\mathbb{R}_+, K_c(\mathbb{R}^n)]$ means that h $\in PC[\mathbb{R}_+,\ K_c(\mathbb{R}^n)]$ and is differentiable in each interval (t_{k-1}, t_k).

Now, consider the impulsive set differential equation (ISDE) given by

$$\left.\begin{array}{rcl} D_H U &=& F(t, U), \quad t \neq t_k, \\ U(t_k^+) &=& I_k(U(t_k)), \quad t = t_k, \\ U(t_0) &=& U_0 \in K_c(\mathbb{R}^n), \end{array}\right\} \tag{5.2.1}$$

where $F \in PC[\mathbb{R}_+ \times K_c(\mathbb{R}^n),\ \ K_c(\mathbb{R}^n)]$, $I_k : K_c(\mathbb{R}^n) \to K_c(\mathbb{R}^n)$ and $\{t_k\}$ is a sequence of points such that $0 \leq t_0 < t_1 < t_2 < \ldots < t_k < \ldots$ with $\lim_{k\to\infty} t_k = \infty$.

Definition 5.2.1 *By a solution $U(t, t_0, U_0)$ of the impulsive set differential equation (5.2.1), we mean a piecewise continuous function on $[t_0, \infty)$ which is left continuous in each subinterval $(t_k, t_{k+1}]$ and is given by*

$$U(t, t_0, U_0) = \begin{cases} U_0(t, t_0, U_0), & t_0 \leq t \leq t_1, \\ U_1(t, t_1, U_1^+), & t_1 < t \leq t_2, \\ \vdots & \vdots \\ U_k(t, t_k, U_k^+), & t_k < t \leq t_{k+1}, \\ \vdots & \vdots \end{cases} \tag{5.2.2}$$

where $U_k(t, t_k, U_k^+)$ *is the solution of the set differential equation*

$$D_H U = F(t, U), \quad U(t_k^+) = U_k^+ \ = \ I_k(U(t_k)).$$

We begin with a basic differential inequality result, which is a useful tool in studying monotone method in the impulsive setup later.

Theorem 5.2.1 *Assume that*

(i) $V, W \in PC^1[\mathbb{R}_+, K_c(\mathbb{R}^n)]$, $F \in PC[\mathbb{R}_+ \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$. $F(t, X)$ *is monotone nondecreasing in* X *for each* $t \in \mathbb{R}_+$ *and*

$$\begin{aligned} & D_H V \leq F(t, V), && t \neq t_k, \\ & V(t_k^+) \leq I_k(V(t_k)), && t = t_k, \\ \text{and} \qquad & D_H W \geq F(t, W), && t \neq t_k, \\ & W(t_k^+) \geq I_k(W(t_k)), && t = t_k, \quad k = 1, 2, \cdots; \end{aligned}$$

(ii) $I_k : \ K_c(\mathbb{R}^n) \to K_c(\mathbb{R}^n)$, $I_k(U)$ *is nondecreasing in* U *for each* k*;*

(iii) for any $X, Y \in K_c(\mathbb{R}^n)$ *such that* $X \geq Y$, $t \in \mathbb{R}_+$,

$$F(t, X) \leq F(t, Y) + L(X - Y) \ \text{ for some } L > 0.$$

Then $V(0) \leq W(0)$ *implies* $V(t) \leq W(t)$, $t \geq 0$.

Proof Consider $J = [0, t_1]$. Let $V(0) \leq W(0)$. Then applying Theorem 2.5.1, we have $V(t) \leq W(t)$ on J. This implies $V(t_1) \leq W(t_1)$ and since $I_1(U)$ is nondecreasing,

$$V(t_1^+) \leq I_1(V(t_1)) \leq I_1(W(t_1)) \leq W(t_1^+).$$

Thus $V(t_1^+) \leq W(t_1^+)$. Next set $J = (t_1, t_2]$, and apply Theorem 2.5.1, to get

$$V(t) \leq W(t) \ \text{ on } \ (t_1, t_2].$$

Proceeding as before, we can obtain the conclusion of the theorem.

Corollary 5.2.1 *Let* $V, W \in PC^1[\mathbb{R}_+, K_c(\mathbb{R}^n)]$, $p, \sigma \in C[\mathbb{R}_+, K_c(\mathbb{R}^n)]$

$$\begin{aligned} & D_H V \leq \sigma, && t \neq t_k \ , \\ & V(t_k^+) \leq p(t_k), && t = t_k \ , \end{aligned}$$

and

$$\begin{aligned} & D_H W \geq \sigma, && t \neq t_k \ , \\ & W(t_k^+) \geq p(t_k), && t = t_k \ . \end{aligned}$$

Then $V(t_0) \leq W(t_0)$ *implies*

$$V(t) \leq W(t), \quad t \geq t_0.$$

We first prove an existence theorem for impulsive set differential equations with fixed moments of impulse.

Theorem 5.2.2 *Suppose that*

(i) $F \in PC[\mathbb{R}_+ \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$,

(ii) $D[F(t,U),\theta] \le g(t, D(U,\theta)]$, $t \ne t_k$, *where* $g \in PC[\mathbb{R}^2_+, \mathbb{R}_+]$, $g(t,w)$ *is nondecreasing in* (t,w),

(iii) $D[U(t_k^+),\theta] \le \psi_k(D[U(t_k),\theta])$, $t = t_k$,

(iv) $\psi_k(w)$ *is a nondecreasing function of* w,

(v) $r(t,t_0,w_0)$ *is the maximal solution of the impulsive scalar differential equation*

$$\left.\begin{array}{rcll} w' & = & g(t,w), & t \ne t_k, \\ w(t_k^+) & = & \psi_k(w(t_k)), & t = t_k, \\ w(t_0) & = & w_0, & \end{array}\right\} \tag{5.2.3}$$

existing on $[0,\infty)$,

(vi) $\{t_k\}$ *is a sequence of points of impulse with* $0 \le t_0 < t_1 < t_2 < \ldots < t_k < \ldots$ *and* $\lim_{k\to\infty} t_k = \infty$.

Then there exists a solution for the ISDE (5.2.1).

Before we proceed with the proof, let us define the notion of a maximal solution of (5.2.3).

Definition 5.2.2 *By a maximal solution* $r(t,t_0,w_0)$ *of the impulsive differential equation (5.2.3), we mean the solution* $r(t,t_0,w_0)$ *defined by By a maximal solution* $r(t,t_0,w_0)$ *of the impulsive differential equation (5.2.3), we mean the solution* $r(t,t_0,w_0)$ *defined by*

$$r(t,t_0,w_0) = \begin{cases} r_0(t,t_0,r_0), & t_0 \le t \le t_1, \\ r_1(t,t_1,r_1^+), & t_1 < t \le t_2, \\ \vdots & \vdots \\ r_k(t,t_k,r_k^+), & t_k < t \le t_{k+1}, \\ \vdots & \vdots \end{cases}$$

satisfying the relation

$$w(t,t_0,w_0) \le r(t,t_0,w_0), \ t \in \mathbb{R}_+,$$

for every solution $w(t,t_0,w_0)$ *of (5.2.3), where* $r_k^+ = \psi_k(r_{k-1}(t_k))$.

We now present the proof of Theorem 5.2.2.

Proof Set $J_0 = [t_0, t_1]$ and restrict F to $J_0 \times K_c(\mathbb{R}^n)$.

Consider the set differential equation given by

$$D_H U = F(t, U)$$

$$U(t_0) = U_0 \quad \text{on} \quad J_0.$$

Then the hypothesis of Theorem 2.8.2 is satisfied with $D[U_0,\ \theta] = w_0$. Hence there exists a solution $U_0(t, t_0, U_0)$, for the set differential equation such that $D[U_0(t),\ \theta] \leq r(t, t_0, w_0)$ on J_0.

For $t = t_1$, $U_0(t_1) = U_0(t_1, t_0, U_0)$. Set $U_1^+ = U_0(t_1^+) = I_1(U_0(t_1))$. From hypothesis,

$$\begin{aligned} D[U_1^+,\ \theta] &= D[I_1(U_0(t_1)),\ \theta] \\ &\leq \psi(D[U_0(t_1),\ \theta]) \\ &\leq \psi(r(t_1)) = r(t_1^+). \end{aligned}$$

Put $J_1 = [t_1, t_2]$ and consider the set differential equation

$$D_H U = F(t, U), \quad t \in J_1,$$

$$U(t_1^+) = U_1^+.$$

Then again, restricting F to the domain $J_1 \times K_c(\mathbb{R}^n)$, the hypothesis of Theorem 2.8.2 is satisfied and thus there exists a solution $U_1(t, t_1, U_1^+)$ for $t \in J_1$ satisfying the set differential equation restricted to J_1. We have

$$U_1(t_2) = U_1(t_2, t_1, U_1^+) \text{ and } U_1(t_2^+) = I_2(U(t_2)).$$

$$\text{Set} \qquad U_2^+ = U_1(t_2^+) \text{and } J_2 = [t_2, t_3]$$

Now repeating the above process, we obtain the existence of a solution of the impulsive set differential equation.

We next give a basic comparison theorem in the impulsive set differential equation set up.

Theorem 5.2.3 *Assume that*

(i) $F \in PC[\mathbb{R}_+ \times K_c(\mathbb{R}^n),\ K_c(\mathbb{R}^n)]$;

(ii) *for* $t \in \mathbb{R}_+$, $t \neq t_k$, $U, V \in K_c(\mathbb{R}^n)$,

$$D[F(t, U), F(t, V)] \leq g(t, D(U, V)], \tag{5.2.4}$$

where $g \in PC[\mathbb{R}_+^2, \mathbb{R}_+]$;

(iii) $D[U(t_k^+),\ V(t_k^+)] \leq \psi_k(D[U(t_k), V(t_k)])$, *where* $\psi_k(w)$ *is a nondecreasing function of* w.

Further, suppose that the maximal solution $r(t, t_0, w_0)$ *of the impulsive scalar differential equation* (5.2.3) *exists on* $\mathbb{R}_+$.

If $U(t)$, $V(t)$ *are any two solutions of ISDE* (5.2.1) *through* $U(t_0) = U_0$ *and* $V(t_0) = V_0$, $U_0, V_0 \in K_c(\mathbb{R}^n)$ *on* J *respectively, then we have*

$$D[U(t), V(t)] \le r(t, t_0, w_0), \quad t \in \mathbb{R}_+,$$

provided $D[U_0, V_0] \le w_0$.

Proof We set $J_0 = [t_0, t_1]$ and restrict the domain of F to $J_0 \times K_c(\mathbb{R}^n)$. Then F is continuous on this domain and further the hypothesis of Theorem 2.2.1 is satisfied. Hence we can conclude that

$$D[U(t), V(t)] \le r(t, t_0, w_0], \quad t \in J_0,$$

which implies $D[U(t_1), V(t_1)] \le r(t_1, t_0, w_0]$.

Now using the hypothesis for $t = t_1^+$, we have

$$\begin{aligned} D[U(t_1^+), V(t_1^+)] &\le \psi_1(D[U(t_1), V(t_1)]) \\ &\le \psi_1[r(t_1, t_0, w_0)] \\ &= r(t_1^+), \end{aligned}$$

since ψ_1 is a nondecreasing function. Thus

$$D[U(t_1^+), V(t_1^+)] \le r(t_1^+). \tag{5.2.5}$$

Next, set $J_1 = [t_1, t_2]$, $domF = J_1 \times K_c(\mathbb{R}^n)$. Then using the inequalities (5.2.4), (5.2.5) and applying Theorem 2.2.1, we conclude that

$$D[U(t), V(t)] \le r(t, t_0, w_0), \quad t \in J_1.$$

Repeating the above process, the conclusion of the theorem is obtained.

We now state a corollary which will be a useful tool in our work.

Corollary 5.2.2 *Assume that,*

(i) $F \in PC[\mathbb{R}_+ \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$;

(ii) $D[F(t, U), \theta] \le g(t, D[U, \theta])$, $t \ne t_k$, *where* $g \in PC[\mathbb{R}_+^2, R]$;

(iii) $D[U(t_k^+), \theta] \le \psi_k(D[U(t_k), \theta])$, $t = t_k$, *and* $\psi_k(w)$ *is nondecreasing in* w;

(iv) $r(t, t_0, w_0)$ *is the maximal solution of the impulsive scalar differential equation* (5.2.3).

Then, if $D[U_0, \theta] \le w_0$, *we have*

$$D[U(t), \theta] \le r(t, t_0, w_0), \quad t \in J.$$

We next prove the comparison theorem using Lyapunov-like functions. This will help us to study stability criteria for ISDE (5.2.1). Before proceeding further, we make the following assumption.

Let $V : \mathbb{R}_+ \times K_c(\mathbb{R}^n) \to \mathbb{R}_+$. We say that V belongs to the class V_0 if

(i) V is continuous in $(t_{k-1}, t_k] \times K_c(\mathbb{R}^n)$ and for each $U \in K_c(\mathbb{R}^n), k = 1, 2, \ldots,$ $\lim_{(t,Y)\to(t_k^+,U)} V(t,Y) = V(t_k^+, U)$ exists,

(ii) $|V(t, A) - V(t, B)| \leq LD[A, B]$ for $A, B \in K_c(\mathbb{R}^n),\ t \in \mathbb{R}_+$, where L is the local Lipschitz constant.

Theorem 5.2.4 *Assume that,*

(i) $V \in V_0$*;*

(ii) *for* $t \in \mathbb{R}_+$, $U \in K_c(\mathbb{R}^n)$,

$$D^+V(t,U) \leq g(t, V(t,U)), \quad t \neq t_k,$$

where $g \in C[\mathbb{R}_+^2, R]$*;*

(iii) $V(t_k^+, U(t_k^+)] \leq \psi_k(V(t_k, U(t_k)))$, $t = t_k$, *where* $\psi_k(w)$ *is nondecreasing in* w.

Further, suppose that $r(t, t_0, w_0)$ *is the maximal solution of the impulsive scalar differential equation (5.2.3) existing on* $\mathbb{R}_+$*. Then, if* $U(t) = U(t, t_0, U_0)$ *is any solution of* (5.2.1) *existing on* $\mathbb{R}_+$ *such that* $V(t_0, U_0) \leq w_0$*, we have*

$$V(t, U(t)) \leq r(t, t_0, w_0), \quad t \in \mathbb{R}_+.$$

Proof Set $J_0 = [t_0, t_1]$. Applying Theorem 3.2.1 on $J_0 \times K_c(\mathbb{R}^n)$, we obtain

$$V(t, U(t)) \leq r(t, t_0, w_0), \quad t \in J_0.$$

At $t = t_1$, $V(t_1, U(t_1)) \leq r(t_1, t_0, w_0)$. Since ψ_1 is a nondecreasing function

$$\psi_1(V(t_1, U(t_1))) \leq \psi_1(r(t_1, t_0, w_0) = r(t_1^+).$$

Hence proceeding as in the earlier theorems, step by step, we can prove the conclusion of the theorem.

We now define stability properties of the null solution of an impulsive set differential equation (5.2.1).

Definition 5.2.3 *Let* $U(t) = U(t, t_0, U_0)$ *be any solution of ISDE (5.2.1). Then the trivial solution* $U(t) \equiv \theta$ *is said to be*

(S_1) stable, if for each $\epsilon > 0$ *and* $t_0 \in \mathbb{R}_+$*, there exists a* $\delta = \delta(t_0, \epsilon) > 0$ *such that* $D[U_0, \theta] < \delta$ *implies* $D[U(t), \theta] < \epsilon$*, for* $t \geq t_0$*.*

(S_2) attractive, if for each $\epsilon > 0$ *and* $t_0 \in \mathbb{R}_+$ *there exist* $\delta_0 = \delta_0(t_0) > 0$ *and a* $T = T(t_0, \epsilon) > 0$ *such that* $D[U_0, \theta] < \delta_0$ *implies*

$$D[U(t), \theta] < \epsilon, \quad \textit{for} \quad t \geq t_0 + T.$$

The other definitions can be formulated similarly.

We denote $B(U_0, b) = \{U \in K_c(\mathbb{R}^n) : D[U, U_0] \leq b\}$,

$$\mathcal{K} = \{\sigma \in C[\mathbb{R}_+, \mathbb{R}_+] : \sigma(0) = 0 \text{ and } \sigma(t) \text{ is strictly increasing in } t\}.$$

The following theorem connects the stability properties of the trivial solution of ISDE (5.2.1) with the stability properties of the trivial solution of the impulsive scalar differential equation (5.2.3) through the Lyapunov-like function.

In order to obtain the trivial solution for ISDE (5.2.1) we assume that $F(t, \theta) \equiv \theta$ and $I_k(\theta) = \theta$ for all k.

Theorem 5.2.5 *Assume that*

(i) $V : \mathbb{R}_+ \times B(\theta, b) \to \mathbb{R}_+$, $V \in V_0$,

$$D^+V(t,U) \leq g(t, V(t,U)), \quad t \neq t_k,$$

where $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, $g(t,0) \equiv 0$ *and* g *satisfies* $A_0(ii)$.

(ii) *there exists* $b_0 > 0$ *such that* $U \in B(\theta, b_0)$ *implies that* $I_k(U) \in B(\theta, b)$ *for all* k, *and*

$$V(t_k, I_k(U(t_k))) \leq \psi_k(V(t_k, U(t_k))), \;\; U \in B(\theta, b_0)$$

and $\psi_k : \mathbb{R}_+ \to \mathbb{R}_+$ *is nondecreasing,* $\psi_k(0) = 0$;

(iii) $b(D[U,\theta]) \leq V(t,U) \leq a(D[U,\theta])$, *where* $a, b \in \mathcal{K}$.

Then the stability properties of the trivial solution of the impulsive scalar differential equation (5.2.3) *imply the corresponding stability properties of the trivial solution of ISDE* (5.2.1).

Proof Let $0 < \epsilon < b^* = \min(b_0, b)$ and $t_0 \in \mathbb{R}_+$ be given. Suppose the trivial solution of (5.2.3) is stable. Then given $b(\epsilon) > 0$ there exists a $\delta_1(t_0, \epsilon) > 0$ such that

$$0 \leq w_0 < \delta_1 \text{ implies } w(t, t_0, u_0) < b(\epsilon), \quad t \geq t_0,$$

where $w(t, t_0, u_0)$ is any solution of (5.2.3).

Let $w_0 = a(D[U_0, \theta])$ and choose $\delta_2 = \delta_2(\epsilon)$ such that $a(\delta_2) < \delta_1$.

Define $\delta = \min(\delta_1, \delta_2)$. With this δ, we claim that if $D[U_0, \theta] < \delta$ then $D[U(t), \theta] < \epsilon$, $t \geq t_0$, where $U(t) = U(t, t_0, U_0)$ is any solution of ISDE (5.2.1). Suppose this does not hold.

Then there exists a solution $U(t) = U(t, t_0, U_0)$ of ISDE (5.2.1) with $D[U_0, \theta] < \delta$ and a $t^* > t_0$ such that $t_k < t^* \leq t_{k+1}$ for some k satisfying $\epsilon \leq D[U(t^*), \theta]$ and $D[U(t), \theta] < \epsilon$ for $t_0 \leq t \leq t_k$. Since $0 < \epsilon < b_0$ from condition (ii) we have

$$D[U(t_k^+), \theta] = D[I_k(U(t_k)), \theta] < b,$$

and $D[U(t_k), \theta] < \epsilon$.

Hence, we can find a t^0 such that $t_k < t^0 \leq t^*$ satisfying

$$\epsilon \leq D[U(t^0), \theta] < b.$$

Setting $m(t) = V(t, U(t))$ for $t_0 \leq t \leq t^0$, and using the hypothesis (i) and (ii), we get from Theorem 5.2.4, the estimate

$$V(t, U(t)) \leq r(t, t_0, a(D[U_0, \theta])), \quad t_0 \leq t \leq t^0,$$

where $r(t, t_0, w_0)$ is the maximal solution of impulsive scalar differential equation (5.2.3).

Now consider

$$\begin{aligned} b(\epsilon) \le b(D[U(t^0), \theta]) &\le V(t^0, U(t^0)) \\ &\le r(t^0, t_0, a(D[U_0, \theta])) < b(\epsilon), \end{aligned}$$

which is a contradiction. This proves that $U(t) \equiv \theta$ of ISDE (5.2.1) is stable.

Next, if we suppose that $w \equiv 0$ of (5.2.3) is uniformly stable. Then clearly δ is independent of t_0 and this gives the uniform stability of $U \equiv \theta$ of the ISDE (5.2.1).

Let us suppose that $w \equiv 0$ of (5.2.3) is asymptotically stable. This implies that $U \equiv \theta$ of ISDE (5.2.1) is stable. Hence, set $\epsilon = b^*$ and $\delta_0^* = \delta(t_0, b^*)$, we have

$$D[U_0, \theta] < \delta_0^* \text{ implies } D[U(t), \theta] < b^*, \ t \ge t_0. \tag{5.2.6}$$

To prove attractivity, we let $0 < \epsilon < b^*$ and $t_0 \in \mathbb{R}_+$. Since $w \equiv 0$ of (5.2.3) is attractive, given $b(\epsilon) > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta_{10} = \delta_{10}(t_0) > 0$ and a $T = T(t_0, \epsilon) > 0$ such that $0 \le w_0 < \delta_{10}$ implies

$$w(t, t_0, w_0) < b(\epsilon) \text{ for } t \ge t_0 + T.$$

Then using (5.2.6) and reasoning as in the earlier case, we get

$$V(t, U(t)) \le r(t, t_0, a(D[U_0, \theta])).$$

Thus we get

$$\begin{aligned} b(D[U(t), \theta]) \le V(t, U(t)) &\le r(t, t_0, a(D[U_0, \theta])) \\ &< b(\epsilon), \quad t \ge t_0 + T, \end{aligned}$$

which implies

$$D[U(t), \theta] < \epsilon, \quad t \ge t_0 + T.$$

Thus $U \equiv \theta$ is attractive and hence asymptotically stable.

We now consider the example given in 3.4.3 and illustrate how impulses control the behavior of the solutions and as such the notion of using Hukuhara difference in initial values becomes redundant in this case.

Example 5.2.1 *Consider the set differential equation,*

$$D_H U = (-1)U, \ U(0) = U_0 \in K_c(\mathbb{R}). \tag{5.2.7}$$

Since the values of the solution $U(t)$ of (5.2.7) are intervals, the equation (5.2.7) can be written as

$$[u_1', u_2'] = (-1)U = [-u_2, -u_1], \tag{5.2.8}$$

where $U = [u_1, u_2]$ and $U_0 = [u_{10}, u_{20}]$. Recall that the solution is given by

$$\left.\begin{aligned} u_1(t) &= \tfrac{1}{2}[u_{10} + u_{20}]e^{-t} + \tfrac{1}{2}[u_{10} - u_{20}]e^{t}, \\ u_2(t) &= \tfrac{1}{2}[u_{20} + u_{10}]e^{-t} + \tfrac{1}{2}[u_{20} - u_{10}]e^{t}. \end{aligned}\right\} \tag{5.2.9}$$

If $U_0 = [u_0, u_0]$, that is U_0 is a singleton, we get from (5.2.9),

$$U(t) = [u_1(t), u_2(t)] = u_0 e^{-t}, \quad t \geq 0.$$

In this situation, the impulses have no role to play and hence we can take

$$u(t_k^+) = u(t_k), \quad k = 1, 2, \ldots.$$

If, on the other hand, we take $U_0 = [-u_0, u_0]$, then (5.2.9)reduces to

$$U(t) = [-u_0, u_0]e^t, \quad t \geq 0.$$

Suppose that we choose the impulses as

$$U(t_k^+) = d_k U(t_k), \quad \text{for } t = t_k, \tag{5.2.10}$$

where the d_k's satisfy $0 < d_k < 1$ and

$$t_{k+1} + ln\ d_k \leq t_k \quad \text{for all } k, \tag{5.2.11}$$

then the solution of the corresponding ISDE (5.2.7),(5.2.10) is given by

$$U(t) = U_0\, \Pi_{0<t_k<t}\ d_k\ e^t, \quad t \geq 0, \tag{5.2.12}$$

we know that $D[U(t), \theta] = \|U(t)\|$, therefore

$$\|U(t)\| \leq \|U_0\| \Pi_{0<t_k<t}\ d_k\ e^t, \quad t \geq 0. \tag{5.2.13}$$

Choosing $\delta = \frac{\epsilon}{2}\ e^{-t_1}$ and using (5.2.11) it follows that $\|U(t)\| < \epsilon$, $t \geq 0$ provided that $||U_0|| < \delta$. Hence the stability of the trivial solution of (5.2.7), (5.2.10) follows.

To prove asymptotic stability, we strengthen the assumption (5.2.11) to

$$t_{k+1} + ln\ \alpha\ d_k\ \leq\ t_k \text{ for all } k, \quad \text{where } \alpha > 1.$$

Then $d_k \leq \frac{1}{\alpha}\ exp[t_k - t_{k+1}]$. Using this estimate on d_k in (5.2.13), we see from the relation

$$\lim_{k \to \infty} \|U(t)\| = 0.$$

Thus, the trivial solution $U \equiv \theta$ of the ISDE (5.2.7), (5.2.10) is asymptotically stable.

Remark 5.2.1 *If U, F and I in (5.2.1) are single-valued mappings then the Hukuhara derivative and integral reduce to the ordinary derivative and integral. Consequently, the impulsive set differential equation (5.2.1) reduces to the corresponding ordinary impulsive differential system. Thus the results obtained in this section include the corresponding results of such equations as a very special case.*

5.3 Monotone Iterative Technique

We develop the monotone technique for ISDE corresponding to the various notions of upper and lower solutions of SDE 2.5.3. We can define similar concepts for the ISDE

$$\begin{aligned} D_H U &= F(t,U) + G(t,U), \qquad t \neq t_k, \\ U(t_k^+) &= I_k(U(t_k)) + J_k(U(t_k)),\ t = t_k, \\ U(0) &= U_0 \in K_c(\mathbb{R}^n), \end{aligned} \tag{5.3.1}$$

where $F, G \in PC[J \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$, $I_k, J_k : K_c(\mathbb{R}^n) \to K_c(\mathbb{R}^n)$ for each k and $0 < t_1 < t_2 < \cdots < t_k < \cdots$, with $\lim_{k\to\infty} t_k = T$ with $J = [0, T]$.

Definition 5.3.1 *Let $V, W \in PC^1[J, K_c(\mathbb{R}^n)]$. Then V, W are said to be*

(a) coupled lower and upper solutions of type I of (5.3.1) *if*

$$\begin{aligned} D_H V &\leq F(t,V) + G(t,W), \qquad t \neq t_k, \\ V(t_k^+) &\leq I_k(V(t_k)) + J_k(W(t_k)),\ t = t_k, \\ V(0) &\leq U_0, \end{aligned} \tag{5.3.2}$$

and

$$\begin{aligned} D_H W &\geq F(t,W) + G(t,V), \qquad t \neq t_k, \\ W(t_k^+) &\geq I_k(W(t_k)) + J_k(V(t_k)),\ t = t_k, \\ W(0) &\geq U_0, \end{aligned} \tag{5.3.3}$$

(b) coupled lower and upper solutions of type II of (5.3.1) *if*

$$\begin{aligned} D_H V &\leq F(t,W) + G(t,V), \qquad t \neq t_k, \\ V(t_k^+) &\leq I_k(W(t_k)) + J_k(V(t_k)),\ t = t_k, \\ V(0) &\leq U_0, \end{aligned} \tag{5.3.4}$$

and

$$\begin{aligned} D_H W &\geq F(t,V) + G(t,W), \qquad t \neq t_k, \\ W(t_k^+) &\geq I_k(V(t_k)) + J_k(W(t_k)),\ t = t_k, \\ W(0) &\geq U_0. \end{aligned} \tag{5.3.5}$$

Theorem 5.3.1 *Assume that*

(A1) $V, W \in PC^1[J, K_c(\mathbb{R}^n)]$ *are coupled lower and upper solutions of type I relative to* (5.3.1) *with* $V(t) \le W(t),\ t \in J$;

(A2) $F, G \in C[J \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$, $F(t, X)$ *is nondecreasing in* X *and* $G(t, Y)$ *is nonincreasing in* Y, *for each* $t \in J$, F, G *map bounded sets into bounded sets;*

(A3) $I_k(U)$ *is continuous and nondecreasing in* U, $J_k(U)$ *is continuous and nonincreasing in* U, *for each* $k = 1, 2, \cdots$.

Then there exist monotone sequences $\{V_n(t)\}, \{W_n(t)\}$ *in* $K_c(\mathbb{R}^n)$ *such that* $V_n(t) \to \rho(t)$, $W_n(t) \to R(t)$ *in* $K_c(\mathbb{R}^n)$ *and* (ρ, R) *are the coupled minimal and maximal solutions of type I of* (5.3.1) *respectively, that is, they satisfy the relations*

$$\begin{aligned} D_H \rho &= F(t, \rho) + G(t, R), \qquad t \neq t_k,\\ \rho(t_k^+) &= I_k(\rho(t_k)) + J_k(R(t_k)),\ t = t_k,\\ \rho(0) &= U_0, \end{aligned} \tag{5.3.6}$$

and

$$\begin{aligned} D_H R &= F(t, R) + G(t, \rho), \qquad t \neq t_k,\\ R(t_k^+) &= I_k(R(t_k)) + J_k(\rho(t_k)),\ t = t_k,\\ R(0) &= U_0, \end{aligned} \tag{5.3.7}$$

for $t \in J$.

Proof Consider, for each $n \ge 0$, the ISDEs given by

$$\begin{aligned} D_H V_{n+1} &= F(t, V_n) + G(t, W_n), \qquad t \neq t_k,\\ V_{n+1}(t_k^+) &= I_k(V_n(t_k)) + J_k(W_n(t_k)),\ t = t_k,\\ V_{n+1}(0) &= U_0, \end{aligned} \tag{5.3.8}$$

and

$$\begin{aligned} D_H W_{n+1} &= F(t, W_n) + G(t, V_n), \qquad t \neq t_k,\\ W_{n+1}(t_k^+) &= I_k(W_n(t_k)) + J_k(V_n(t_k)),\ t = t_k,\\ W_{n+1}(0) &= U_0, \end{aligned} \tag{5.3.9}$$

where $V(0) \le U_0 \le W(0)$.

It is clear that the equations (5.3.8) and (5.3.9) have unique solutions say $V_{n+1}(t)$ and $W_{n+1}(t)$, $t \in J$. We set $V_0(t) = V(t)$ and $W_0(t) = W(t)$, $t \in J$.

Our aim is to prove

$$V_0 \le V_1 \le V_2 \le \cdots \le V_n \le W_n \le \cdots \le W_2 \le W_1 \le W_0,\ \ t \in J\ . \tag{5.3.10}$$

From the hypothesis, we have that V_0 and W_0 are coupled lower and upper solutions of type I of (5.3.1). Setting, $V_n = V_0$ and $W_n = W_0$ in (5.3.8) and (5.3.9), we get, $V_1(t)$ and $W_1(t)$, $t \in J$, which are unique solutions of (5.3.8) and (5.3.9). We now claim

(i) $V_0 \leq V_1$,

(ii) $V_1 \leq W_1$, and

(iii) $W_1 \leq W_0$ for $t \in J$.

To prove (i), consider the equation (5.3.8) with $n = 0$,then

$$\begin{aligned} D_H V_1 &= F(t, V_0) + G(t, W_0), \qquad t \neq t_k, \\ V_1(t_k^+) &= I_k(V_0(t_k)) + J_k(W_0(t_k)), \; t = t_k, \\ V_1(0) &= U_0, \end{aligned} \tag{5.3.11}$$

and from hypothesis $(A1)$, we have,

$$\begin{aligned} D_H V_0 &\leq F(t, V_0) + G(t, W_0), \qquad t \neq t_k, \\ V_0(t_k^+) &\leq I_k(V_0(t_k)) + J_k(W_0(t_k)), \; t = t_k, \\ V_0(0) &\leq U_0. \end{aligned} \tag{5.3.12}$$

Now arguing as in Theorem 2.5.1, we get

$$V_0(t) \leq V_1(t), \quad t \in (t_{k-1}, t_k].$$

Now using the fact that

$$V_0(t_k^+) \leq V_1(t_k^+), \;\text{ at each } t = t_k,$$

we get $V_0(t) \leq V_1(t), \quad t \in J$.

Next, to prove (ii), we consider the relations (5.3.8), (5.3.9) with $n = 0$. We use the monotone properties of F and G, and I_k and J_k for each $k = 1, 2, \cdots$. Then we arrive at the following equations:

$$\begin{aligned} D_H V_1 &\leq F(t, W_0) + G(t, W_0), \qquad t \neq t_k, \\ V_1(t_k^+) &\leq I_k(W_0(t_k)) + J_k(W_0(t_k)), \; t = t_k, \\ V_1(0) &= U_0, \end{aligned}$$

and

$$\begin{aligned} D_H W_1 &\geq F(t, W_0) + G(t, W_0), \qquad t \neq t_k, \\ W_1(t_k^+) &\geq I_k(W_0(t_k)) + J_k(W_0(t_k)), \; t = t_k, \\ W_1(0) &= U_0, \end{aligned}$$

which yield from Corollary 5.2.1,

$$V_1(t) \leq W_1(t), \quad t \in J.$$

Proceeding as in the proof of (i), we obtain, $W_1(t) \leq W_0(t), \; t \in J$, which is (iii).

Thus, we have

$$V_0(t) \le V_1(t) \le W_1(t) \le W_0(t) \quad \text{for} \quad t \in J.$$

Assume that for some $j > 1$, we have

$$V_{j-1}(t) \le V_j(t) \le W_j(t) \le W_{j-1}(t) \quad \text{for} \quad t \in J. \tag{5.3.13}$$

Then we prove $V_j(t) \le V_{j+1}(t) \le W_{j+1}(t) \le W_j(t) \quad \text{for} \quad t \in J$.

Consider

$$\begin{aligned} D_H V_j &= F(t, V_{j-1}) + G(t, W_{j-1}), \qquad t \ne t_k, \\ V_j(t_k^+) &= I_k(V_{j-1}(t_k)) + J_k(W_{j-1}(t_k)),\ t = t_k, \\ V_j(0) &= U_0, \end{aligned} \tag{5.3.14}$$

and

$$\begin{aligned} D_H V_{j+1} &= F(t, V_j) + G(t, W_j), \qquad t \ne t_k, \\ V_{j+1}(t_k^+) &= I_k(V_j(t_k)) + J_k(W_j(t_k)),\ t = t_k, \\ V_{j+1}(0) &= U_0. \end{aligned} \tag{5.3.15}$$

Using the nondecreasing nature of F, for each $t \in J$ and the nonincreasing nature of G, for each $t \in J$, and also using the nondecreasing nature of I_k and the nonincreasing nature of J_k, for each k, along with the relation (5.3.13) we arrive at

$$\begin{aligned} D_H V_{j+1} &\ge F(t, V_{j-1}) + G(t, W_{j-1}), \qquad t \ne t_k, \\ V_{j+1}(t_k^+) &\ge I_k(V_{j-1}(t_k)) + J_k(W_{j-1}(t_k)),\ t = t_k, \\ V_{j+1}(0) &\ge U_0. \end{aligned} \tag{5.3.16}$$

By applying Corollary 5.2.1, to the equations (5.3.14) and (5.3.16) we get,

$$V_j(t) \le V_{j+1}(t), \quad t \in J.$$

Similarly, we can show that $W_{j+1}(t) \le W_j(t), \quad t \in J$.

We next prove that $V_{j+1}(t) \le W_{j+1}(t), \quad t \in J$. Taking $n = j$ in (5.3.8) and (5.3.9), we have

$$\begin{aligned} D_H V_{j+1} &= F(t, V_j) + G(t, W_j), \qquad t \ne t_k, \\ V_{j+1}(t_k^+) &= I_k(V_j(t_k)) + J_k(W_j(t_k)),\ t = t_k, \\ V_{j+1}(0) &= U_0, \end{aligned} \tag{5.3.17}$$

and

$$\begin{aligned} D_H W_{j+1} &= F(t, W_j) + G(t, V_j), \qquad t \ne t_k, \\ W_{j+1}(t_k^+) &= I_k(W_j(t_k)) + J_k(V_j(t_k)),\ t = t_k, \\ W_{j+1}(0) &= U_0. \end{aligned} \tag{5.3.18}$$

Again, using the fact that G is nonincreasing in Y for each t, and $J_k(U)$ is nonincreasing in U in the relation (5.3.18), and F is nondecreasing in X for each t and $I_k(U)$ is nondecreasing in U for each $k = 1, 2, \cdots$, in the relations (5.3.17),(5.3.18) we have,

$$\begin{aligned} D_H V_{j+1} &\le F(t, W_j) + G(t, W_j), \qquad t \neq t_k, \\ V_{j+1}(t_k^+) &\le I_k(W_j(t_k)) + J_k(W_j(t_k)),\ t = t_k, \\ V_{j+1}(0) &\le U_0, \end{aligned}$$

and

$$\begin{aligned} D_H W_{j+1} &\ge F(t, W_j) + G(t, W_j), \qquad t \neq t_k, \\ W_{j+1}(t_k^+) &\ge I_k(W_j(t_k)) + J_k(W_j(t_k)),\ t = t_k, \\ W_{j+1}(0) &\ge U_0, \quad t \in J, \end{aligned}$$

which on using Corollary 5.2.1, yields

$$V_{j+1} \le W_{j+1} \quad \text{for} \quad t \in J.$$

Thus we have the sequences of functions $\{V_n\}, \{W_n\}$ which are piecewise continuous functions and also satisfy the relation (5.3.10). Clearly these sequences are uniformly bounded on J. In each subinterval $[t_k, t_{k+1}]$, the sequence of functions $\{V_n\}$ and $\{W_n\}$ are equi-continuous; hence using Arzela–Ascoli Theorem on each subinterval, we show that the entire sequence $\{V_n(t)\}$ converges uniformly to $\rho(t)$ on $[t_k, t_{k+1}]$ and $\{W_n\}$ converges uniformly to $R(t)$ on $[t_k, t_{k+1}]$. Since I_k, J_k are continuous functions for each $k = 1, 2, \cdots$, we obtain from

$$\lim_{n\to\infty} V_n(t_k^+) = \lim_{n\to\infty} [I_k(V_{n-1}(t_k)) + J_k(W_{n-1}(t_k))]$$

that $\rho(t_k^+) = I_k(\rho(t_k)) + J_k(R(t_k))$, similarly $R(t_k^+) = I_k(R(t_k)) + J_k(\rho(t_k))$.

We now consider the integral equations,

$$\begin{aligned} V_{n+1}(t) &= U_0 + \int_0^t [F(s, V_n(s)) + G(s, W_n(s))]\, ds \\ W_{n+1}(t) &= U_0 + \int_0^t [F(s, W_n(s)) + G(s, V_n(s))]\, ds \end{aligned}$$

Taking limits as $n \to \infty$, using the uniform continuity of F and G on each subinterval $[t_k, t_{k+1}]$, we get (5.3.6) and (5.3.7). Further, $V_0 \le \rho \le R \le W_0$ for $t \in J$.

Next, we claim that (ρ, R) are coupled minimal and maximal solutions of ISDE(5.3.1). For proof, we show that if $U(t)$ is any solution of (5.3.1) such that $V_0 \le U \le W_0$ for $t \in J$ then,

$$V_0 \le \rho \le U \le R \le W_0, \quad \text{for} \quad t \in J. \tag{5.3.19}$$

Suppose for some n,

$$V_n \leq U \leq W_n, \quad t \in J. \tag{5.3.20}$$

Using (5.3.20) along with the monotone properties of F, G for each t, and I_k, J_k for each k, we arrive at

$$\begin{aligned} D_H U &\geq F(t, V_n) + G(t, W_n), \qquad t \neq t_k, \\ U(t_k^+) &\geq I_k(V_n(t_k)) + J_k(W_n(t_k)),\ t = t_k, \\ U(0) &\geq U_0, \end{aligned}$$

and

$$\begin{aligned} D_H V_{n+1} &= F(t, V_n) + G(t, W_n), \qquad t \neq t_k, \\ V_{n+1}(t_k^+) &= I_k(V_n(t_k)) + J_k(W_n(t_k)),\ t = t_k, \\ V_n(0) &= U_0, \end{aligned}$$

which yields, on using Corollary 5.2.1, $V_{n+1}(t) \leq U(t),\ \ t \in J$.

Similarly $W_{n+1}(t) \geq U(t)$ for $t \in J$. This holds for all n. Hence taking limits as $n \to \infty$, we come up with the relation (5.3.19), thus proving our claim.

Corollary 5.3.1 *If in addition to the assumptions of Theorem* 5.3.1*, suppose that the following hold.*

(i) F and G satisfy the relations, for $X, Y \in K_c(\mathbb{R}^n)$, whenever $X \geq Y$,

$$\begin{aligned} F(t, X) &\leq F(t, Y) + N_1(X - Y), \quad N_1 \geq 0 \\ \textit{and} \quad G(t, X) + N_2(X - Y) &\geq G(t, Y), \quad N_2 \geq 0; \end{aligned}$$

(ii) for each k, I_k, J_k satisfy the relations

$$\begin{aligned} I_k(X) &\leq I_k(Y) + M_{1k}(X - Y), \quad M_{1k} \geq 0 \\ \textit{and} \quad J_k(X) + M_{2k}(X - Y) &\geq J_k(Y), \quad M_{2k} \geq 0, \end{aligned}$$

such that $M_{1k} + M_{2k} < 1$.

Then $\rho = U = R$ is the unique solution of the ISDE (5.3.1).

Proof Since $\rho \leq R$, we have $R = \rho + m$ or $m = R - \rho$. Now

$$\begin{aligned} D_H \rho + D_H m = D_H R &= F(t, R) + G(t, \rho),\ \ t \neq t_k, \\ &\leq F(t, \rho) + N_1 m + G(t, R) + N_2 m,\ \ t \neq t_k, \\ &= D_H \rho + (N_1 + N_2) m,\ \ t \neq t_k, \end{aligned}$$

and for $t = t_k$,

$$\begin{aligned} m(t_k^+) + \rho(t_k^+) &= R(t_k^+) \\ &= I_k(R(t_k)) + J_k(\rho(t_k)) \\ &\leq I_k(\rho(t_k)) + J_k(R(t_k)) + M_{1k} m(t_k) + M_{2k} m(t_k) \\ &= \rho(t_k^+) + (M_{1k} + M_{2k}) m(t_k). \end{aligned}$$

Thus we have,

$$\begin{aligned} D_H m &\leq Nm, \quad t \neq t_k, \\ m(t_k^+) &\leq M_k m(t_k), \quad t = t_k, \\ m(0) &= 0, \end{aligned}$$

with $N = N_1 + N_2 > 0$ and $M_k = M_{1k} + M_{2k}$ with $0 < M_k < 1$ for each k.

Using a special case of Theorem 1.4.1 in Lakshmikantham, Bainov, Simeonov [1], we obtain $m(t) \leq 0$, that is $R \leq \rho$. Hence $\rho = U = R$ is the unique solution of the ISDE (5.3.1).

Remark 5.3.1

(1) In Theorem 5.3.1*, if* $G(t, Y) \equiv 0$, $J_k(U) \equiv 0$ *for every* k*, then we get the result when* F *is nondecreasing in* X *for each* t *and* $I_k(U)$ *is nondecreasing in* U *for every* k*.*

(2) In (1) *above, suppose that* F *is not nondecreasing in* X *for every* t *and for every* k*,* $I_k(X)$ *is not nondecreasing in* X*, but* $\tilde{F}(t, X) = F(t, X) + MX$*,* $M > 0$ *is nondecreasing in* X *and* $\tilde{I}(X) = I_k(X) + N_k X$ *is nondecreasing in* X*, for* $N_k > 0$*.*

Now we consider the IVP of ISDE

$$\begin{gathered} D_H U + MU = \tilde{F}(t, U), \quad t \neq t_k, \\ U(t_k^+) + N_k U(t_k) = \tilde{I}_k(U(t_k)), \quad t = t_k, \\ U(0) = U_0. \end{gathered}$$

Then we obtain the same conclusion as in (1)*. To see this, consider the transformation*

$$\tilde{U}(t) = \begin{cases} U(t)e^{Mt}, & t \neq t_k, \\ \frac{1}{1+N_k}\tilde{U}(t), & t = t_k, \end{cases}$$

then

$$\begin{aligned} D_H \tilde{U} &= \tilde{F}(t, \tilde{U}e^{-Mt})e^{Mt} = F_0(t, \tilde{U}), \quad t \neq t_k, \\ \tilde{U}(t_k^+) + N_k\, \tilde{U}(t_k) &= [1 + N_k]\tilde{I}_k\left(\frac{\tilde{U}(t_k)}{1 + N_k}\right) = \mathring{I}_k(\tilde{U}(t_k)), \quad t = t_k, \\ \tilde{U}(0) &= U_0. \end{aligned}$$

For this system

$$\tilde{V}(t) = \begin{cases} V(t)e^{Mt}, & t \neq t_k, \\ (1 + N_k)V(t), & t = t_k, \end{cases}$$

and

$$\tilde{W}(t) = \begin{cases} W(t)e^{Mt}, & t \neq t_k, \\ (1 + N_k)W(t), & t = t_k, \end{cases}$$

are lower and upper solutions. Here we have assumed that $D_H\tilde{U}$ *exists.*

(3) If $F(t, X) \equiv 0$, $I_k(X) \equiv 0$, $k = 1, 2, \cdots$ in Theorem 5.3.1, then we obtain the result for $G(t, Y)$ nonincreasing in Y for each t and $J_k(Y)$ nonincreasing in Y, for each $k = 1, 2, \cdots$.

(4) If in (3) above, G and J_k, for each k, are not monotone but

(i) there exists a function $\tilde{G}(t, Y)$ which is nonincreasing in Y for each $t \in J$, and a constant $M > 0$ such that $G(t, Y) = MY + \tilde{G}(t, Y)$, that is $\tilde{G}(t, Y) = G(t, Y) - MY$, and

(ii) there exists a function $\tilde{J}_k(Y)$ which is nonincreasing in Y for each k and a constant $N_k > 0$, with $0 < N_k < 1$, for each k such that $J_k(Y) = N_k Y + \tilde{J}_k(Y)$

Then using the transformation

$$U(t) = \begin{cases} \tilde{U}(t)e^{Mt}, & t \neq t_k, \\ \frac{1}{1-N_k}\tilde{U}(t), & t = t_k, \end{cases}$$

we obtain

$$\begin{aligned} D_H\tilde{U} &= G_0(t, \tilde{U}), & t \neq t_k, \\ \tilde{U}(t_k^+) &= \overset{\circ}{J}_k[\tilde{U}(t_k)], & t = t_k, \\ \tilde{U}(0) &= U_0, \end{aligned} \tag{5.3.21}$$

where $G_0[t, \tilde{U}] = \tilde{G}(t, \tilde{U}e^{Mt})e^{-Mt}$ and $\overset{\circ}{J}_k[\tilde{U}(t_k)] = [1 - N_k][\tilde{J}_k[\frac{1}{1-N_k}\tilde{U}(t_k)] + N_k\tilde{U}(t_k)]$.

In this case we need to assume that (5.3.21) has coupled lower and upper solutions of type I, to get the same conclusion as in (3).

(5) Suppose that in Theorem 5.3.1, $G(t, Y)$ is nonincreasing in Y and $F(t, X)$ is not monotone but $\tilde{F}(t, X) = F(t, X) + MX$, $M > 0$ is nondecreasing in X. Further, suppose that $J_k(Y)$ is nonincreasing in Y, for each k and $I_k(U)$ is not monotone but $\tilde{I}_k(U) = I_k(U) + N_kU$, $N_k > 0$, is nondecreasing in U. Then, consider the IVP of ISDE

$$\begin{aligned} D_HU + MU &= \tilde{F}(t, U) + G(t, U), & t \neq t_k, \\ U(t_k^+) + N_kU(t_k) &= \tilde{I}_k(U(t_k)) + J_k(U(t_k)), & t = t_k, \\ U(0) &= U_0, \end{aligned} \tag{5.3.22}$$

In this case also, we obtain the same conclusion of Theorem 5.3.1, by utilizing the transformation used in (2).

(6) If $F(t, X)$ is nondecreasing in X, and I_k is nondecreasing for each k. But $G(t, Y)$ is not monotone in Y for each $t \in J$, and J_k is not nonincreasing for

each k. Then we assume there exist functions $\tilde{G}(t, Y)$ and $\tilde{J}_k(Y)$, and constants $M, N_k > 0$ as in (4). Now, we consider the IVP

$$\begin{aligned} D_H U &= F(t,U) + \tilde{G}(t,U) + MU, && t \neq t_k, \\ U(t_k^+) &= I_k(U) + \tilde{J}_k(U) + N_k U, && t = t_k, \\ U(0) &= U_0. \end{aligned} \tag{5.3.23}$$

Then using the transformation

$$U(t) = \begin{cases} \tilde{U}(t) e^{Mt}, & t \neq t_k, \\ \frac{1}{1-N_k} \tilde{U}(t), & t = t_k, \end{cases}$$

we get

$$\begin{aligned} D_H \tilde{U} &= F_0(t, \tilde{U}) + G_0(t, \tilde{U}), && t \neq t_k, \\ \tilde{U}(t_k^+) &= \mathring{I}_k(\tilde{U}(t_k)) + \mathring{J}_k(\tilde{U}(t_k)), && t = t_k, \\ \tilde{U}(0) &= U_0, \end{aligned} \tag{5.3.24}$$

where for $t \neq t_k$, $F_0[t, \tilde{U}] = F(t, \tilde{U} e^{Mt}) e^{-Mt}$
and $G_0[t, \tilde{U}] = \tilde{G}(t, \tilde{U} e^{Mt}) e^{-Mt}$

$\mathring{J}_k[\tilde{U}(t_k)] = [1 - N_k][\tilde{J}_k[\frac{1}{1-N_k}\tilde{U}(t_k)] + N_k \tilde{U}(t_k)$,

and$\mathring{I}_k[\tilde{U}(t_k)] = I_k[\frac{1}{1-N_k}\tilde{U}(t_k)]$.

If we assume that the system (5.3.24) has coupled lower and upper solutions of type I, then we get by Theorem 5.3.1 the same conclusion.

(7) If both F and G are not monotone and also I_k, J_k for each k are not monotone in Theorem 5.3.1, then we suppose

(i) there exist functions $\tilde{F}(t, U), \tilde{G}(t, U)$ and a constant $M > 0$ such that

$$\tilde{F}(t,U) + \tilde{G}(t,U) = F(t,U) + G(t,U) + MU,$$

exists, and $\tilde{F}(t, U)$ is nondecreasing in U and $\tilde{G}(t, U)$ is nonincreasing in U.

(ii) there exist functions $\tilde{I}_k$ and $\tilde{J}_k$ and $N_k U$, with $0 < N_k < 1$ for each k such that

$$\tilde{I}_k(U) + \tilde{J}_k(U) = I_k(U) + J_k(U) + N_k U,$$

where $\tilde{I}_k$ is nondecreasing in U and $\tilde{J}_k$ is nonincreasing in U for each k.

Now using the transformation

$$U(t) = \begin{cases} \tilde{U}(t) e^{Mt}, & t \neq t_k, \\ \frac{1}{1-N_k} \tilde{U}(t), & t = t_k, \end{cases}$$

we get,

$$\begin{aligned} D_H\tilde{U} &= F_0(t,\tilde{U}) + G_0(t,\tilde{U}), \qquad t \neq t_k,\\ \tilde{U}(t_k^+) &= \overset{\circ}{J}_k(\tilde{U}(t_k)), \qquad\qquad t = t_k,\\ \tilde{U}(0) &= U_0. \end{aligned}$$

Assuming that the above ISDE has coupled lower and upper solutions of type I, we conclude Theorem 5.3.1.

Next, we try to utilize the coupled lower and upper solution of type II in our study. In this case, we need not assume the existence of coupled lower and upper solutions of type II of (5.3.1) as we can construct them under the given assumptions. But this leads to assumptions on second iterates. Further, we get complicated alternative sequences which are monotone.

Theorem 5.3.2 *Assume that* ($A2$) *and* ($A3$) *of Theorem* 5.3.1 *hold. Then for any solution $U(t)$ of (5.3.1) with $V_0 \leq U \leq W_0$, $t \geq 0$, we have the iterates $\{V_n\}$, $\{W_n\}$ satisfying*

$$V_0 \leq V_2 \leq \cdots \leq V_{2n} \leq U \leq V_{2n+1} \leq \cdots \leq V_3 \leq V_1 \ \ on\ \mathbb{R}_+ \tag{5.3.25}$$

$$and \quad W_1 \leq W_3 \leq \cdots \leq W_{2n+1} \leq U \leq W_{2n} \leq \cdots \leq W_2 \leq W_0 \ \ on\ \mathbb{R}_+, \tag{5.3.26}$$

provided $V_0 \leq V_2$, $W_2 \leq W_0$ on J, where the iterative schemes are given by

$$\begin{aligned} D_H V_{n+1} &= F(t,W_n) + G(t,V_n), \qquad t \neq t_k,\\ V_{n+1}(t_k^+) &= I_k(W_n(t_k)) + J_k(V_n(t_k)), \ \ t = t_k,\\ V_{n+1}(0) &= 0, \end{aligned} \tag{5.3.27}$$

and

$$\begin{aligned} D_H W_{n+1} &= F(t,V_n) + G(t,W_n), \qquad t \neq t_k,\\ W_{n+1}(t_k^+) &= I_k(V_n(t_k)) + J_k(W_n(t_k)), \ \ t = t_k,\\ W_{n+1}(0) &= U_0 \quad on\ \ J. \end{aligned} \tag{5.3.28}$$

Moreover, the monotone sequences $\{V_{2n}\}$, $\{V_{2n+1}\}$, $\{W_{2n}\}$, $\{W_{2n+1}\} in K_c(\mathbb{R}^n)$ converge to $\rho, R, \rho*, R^*$ in $K_c(\mathbb{R}^n)$ respectively and verify,

$$\begin{cases} D_H R = F(t,R^*) + G(t,\rho), \qquad t \neq t_k,\\ R(t_k^+) = I_k(R^*(t_k)) + J_k(\rho(t_k)), \ \ t = t_k,\\ R(0) = U_0; \end{cases}$$

$$\begin{cases} D_H \rho = F(t,\rho^*) + G(t,R), \qquad t \neq t_k,\\ \rho(t_k^+) = I_k(\rho^*(t_k)) + J_k(R(t_k)), \ \ t = t_k,\\ \rho(0) = U_0; \end{cases}$$

$$\begin{cases} D_H R^* = F(t,R) + G(t,\rho^*), & t \neq t_k, \\ R^*(t_k^+) = I_k(R(t_k)) + J_k(\rho^*(t_k)), & t = t_k, \\ R^*(0) = U_0; \end{cases}$$

and

$$\begin{cases} D_H \rho^* = F(t,\rho) + G(t,R^*), & t \neq t_k, \\ \rho^*(t_k^+) = I_k(\rho(t_k)) + J_k(R^*(t_k)), & t = t_k, \\ \rho^*(0) = U_0, \end{cases}$$

respectively on J.

Proof First, we prove the existence of coupled lower and upper solutions V_0, W_0 of type II, satisfying $V_0(t) \leq W_0(t),\ t \in J$.

To achieve this, consider,

$$\begin{aligned} D_H Z &= F(t,\theta) + G(t,\theta), & t \neq t_k, \\ Z(t_k^+) &= I_k(\theta) + J_k(\theta), & t = t_k, \\ Z(0) &= U_0; \end{aligned}$$

Let $Z(t)$ be the unique solution which exists on J. Define V_0 and W_0 by

$$R_0 + V_0 = Z \quad \text{and} \quad W_0 = Z + R_0,$$

where the positive vector $R_0 = (R_{01}, \cdots, R_{0n})$ is chosen sufficiently large so that we have $V_0 \leq \theta \leq W_0$ on J.

Next, using the monotone character of F, G, I_k and J_k, for each k, we get for $t \neq t_k$

$$\begin{aligned} D_H V_0 &= D_H Z \\ &= F(t,\theta) + G(t,\theta) \\ &\leq F(t,W_0) + G(t,V_0), \\ \text{and} V_0(t_k^+) &\leq Z(t_k^+), \\ &= I_k(\theta) + J_k(\theta), \\ &\leq I_k(W_0(t_k)) + J_k(V_0(t_k)), \end{aligned}$$

and $V_0(0) \leq U_0$.

Similarly,

$$\begin{aligned} D_H W_0 &\geq F(t,V_0) + G(t,W_0), & t \neq t_k, \\ W_0(t_k^+) &\geq I_k(V_0(t_k)) + J_k(W_0(t_k)), & t = t_k, \\ W_0(0) &\geq U_0, \end{aligned}$$

Thus V_0 and W_0 are coupled lower and upper solutions of type II of (5.3.1).

Let $U(t)$ be any solution of (5.3.1) such that $V_0 \leq U \leq W_0$ on J. We prove that,

$$V_0 \leq V_2 \leq U \leq V_3 \leq V_1 \quad \text{and} \quad W_1 \leq W_3 \leq U \leq W_2 \leq W_0 \quad \text{on} \quad J. \tag{5.3.29}$$

The monotonicity of F and G, I_k and J_k along with the facts $V_0 \le U \le W_0$ and U is a solution of (5.3.1) gives,

$$\begin{aligned} D_H U &= F(t,U) + G(t,U) \le F(t,W_0) + G(t,V_0), && t \ne t_k \\ U(t_k^+) &= I_k(U(t_k)) + J_k(U(t_k)) \le I_k(W_0(t_k)) + J_k(V_0(t_k)), && t = t_k \\ U(0) &\le U_0. \end{aligned}$$

The relations (5.3.27) for $n = 0$ are

$$\begin{aligned} D_H V_1 &= F(t,W_0) + G(t,V_0), && t \ne t_k, \\ V_1(t_k^+) &= I_k(W_0(t_k)) + J_k(V_0(t_k)), && t = t_k, \\ V_1(0) &= U_0. \end{aligned}$$

On using Corollary 5.2.1, for U and V_1, we get,

$$U \le V_1 \text{ on } J.$$

Again, setting $n = 1$ in (5.3.27), we get

$$\begin{aligned} D_H V_2 &= F(t,W_1) + G(t,V_1), \\ &\le F(t,U) + G(t,U), && t \ne t_k, \\ V_2(t_k^+) &= I_k(W_1(t_k)) + J_k(V_1(t_k)), \\ &\le I_k(U(t_k)) + J_k(U(t_k)), && t = t_k, \\ V_2(0) &= U_0. \end{aligned}$$

Now, since U is a solution of (5.3.1), the above differential inequalities of V_2, along with Corollary 5.2.1, imply $V_2 \le U$ on J. Thus we have $V_2 \le U \le V_1$ on J.

Similarly , we can show that $U \le W_2$ on J.

Further, using the fact, $V_0 \le V_2$ and $W_2 \le W_0$ on J along with the properties of F, G, I_k and J_k for each k, we have,

$$\begin{aligned} D_H V_3 &= F(t,W_2) + G(t,V_2) \le F(t,W_0) + G(t,V_0), && t \ne t_k, \\ V_3(t_k^+) &= I_k(W_2(t_k)) + J_k(V_2(t_k)) \le I_k(W_0(t_k)) + J_k(V_0(t_k)), && t = t_k, \\ V_3(0) &= U_0. \end{aligned}$$

Now considering equations (5.3.27) with $n = 0$ and using the Corollary 5.2.1, we get $V_3 \le V_1$ on J. In a similar fashion, we can show that $W_1 \le W_3$ on J. Also, we get $U \le V_3$ and $W_3 \le U$ on J, thus proving the relations (5.3.29).

Now assume for some $n > 2$, the inequalities,

$$V_{2n-4} \le V_{2n-2} \le U \le V_{2n-1} \le V_{2n-3}, \tag{5.3.30}$$

$$W_{2n-3} \le W_{2n-1} \le U \le W_{2n-2} \le W_{2n-4}, \tag{5.3.31}$$

hold on J. We claim that

$$V_{2n-2} \le V_{2n} \le U \le V_{2n+1} \le V_{2n-1}, \tag{5.3.32}$$

$$W_{2n-1} \le W_{2n+1} \le U \le W_{2n} \le W_{2n-2}, \text{ on } J. \tag{5.3.33}$$

Taking $n = 2n-1$ in the equations (5.3.27) and using the relations (5.3.30), (5.3.31) and monotone character of F, G, I_k and J_k for each k, yields

$$\begin{aligned} D_H V_{2n} &= F(t, W_{2n-1}) + G(t, V_{2n-1}) \\ &\le F(t, U) + G(t, U), && t \ne t_k, \\ V_{2n}(t_k^+) &\le I_k(W_{2n-1}(t_k)) + J_k(V_{2n-1}(t_k)) \\ &\le I_k(U(t_k)) + J_k(U(t_k)), && t = t_k, \\ V_{2n}(0) &\le U_0. \end{aligned}$$

Since U is a solution of (5.3.1), U satisfies the reverse inequalities.

Now arguing as in the proof of Theorem 2.5.1, we get $V_{2n} \le U$ on J.

We now show $W_{2n} \le W_{2n-2}$. Setting $n = 2n-1$ in relations (5.3.28), and using the hypothesis,

$$\begin{aligned} D_H W_{2n} &= F(t, V_{2n-1}) + G(t, W_{2n-1}) \\ &\le F(t, V_{2n-3}) + G(t, W_{2n-3}), && t \ne t_k, \\ W_{2n}(t_k^+) &= I_k(V_{2n-1}(t_k)) + J_k(W_{2n-1}(t_k)) \\ &\le I_k(V_{2n-3}(t_k)) + J_k(W_{2n-3}(t_k)), && t = t_k, \\ W_{2n}(0) &= U_0. \end{aligned}$$

Observing that by setting $n = 2n-3$ in (5.3.28), we get the equalities (reverse inequalities) of the above relations with W_{2n} replaced by W_{2n-2}. Now using Corollary 5.2.1, gives $W_{2n} \le W_{2n-2}$ on J.

The proofs of the remaining relations are an exact repetition of the above discussion. Hence we avoid them. Thus we conclude the relations (5.3.32) and (5.3.33). By induction on n, we have the conclusion of the relations (5.3.25) and (5.3.26).

Since $V_n, W_n \in PC^1[J, K_c(\mathbb{R}^n)]$ for all n, reasoning as in Theorem 5.3.1, we get

$$\lim_{n\to\infty} V_{2n} = \rho, \text{ and } \lim_{n\to\infty} V_{2n+1} = R,$$

$$\lim_{n\to\infty} W_{2n+1} = \rho^*, \text{ and } \lim_{n\to\infty} W_{2n} = R^*$$

exist over each subinterval $[t_k, t_{k+1}]$, and the convergence is uniform in each subinterval.

Further, for each $t = t_k$,

$$\lim_{n\to\infty} V_{2n}(t_k^+) = I_k(\lim_{n\to\infty} W_{2n-1}(t_k)) + J_k(\lim_{n\to\infty} V_{2n-1}(t_k))$$

which implies, $\rho(t_k^+) = I_k(\rho^*(t_k)) + J_k(R(t_k))$.

With a similar reasoning, we observe that the functions ρ^*, R, R^*, satisfy their corresponding impulse conditions, at $t = t_k$. Also, using the integral representation for the differential equations in (5.3.27) and (5.3.28) suitably, we obtain that ρ, ρ^*, R, R^* satisfy their corresponding impulsive set differential equations, given in the statement of the theorem.

Also from (5.3.25) and (5.3.26), we get

$$\rho \leq U \leq R \quad \text{and} \quad \rho^* \leq U \leq R^*, \quad \text{on } J.$$

Thus the proof is complete.

Corollary 5.3.2 *Assume that the hypothesis of the Theorem* 5.3.2 *hold. Further, suppose F, G, I_k and J_k satisfy the hypothesis in the corollary 5.3.1. Then $\rho = \rho^* = R = R^* = U$ is the unique solution of the ISDE (5.3.1).*

Proof Since $\rho \leq R$ and $\rho^* \leq R^*$, let $q_1 + \rho = R$ and $q_2 + \rho^* = R^*$. Then considering $D_H(q_1 + q_2)$ and using the hypothesis, we get

$$\begin{aligned} D_H(q_1 + q_2) &\leq (N_1 + N_2)(q_1 + q_2), \qquad t \neq t_k, \\ q_1(t_k^+) + q_2(t_k^+) &\leq (M_{1k} + M_{2k})(q_1 + q_2), \quad t = t_k, \\ \text{and} \qquad (q_1 + q_2)(0) &= 0. \end{aligned}$$

Using Theorem 1.4.1 in Lakshmikantham, Bainov, Simeonov [1] in this context, since $N_1 + N_2 \geq 0$ and $0 < M_{1k} + M_{2k} < 1$, we get

$$(q_1 + q_2)(t) \leq 0, \quad t \in J,$$

which means $R + R^* \leq \rho + \rho^* \leq R + R^*$,. This gives $U = \rho = R = \rho^* = R^*$, and hence the solution is unique.

Remark 5.3.2 *Corresponding to the Remark* 5.3.1*, we can make similar remark following from Theorem* 5.3.2*. To avoid monotony we do not list them.*

Remark 5.3.3 *The impulsive set differential equation* (5.3.1) *reduces to an ordinary impulsive differential equation if F, G, I_k, J_k are all single valued mappings. In this case, Theorem* 5.3.1 *and* 5.3.2 *along with the remarks give rise to many new results in the theory of impulsive differential equations.*

5.4 Set Differential Equations with Delay

In ordinary differential difference equations or, more generally, in differential equations with delay, the history exerts its influence in a significant way on the future of solutions. There are several applications in which future depends on the past history (finite or infinite). This area of differential equations with delay or usually known as functional differential equations is investigated as an independent subject and is very interesting.

In this section, we shall incorporate delay in the formulation of set differential equations and provide some basic results of interest. We start by describing the set differential equation with delay.

Given any $\tau > 0$, consider $\mathcal{C}_0 = C[[-\tau, 0], K_c(\mathbb{R}^n)]$. For any $\Phi,\ \Psi \in \mathcal{C}_0$, define the metric

$$D_0[\Phi, \Psi] = \max_{-\tau \le s \le 0} D[\Phi(s), \Psi(s)]$$

. Also, we write $\|\Phi\|_0 = D_0[\Phi, \theta]$.

Suppose that $J_0 = [t_0 - \tau, t_0 + a],\ a > 0$. Let $U \in C[J_0, K_c(\mathbb{R}^n)]$. For any $t \ge t_0,\ t \in J_0$, let U_t denote a translation of the restriction of U to the interval $[t - \tau, t]$. That is, $U_t \in \mathcal{C}_0$ is defined by $U_t(s) = U(t + s),\ -\tau \le s \le 0$.

Consider the set differential equation with finite delay given by

$$D_H U = F(t, U_t), \quad U_{t_0} = \Phi_0 \ \in \mathcal{C}_0 \tag{5.4.1}$$

where $F \in C[J \times \mathcal{C}_0, K_c(\mathbb{R}^n)]$ and $J = [t_0, t_0 + a]$.

The following existence result is obtained using the contraction principle.

Theorem 5.4.1 *Assume that*

$$D[F(t, \Phi), F(t, \Psi)] \le K D_0[\Phi, \Psi], \quad K > 0 \tag{5.4.2}$$

for $t \in J, \quad \Phi, \Psi \in \mathcal{C}_0$. Then the IVP (5.4.1) possesses a unique solution $U(t)$ on J_0.

Proof Consider the set of functions $U \in C[J_0, K_c(\mathbb{R}^n)]$ such that $U(t) = \Phi_0(t),\ t_0 - \tau \le t \le t_0$ and $U \in C[J, K_c(\mathbb{R}^n)]$ with $U(t_0) = \Phi_0(0)$ with $\Phi_0(t) \in K_c(\mathbb{R}^n),\ -\tau \le t \le 0$.

Define the metric on $C[J_0, K_c(\mathbb{R}^n)]$ by

$$D_1(U, V) = \max_{t_0 - \tau \le t \le t_0 + a} D[U(t), V(t)] e^{-\lambda t}, \quad \lambda > 0, \text{is chosen suitably later.} \tag{5.4.3}$$

Next, define the operator T on $C[J_0, K_c(\mathbb{R}^n)]$ by

$$\begin{aligned} TU(t) &= \Phi_0(t), \quad t_0 - \tau \le t \le t_0 \\ TU(t) &= \Phi_0(0) + \int_{t_0}^{t} F(s, U_s)\, ds, \quad t \in J. \end{aligned} \tag{5.4.4}$$

Then, for $-\tau \le s \le 0$,

$$\begin{aligned} &D[TU(t_0 + s), TV(t_0 + s)], \\ &= D[\Phi_0(t_0 + s), \Phi_0(t_0 + s)], \\ &= 0\ . \end{aligned}$$

We get, using the properties of Hausdorff metric (1.3.9) and (1.7.11), for

$t \in J$,

$$\begin{aligned}
&D[TU(t), TV(t)] \\
&= D[\Phi_0(0) + \int_{t_0}^{t} F(\xi, U_\xi)\, d\xi, \Phi_0(0) + \int_{t_0}^{t} F(\xi, V_\xi)\, d\xi] \\
&= D[\int_{t_0}^{t} F(\xi, U_\xi)\, d\xi, \int_{t_0}^{t} F(\xi, V_\xi)\, d\xi] \\
&\leq \int_{t_0}^{t} D[F(\xi, U_\xi), F(\xi, V_\xi)]\, d\xi \\
&\leq K \int_{t_0}^{t} D_0[(U_\xi, V_\xi)]\, d\xi.
\end{aligned}$$

Consider

$$\begin{aligned}
\int_{t_0}^{t} D[U(\xi+s), V(\xi+s)]\, d\xi &\leq \int_{t_0-\tau}^{t} D[U(\sigma), V(\sigma)]\, d\sigma \\
&\leq \int_{t_0-\tau}^{t} \max_{t_0-\tau \leq \sigma \leq t_0+a} [D[U(\sigma), V(\sigma)] e^{-\lambda\sigma}]\, e^{\lambda\sigma}\, d\sigma \\
&= D_1[U, V] \int_{t_0-\tau}^{t} e^{\lambda\sigma} d\sigma \\
&= D_1[U, V] \frac{1}{\lambda} [e^{\lambda t} - e^{\lambda(t_0-\tau)}] \\
&\leq \frac{1}{\lambda} D_1[U, V] e^{\lambda t}.
\end{aligned}$$

Thus we obtain

$$K \int_{t_0}^{t} D_0[U_\xi, V_\xi]\, d\xi \leq \frac{K}{\lambda} D_1[U, V]\, e^{\lambda t},$$

which implies that, on J_0,

$$e^{-\lambda t}\, D[TU(t), TV(t)] \leq \frac{K}{\lambda} D_1[U, V].$$

Choosing $\lambda = 2K$ and taking maximum over t, on J_0, we have

$$D_1[TU, TV] \leq \frac{1}{2} D_1[U, V],$$

which means that the operator T on $C[J_0, K_c(\mathbb{R}^n)]$ is a contraction. Thus there exists a unique fixed point $U \in C[J_0, K_c(\mathbb{R}^n)]$ of T by the contraction principle. Hence $U(t) = U(t_0, \Phi_0)(t)$ is the unique solution of the IVP (5.4.1).

We now prove a comparison theorem in this context, which is a useful tool in proving the global existence theorem.

Theorem 5.4.2 *Assume that* $F \in C[\mathbb{R}_+ \times \mathcal{C}_0, K_c(\mathbb{R}^n)]$ *and* $D[F(t,\Phi), F(t,\Psi)] \le g(t, D_0[\Phi,\Psi])$ *for* $t \in \mathbb{R}_+$, $\Phi_0, \Psi_0 \in \mathcal{C}_0$, *where* $g \in C[\mathbb{R}_+^2, \mathbb{R}_+]$. *Let* $r(t) = r(t, t_0, w_0)$ *be the maximal solution of*

$$\begin{aligned} w' &= g(t,w) \\ w(t_0) &= w_0 \ge 0 \quad \textit{existing for } t \ge t_0. \end{aligned}$$

Then $D_0[\Phi_0, \Psi_0] \le w_0$ *implies*

$$D[U(t), V(t)] \le r(t), \quad t \ge t_0,$$

where $U(t) = U(t_0, \Phi_0)(t)$ *and* $V(t) = V(t_0, \Psi_0)(t)$ *are the solutions of (5.4.1).*

Proof Since $U(t), V(t)$ are solutions of (5.4.1) for small $h > 0$, the differences $U(t+h) - U(t), V(t+h) - V(t)$ exist. Now for $t \in \mathbb{R}_+$, set $m(t) = D[U(t), V(t)]$. Then using the properties of Hausdorff metric, (1.3.8) and (1.3.9), we have

$$\begin{aligned} m(t+h) - m(t) &= D\left[U(t+h), V(t+h)\right] - D[U(t), V(t)] \\ \le\ & D[U(t+h), U(t) + hF(t, U_t)] + D[U(t) + hF(t,U_t), V(t) + hF(t, V_t)] \\ & + D[V(t) + hF(t, V_t), V(t+h)] - D[U(t), V(t)] \\ \le\ & D[U(t+h), U(t) + hF(t,U_t)] + D[V(t) + hF(t,V_t), V(t+h)] \\ & + hD[F(t,U_t), F(t,V_t)] \quad , \end{aligned}$$

from which we get, using (1.3.8) and (1.3.9) again,

$$\begin{aligned} \frac{m(t+h) - m(t)}{h} &\le D\left[\frac{U(t+h) - U(t)}{h}, F(t, U_t)\right] \\ &\quad + D\left[F(t, V_t), \frac{V(t+h) - V(t)}{h}\right] + D[F(t,U_t), F(t,V_t)]. \end{aligned}$$

Taking limit supremum as $h \to 0^+$, gives,

$$\begin{aligned} D^+ m(t) &= \limsup_{h \to 0^+} \frac{m(t+h) - m(t)}{h} \\ &\le D[F(t,U_t), F(t,V_t)] \\ &\le g(t, D_0[U_t, V_t]) = g(t, |m_t|_0). \end{aligned}$$

The above inequality, along with the fact that $|m_{t_0}|_0 = D_0[\Phi_0, \Psi_0] \le w_0$, implies from the comparison theorem for ordinary delay differential equations (Lakshmikantham and Leela [1]), that

$$D[U(t), V(t)] \le r(t), \qquad t \ge t_0.$$

We are now ready to prove the global existence theorem.

Theorem 5.4.3 *Let $F \in C[\mathbb{R}_+ \times \mathcal{C}_0, K_c(\mathbb{R}^n)]$ and for $(t, \Phi) \in \mathbb{R}_+ \times \mathcal{C}_0$,*

$$D[F(t, \Phi), \theta] \leq g(t, D_0[\Phi, \theta]),$$

where $g \in C[\mathbb{R}_+^2, \mathbb{R}_+]$, $g(t, w)$ is nondecreasing in w for each $t \in \mathbb{R}_+$. Assume that the solutions $w(t, t_0, w_0)$ of $w' = g(t, w)$, $w(t_0) = w_0$ exist for $t \geq t_0$, and F is smooth enough to assure local existence. Then the largest interval of existence of any solution $U(t_0, \Phi_0)(t)$ of (5.4.1) is $[t_0, \infty)$.

Proof Suppose that $U(t_0, \Phi_0)(t)$ is a solution of (5.4.1) existing on some interval $[t_0 - \tau, \beta)$, where $t_0 < \beta < \infty$. Assume that β cannot be increased. Define for $t \in [t_0 - \tau, \beta)$,

$$m(t) = D[U(t_0, \Phi_0)(t), \theta]$$

$$m_t = D[U_t(t_0, \Phi_0), \theta] \quad \text{and} \quad |m_t|_0 = D_0[U_t(t_0, \Phi_0), \theta]$$

Then reasoning and proceeding as in the comparison theorem, we get the differential inequality

$$D^+ m(t) \leq g(t, |m_t|_0), \quad t_0 \leq t < \beta.$$

Choosing $|m_{t_0}|_0 = D_0[\Phi_0, \theta] \leq w_0$, we arrive at

$$D[U(t_0, \Phi_0)(t), \theta] \leq r(t, t_0, w_0), \quad t_0 \leq t < \beta.$$

Now $g(t, w) \geq 0$ implies that $r(t, t_0, w_0)$ is nondecreasing in t, which further yields

$$D[U_t(t_0, \Phi_0), \theta] \leq r(t, t_0, w_0), \quad t_0 \leq t < \beta. \tag{5.4.5}$$

Consider t_1, t_2 such that $t_0 < t_1 < t_2 < \beta$, then using (1.3.8) and (1.7.11), we get

$$\begin{aligned} &D[U(t_0, \Phi_0)(t_1), U(t_0, \Phi_0)(t_2)] \\ &= D[U(t_0, \Phi_0)(t_1), U(t_0, \Phi_0)(t_1) + \int_{t_1}^{t_2} F(s, U_s)\, ds] \\ &= D[\theta, \int_{t_1}^{t_2} F(s, U_s)\, ds] \\ &\leq \int_{t_1}^{t_2} D[F(s, U_s), \theta]\, ds \\ &\leq \int_{t_1}^{t_2} g(s, D_0[U_s(t_0, \Phi_0), \theta])\, ds. \end{aligned}$$

Next, using the fact that g is monotonically nondecreasing in w and the relation (5.4.5) in the above inequality, we obtain

$$\begin{aligned} D[U(t_0, \Phi_0)(t_1), U(t_0, \Phi_0)(t_2)] &\leq \int_{t_1}^{t_2} g(s, r(s, t_0, w_0))\, ds \\ &= r(t_2, t_0, w_0) - r(t_1, t_0, w_0). \end{aligned} \tag{5.4.6}$$

If we let $t_1, t_2 \to \beta$ in the above relation (5.4.6), then $\lim_{t\to\beta^-} U(t_0,\Phi_0)(t)$ exists, because of Cauchy's criterion for convergence.

We now define $U(t_0,\Phi_0)(\beta) = \lim_{t\to\beta^-} U(t_0,\Phi_0)(t)$ and consider $\Psi_0 = U_\beta(t_0,\Phi_0)$

as the new initial function at $t = \beta$. The assumption of local existence implies that there exists a solution $U(\beta,\Psi_0)(t)$ of (5.4.1) on $[\beta-\tau, \beta+\alpha]$, $\alpha > 0$. This means that the solution $U(t_0,\Phi_0)(t)$ can be continued beyond β, which is contrary to our assumption that the value of β cannot be increased. Hence the theorem.

Next, we present a result on nonuniform practical stability of (5.4.1) using perturbing Lyapunov functions. See (Lakshmikantham Leela and Martynyuk [1]) for details.

Before proceeding further, we need the following classes of functions

$$\mathcal{K} = \{a \in C[[0,A], \mathbb{R}_+], a(0) = 0 \text{ and } a(u) \text{ is strictly increasing}\}$$

$$C\mathcal{K} = \{\sigma \in C[\mathbb{R}_+ \times [0,A], \mathbb{R}_+] : \sigma(t,.) \in \mathcal{K} \text{ for each } t \in \mathbb{R}_+\}$$

We now define practical stability in this context.

Definition 5.4.1 *The system (5.4.1) is practically stable, given* (λ, A) *with* $0 < \lambda < A$, *we have* $D_0[\Phi_0, \theta] < \lambda$ *implies* $D[U(t), \theta] < A$, $t \geq t_0$ *for some* $t_0 \in \mathbb{R}_+$.

We set

$$S(A) = \{\, U \in K_c(\mathbb{R}^n) : D[U,\theta] < A\}$$

and

$$\Omega(A) = \{\Phi \in \mathcal{C}_0 : \;\; D_0[\Phi,\theta] < A\}.$$

We are now in a position to prove the following result on practical stability.

Theorem 5.4.4 *Assume that* (*i*) $0 < \lambda < A$;
(*ii*) $V_1 \in C[\, \mathbb{R}_+ \times S(A) \times \Omega(A), \mathbb{R}_+]$,
for $(t, U_1, \Phi), (t, U_2, \Phi) \in \mathbb{R}_+ \times S(A) \times \Omega(A)$

$$|V_1(t,U_1,\Phi) - V_1(t,U_2,\Phi)| \leq L_1 \; D[U_1,U_2], \;\; L_1 > 0;$$

for each $(t,U,\Phi) \in \mathbb{R}_+ \times S(A) \times \Omega(A)$,

$$V_1(t,U,\Phi) \leq a_1(t, D_0[\Phi,\theta]), \; a_1 \in C\mathcal{K},$$

and

$$\begin{aligned} D^+V_1(t,U,\Phi) &\equiv \limsup_{h\to 0^+} \frac{1}{h}[V_1(t+h, U+hF(t,U_t), U_{t+h}) - V_1(t,U,U_t)] \\ &\leq g_1(t, V_1(t,U,\Phi)), \end{aligned}$$

where $g_1 \in C[\mathbb{R}_+^2, \mathbb{R}]$;
(*iii*) $V_2 \in C[\, \mathbb{R}_+ \times S(A) \times \Omega(A), \mathbb{R}_+]$,
for $(t, U_1, \Phi), (t, U_2, \Phi) \in \mathbb{R}_+ \times S(A) \times \Omega(A)$,

$$|V_2(t,U_1,\Phi) - V_2(t,U_2,\Phi)| \leq L_2 \; D[U_1,U_2], \; L_2 > 0;$$

for each $(t, U, \Phi) \in \mathbb{R}_+ \times S(A) \times \Omega(A)$,

$$b(D[U,\theta]) \le V_2(t,U,\Phi) \le a_2(D_0[\Phi,\theta]),$$
$$D^+V_1(t,U,\Phi) + D^+V_2(t,U,\Phi) \le g_2(t, V_1(t,U,\Phi) + V_2(t,U,\Phi)),$$

where $a_2, b \in \mathcal{K}$ *and* $g_2 \in C[\mathbb{R}_+^2, \mathbb{R}]$*;*
(iv) $a_1(t_0,\lambda) + a_2(\lambda) < b(A)$ *for some* $t_0 \in \mathbb{R}_+$*;*
(v) $u_0 < a_1(t_0,\lambda)$ *implies* $u(t,t_0,u_0) < a_1(t_0,\lambda)$ *for* $t \ge t_0$ *where* $u(t,t_0,u_0)$ *is any solution of*

$$u' = g_1(t,u), \quad u(t_0) = u_0, \tag{5.4.7}$$

and $v_0 < a_1(t_0,\lambda) + a_2(\lambda)$ *implies*

$$v(t,t_0,v_0) < b(A), \quad t \ge t_0,$$

for every $t_0 \in \mathbb{R}_+$*, where* $v(t,t_0,v_0)$ *is any solution of*

$$v' = g_2(t,v), \quad v(t_0) = v_0 \ge 0. \tag{5.4.8}$$

Then the system (5.4.1) is practically stable.

Proof We have to prove that given $0 < \lambda < A$, $D_0[\Phi,\theta] < \lambda$ then $D[U(t),\theta] < A$ where $U(t) = U(t_0,\Phi_0)(t)$ is any solution of (5.4.1), for $t \ge t_0$. Suppose it is not true, then there exists $t_2 > t_1 > t_0$ and a solution $U(t_0,\Phi_0)(t)$ of (5.4.1) such that

$$D[U(t_1),\theta] = \lambda \quad \text{and} \quad D[U(t_2),\theta] = A \tag{5.4.9}$$

and $\lambda \le D[U(t),\theta] \le A$, for $t_1 \le t \le t_2$. Now using the hypothesis *(iii)* and *(v)* and the standard arguments, we get

$$\begin{aligned} V_1(t, U(t_0,\Phi_0)(t), U_t(t_0,\Phi_0)) &+ V_2(t, U(t_0,\Phi_0)(t), U_t(t_0,\Phi_0)) \\ &\le r_2(t, t_1, V_1(t, U(t_0,\Phi_0)(t_1), U_{t_1}(t_0,\Phi_0)) \\ &\quad + V_2(t_1, U(t_0,\Phi_0)(t_1), U_{t_1}(t_0,\Phi_0))) \end{aligned} \tag{5.4.10}$$

for $t_1 \le t \le t_2$, where $r_2(t,t_1,v_0)$ is the maximal solution of (5.4.8) through (t_1,v_0), and $v_0 = V_1(t_1, U(t_0,\Phi_0)(t_1), U_{t_1}(t_0,\Phi_0)) + V_2(t_1, U(t_0,\Phi_0)(t_1), U_{t_1}(t_0,\Phi_0))$.

Similarly, condition *(ii)* gives the estimate

$$V_1(t, U(t_0,\Phi_0)(t), U_t(t_0,\Phi_0)) \le r_1(t, t_0, V_1(t_0, \Phi_0(0), \Phi_0)), \quad t_0 \le t \le t_1,$$

where $r_1(t,t_0,u_0)$ is the maximal solution of (5.4.7) with $u_0 = V_1(t_0,\Phi_0(0),\Phi_0)$.

Since $D_0[\Phi_0,\theta] < \lambda$, using hypothesis *(ii)*

$$V_1(t_0,\Phi_0(0),\Phi_0) \le a_1(t_0, D_0[\Phi_0,\theta]) \le a_1(t_0,\lambda).$$

Also, we have

$$V_2(t_1, U(t_0,\Phi_0)(t_1), U_{t_1}(t_0,\Phi_0)) \le a_2(D_0[U_{t_1},\theta]) \le a_2(\lambda).$$

Thus we get

$$\begin{aligned}&V_1(t_1, U(t_0,\Phi_0)(t_1), U_{t_1}(t_0,\Phi_0)) + V_2(t_1, U(t_0,\Phi_0)(t_1), U_{t_1}(t_0,\Phi_0))\\ &\quad \le a_1(t_0,\lambda) + a_2(\lambda).\end{aligned}$$

Now using the relation (5.4.10) and the hypothesis (v), we obtain

$$\begin{aligned}&V_1(t_2, U(t_0,\Phi_0)(t_2), U_{t_2}(t_0,\Phi_0)) + V_2(t_2, U(t_0,\Phi_0)(t_2), U_{t_2}(t_0,\Phi_0))\\ &\quad \le r_2(t_2, t_1, a_1(t_0,\lambda) + a_2(\lambda)) \ < b(A).\end{aligned} \tag{5.4.11}$$

However, using the relation (5.4.9) and the hypothesis (ii) and (iii), we get

$$\begin{aligned}&V_1(t_2, U(t_0,\Phi_0)(t_2), U_{t_2}(t_0,\Phi_0)) + V_2(t_2, U(t_0,\Phi_0)(t_2), U_{t_2}(t_0,\Phi_0))\\ &\quad \ge V_2(t_2, U(t_0,\Phi_0)(t_2), U_{t_2}(t_0,\Phi_0))\\ &\quad \ge b(D[U(t_2),\theta]) = b(A),\end{aligned}$$

which contradicts (5.4.11). Thus the proof of our claim.

We next study the nonuniform boundedness property for the system (5.4.1). We define the concept of boundedness as follows.

Definition 5.4.2 *The differential system (5.4.1) is said to be*

(1) Equibounded, if for any $\alpha > 0$ and $t_0 \in \mathbb{R}_+$, there exists a β where $\beta = \beta(t_0,\alpha) > 0$ such that

$$D_0[\Phi_0,\theta] < \alpha \ \textit{implies} \ D[U(t),\theta] < \beta, \ t \ge t_0,$$

where $U(t) = U(t_0,\Phi_0)(t)$ is any solution of (5.4.1).

(2) Uniform Bounded, if β in (1) *does not depend on t_0.*

The following theorem uses the method of perturbing Lyapunov functions to obtain nonuniform boundedness property.

For that purpose, set
$S(\rho) = \{U \in K_c(\mathbb{R}^n) : D[U,\theta] < \rho\}$ and $\widetilde{S}(\rho) = \{\Phi \in \mathcal{C} : D_0[\Phi,\theta] < \rho\}$.

Theorem 5.4.5 *Assume that*

(i) $\rho > 0, V_1 \in C[\mathbb{R}_+ \times S(\rho) \times \widetilde{S}(\rho), \mathbb{R}_+], V_1$ is bounded for $(t, U, \Phi) \in \mathbb{R}_+ \times \partial S(\rho) \times \partial\widetilde{S}(\rho)$;

$$|V_1(t, U_1, \Phi) - V_1(t, U_2, \Phi)| \le L_1 D[U_1, U_2], \quad L_1 > 0,$$

and for $(t, U, \Phi) \in \mathbb{R}_+ \times S^c(\rho) \times \widetilde{S}^c(\rho)$,

$$\begin{aligned}D^+V_1(t, U, \Phi) &\equiv \limsup_{h\to 0^+} \frac{1}{h}[V_1(t+h, U + hF(t,U_t), U_{t+h}) - V_1(t, U, U_t)]\\ &\le g_1(t, V_1(t, U, \Phi)),\end{aligned}$$

where $g_1 \in C[\mathbb{R}_+^2, \mathbb{R}]$;

(ii) $V_2 \in C[\mathbb{R}_+ \times S^c(\rho) \times \widetilde{S}^c(\rho), \mathbb{R}_+]$,

$$b(D[U,\theta]) \le V_2(t,U,\Phi) \le a(D_0[\Phi,\theta]),$$
$$D^+V_1(t,U,\Phi) + D^+V_2(t,U,\Phi) \le g_2(t, V_1(t,U,\Phi) + V_2(t,U,\Phi)),$$

where $a, b \in K$ *and* $g_2 \in C[\mathbb{R}_+^2, \mathbb{R}]$;

(iii) The scalar differential equations

$$w_1' = g_1(t, w_1), \quad w_1(t_0) = w_{10} \ge 0, \tag{5.4.12}$$

$$w_2' = g_2(t, w_2), \quad w_2(t_0) = w_{20} \ge 0, \tag{5.4.13}$$

are equibounded and uniformly bounded respectively.

Then the system (5.4.1) is equibounded.

Proof Let $B_1 > \rho$ and $t_0 \in \mathbb{R}_+$ be given. Let

$$\begin{aligned} \alpha_0 &= \max\{V_1(t_0,U_0,\Phi_0) : U_0 = \Phi_0(0) \in cl\{S(B_1) \cap S^c(\rho)\}, \\ &\qquad \Phi_0 \in cl\{\widetilde{S}(B_1) \cap \widetilde{S}^c(\rho)\}\} \\ \alpha^* &\ge V_1(t,U,\Phi) \text{ for } (t,U,\Phi) \in \mathbb{R}_+ \times \partial S(\rho) \times \partial\widetilde{S}(\rho), \end{aligned}$$

and set $\alpha_1 = \alpha_1(t_0, B_1) = \max(\alpha_0, \alpha^*)$.

Since the scalar differential equation (5.4.12) is equibounded, given $\alpha_1 > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\beta_0 = \beta_0(t_0, \alpha_1)$ such that

$$w_1(t, t_0, w_0) < \beta_0, \quad t \ge t_0\ , \tag{5.4.14}$$

provided $w_{10} < \alpha_1$, where $w_1(t, t_0, w_{10})$ is any solution of (5.4.12).

Let $\alpha_2 = a(B_1) + \beta_0$. Then the uniform boundedness of the equation (5.4.13) yields

$$w_2(t, t_0, w_{20}) < \beta_1(\alpha_2), \quad t \ge t_0 \tag{5.4.15}$$

provided $w_{20} < \alpha_2$, where $w_2(t, t_0, w_{20})$ is any solution of (5.4.13).

Choose B_2 satisfying

$$b(B_2) > \beta_1(\alpha_2). \tag{5.4.16}$$

We now claim that $\Phi_0 \in \widetilde{S}(B_1)$ implies that $U(t) \in S(B_2)$ for $t \ge t_0$, where $U(t) = U(t_0, \Phi_0)(t)$ is any solution of (5.4.1).

If it is not true, there exists a solution $U(t_0, \Phi_0)(t)$ of (5.4.1) with $\Phi_0 \in \widetilde{S}(B_1)$, such that, for some $t^* > t_0$, $D[U(t_0, \Phi_0)(t^*), \theta] = B_2$. Since $B_1 > \rho$, there are two possibilities to consider,

(i) $U(t_0, \Phi_0)(t) \in S^c(\rho)$ for $t \in [t_0, t^*]$,

(ii) there exists a $\bar{t} \ge t_0$ such that $U(t_0, \Phi_0)(\bar{t}) \in \partial S(\rho)$ and
$U(t_0, \Phi_0)(t) \in S^c(\rho)$ for $t \in [\bar{t}, t^*]$.

If (i) holds, we can find a $t_1 > t_0$ such that:

$$\begin{aligned} &U(t_0, \Phi_0)(t_1) \in \partial S(B_1), \\ &U(t_0, \Phi_0)(t^*) \in \partial S(B_2), \\ &U(t_0, \Phi_0)(t) \in S^c(B_1), \quad t \in [t_1, t^*]. \end{aligned} \tag{5.4.17}$$

Setting $m(t) = V_1(t, U(t_0, \Phi_0)(t), U_t(t_0, \Phi_0)) + V_2(t, U(t_0, \Phi_0)(t), U_t(t_0, \Phi_0))$ for $t \in [t_1, t^*]$ and using the standard arguments, we obtain the differential inequality

$$D^+ m(t) \le g_2(t, m(t)), \quad t \in [t, t^*] \; .$$

It then follows from the Comparison Theorem 1.4.1 of Lakshmikantham and Leela [1]

$$m(t) \le r_2(t, t_1, m(t_1)), \quad t \in [t_1, t^*]$$

where $r_2(t, t_1, w_{20})$ is the maximal solution of (5.4.13) with

$$\begin{aligned} r_2(t_1, t_1, w_{20}) = w_{20} = &V_1(t_1, U(t_0, \Phi_0)(t_1), U_{t_1}(t_0, \Phi_0)) \\ &+ V_2(t_1, U(t_0, \Phi_0)(t_1), U_{t_1}(t_0, \Phi_0)). \end{aligned}$$

Thus,

$$\begin{aligned} V_1(t^*, U(t_0, \Phi_0)(t^*), U_{t^*}(t, \Phi_0)) + V_2(t^*, U(t_0, \Phi_0)(t^*), U_{t^*}(t, \Phi_0)) \\ \le r_2(t^*, t_1, w_{20}). \end{aligned} \tag{5.4.18}$$

Similarly, we get

$$V_1(t, U(t_0, \Phi_0)(t), U_t(t_0, \Phi_0)) \le r_1(t_1, t_0, V_1(t_0, U(t_0, \Phi_0)(t_0), U_{t_0}(t_0, \Phi_0)) \tag{5.4.19}$$

where $r_1(t, t_0, u_0)$ is the maximal solution of (5.4.12) with

$$u_0 = V_1(t_0, U(t_0, \Phi_0)(t_0), U_{t_0}(t_0, \Phi_0)) = V_1(t_0, \Phi_0(0), \Phi_0).$$

Setting $w_{10} = V_1(t_0, \Phi_0(0), \Phi_0) < \alpha_1$, and using the relation (5.4.14) we get,

$$V_1(t_1, U(t_0, \Phi_0)(t_1), U_{t_1}(t_0, \Phi_0)) \le r_1(t_1, t_0, w_{10}) \le \beta_0.$$

Furthermore, $V_2(t_1, U(t_0, \Phi_0)(t_1), U_{t_1}(t_0, \Phi_0)) \le a(B_1)$ and we have $w_{20} \le \beta_0 + a(B_1) = \alpha_2$.

Now combining (5.4.15), (5.4.16),(5.4.17) we have

$$b(B_2) \le m(t^*) \le r(t^*) \le \beta_1(\alpha_2) < b(B_2), \tag{5.4.20}$$

which is a contradiction.

If case (ii) holds, we also come up with the inequality (5.4.18), where $t_1 > \overline{t}$ satisfies (5.4.17). We then obtain, in place of (5.4.19) the relation

$$V_1(t_1, U(t_0, \Phi_0)(t_1), \Phi_{t_1}) \le r_1(t_1, \overline{t}, V_1(\overline{t}, U(t_0, \Phi_0)(\overline{t})), \Phi_{\overline{t}}) \; ,$$

since $U(t_0, \Phi_0)(\bar{t}) \in \partial S(\rho)$ and

$$V_1(\bar{t}, U(t_0, \Phi_0)(\bar{t}), \Phi_{\bar{t}}) \leq \alpha^* \leq \alpha_1 \ ,$$

arguing as before, we get a contradiction.

This proves that, for any given $B_1 > \rho$, $t_0 > 0$ there exists a B_2 such that $\Phi_0 \in \widetilde{S}(B_1)$ implies $U(t_0, \Phi_0)(t) \in S(B_2)$, $t \geq 0$. For the case $B_1 < \rho$, we get $B_2(t_0, B_1) = B_2(t_0, \rho)$ and hence the proof.

5.5 Impulsive Set Differential Equations with Delay

In this section we establish basic results in the theory of impulsive set differential equations with delay.

Consider the impulsive set differential equation with delay

$$\left.\begin{array}{rcl} D_H U & = & F(t, U_t), \ t \neq t_k, \\ U_{t_k^+} & = & I_k(U_{t_k}), \ t = t_k, \\ U_{t_0} & = & \Phi_0 \in K_c(\mathbb{R}^n), \end{array}\right\} \tag{5.5.1}$$

where $F \in PC[\mathbb{R}_+ \times \mathcal{C}, K_c(\mathbb{R}^n)]$, $I_k : \mathcal{C} \longrightarrow \mathcal{C}$ with $\mathcal{C} = C[[-\tau, 0], K_c(\mathbb{R}^n)]$ and $\{t_k\}$ is a sequence of points such that $0 \leq t_0 < t_1 < \cdots < t_k < \ldots$ with $\lim_{k \to \infty} t_k = \infty$.

By a solution of (5.5.1) we mean a piecewise continuous function $U(t_0, \Phi_0)(t)$ on $[t_0, \infty)$ which is left continuous on $(t_k, t_{k+1}]$ and defined by

$$U(t_0, \Phi_0)(t) = \begin{cases} \Phi_0, & t_0 - \tau \leq t \leq t_0, \\ U_0(t_0, \Phi_0)(t), & t_0 \leq t \leq t_1, \\ U_1(t_1, \Phi_1)(t), & t_1 < t \leq t_2, \\ \vdots & \vdots \\ U_k(t_k, \Phi_k)(t), & t_k < t \leq t_{k+1}, \\ \vdots & \vdots \end{cases} \tag{5.5.2}$$

where $U_k(t_k, \Phi_k)(t)$ is the solution of the set differential equation with delay

$$D_H U = F(t, U_t), \quad U_{t_k^+} = \Phi_k, \quad k = 0, \ 1, \ 2, \ \ldots \ .$$

We will first prove an existence theorem for impulsive set differential equations with delay.

Theorem 5.5.1 *Assume that*

(i) $F \in PC[\mathbb{R}_+ \times \mathcal{C}, \ K_c(\mathbb{R}^n]$,

(ii) $D[F(t, \Phi), \theta] \leq g(t, D_0[\Phi, \theta)])$, $t \neq t_k$, *where* $g \in PC[{\mathbb{R}_+}^2, \mathbb{R}_+]$, $g(t, w)$ *is nondecreasing in* w *for each* $t \in \mathbb{R}_+$,

(iii) $D_0[I(U_{t_k}),\theta] \le J_k(D_0[U_{t_k},\theta]), t = t_k$, *where* $J_k(w)$ *is a nondecreasing function of* w*,*

(iv) $r(t,t_0,w_0)$ *is the maximal solution of the impulsive scalar differential equation*

$$\begin{cases} w' &= g(t,w), t \neq t_k, \\ w(t_k^+) &= J_k(w(t_k)), t = t_k, \\ w(t_0) &= w_0, \end{cases} \tag{5.5.3}$$

existing on $[t_0,\infty]$ *and* F *is smooth enough to assure local existence.*

Then there exists a solution for (5.5.1) on $[t_0,\infty)$*.*

Proof Set $J_0 = [t_0,t_1]$ and restrict F to $J_0 \times \mathcal{C}$. Note that F is continuous on $J_0 \times \mathcal{C}$.

Consider on J_0 the set differential equation

$$\begin{cases} D_H U &= F(t,U_t), \\ U_{t_0} &= \Phi_0. \end{cases}$$

Then, the hypothesis of Theorem 5.4.3 is satisfied, and hence there exists a solution $U_0(t_0,\Phi_0)(t),\ t \in J_0$, for the set differential equation with delay on J_0.

Now, for $t = t_1,\ U_0(t_1) = U_0(t_0,\Phi_0)(t_1)$ and $U_{0,t_1^+} = I_1(U_{0,t_1})$. Set $\Phi_1 = U_{0,t_1^+}$. Let $J_1 = (t_1,t_2]$ and consider the set differential equation with delay

$$\begin{cases} D_H U &= F(t,U_t), \quad t \in J_1, \\ U_{t_1^+} &= \Phi_1. \end{cases}$$

Once again, restricting the domain of F to $J_1 \times \mathcal{C}$ and employing the impulsive condition in (iii), the hypothesis of Theorem 5.4.3 is satisfied and thus there exists a solution $U_1(t_1,\Phi_1)(t)$,
$t \in J_1$, satisfying the set differential equation with delay restricted to J_1. We have $U_1(t_2) = U_1(t_1,\Phi_1)(t_2),\ U_{1,t_2^+} = I_2(U_{1,t_2})$. Set $\Phi_2 = U_{1,t_2^+}$ and let $J_2 = (t_2,t_3]$. Repeating the above process, we get the existence of a solution of the impulsive set differential equation with delay on $[t_0,\infty)$.
Next, we give a basic comparison theorem for impulsive set differential equations with delay.

Theorem 5.5.2 *Assume that*

(i) $F \in PC[\mathbb{R}_+ \times \mathcal{C}, K_c(\mathbb{R}^n)]$;

(ii) $D[F(t,\Phi),F(t,\Psi)] \le g(t,D_0[\Phi,\Psi])$ *for* $t \in \mathbb{R}_+, t \neq t_k,\ \Phi,\Psi \in \mathcal{C}$, *and* $g \in PC[\mathbb{R}_2^+,\mathbb{R}_+]$;

(iii) $D_0[I_k(U_{t_k}),I_k(V_{t_k})] \le J_k(D_0[U_{t_k},V_{t_k}]),\quad t = t_k$, *where* $J_k(w)$ *is a nondecreasing function of* w*;*

(iv) $r(t) = r(t,t_0,w_0)$ *is the maximal solution of the scalar impulsive differential equation (5.5.3) existing on* $[t_0,\infty)$*.*

Then, if $U(t) = U(t_0, \Phi_0)(t)$ and $V(t) = V(t_0, \Psi_0)(t)$ are any two solutions of (5.5.1) on $[t_0, \infty)$, we have

$$D[U(t), V(t)] \leq r(t), \quad t \geq t_0,$$

provided $D_0[\Phi_0, \Psi_0] \leq w_0$.

Proof We set $J_0 = [t_0, t_1]$ and restrict the domain of F to $J_0 \times \mathcal{C}$. Then F is continuous on this domain and the hypothesis of Theorem 5.4.2 is satisfied. Hence we have that

$$D[U(t), V(t)] \leq r(t), \quad t \in J_0,$$

which implies that $D[U(t_1), V(t_1)] \leq r(t_1)$. Using hypothesis (iii) for $t = t_1^+$, we get

$$\begin{aligned} D_0[U_{t_1^+}, V_{t_1^+}] &= D_0[I_1(U_{t_1}), I_1(V_{t_1})] \\ &\leq J_1(D_0[U_{t_1}, V_{t_1}]) \\ &\leq J_1(r(t_1)) \equiv r(t_1^+). \end{aligned}$$

Thus

$$D_0[U_{t_1}^+, V_{t_1}^+] \leq r(t_1^+). \tag{5.5.4}$$

Next, restrict the domain of F to $J_1 \times \mathcal{C}$, where $J_1 = (t_1, t_2]$. Then using (ii), (5.5.4) and Theorem 5.4.2 we can conclude that

$$D[U(t), V(t)] \leq r(t, t_0, w_0), \quad t \in J_1.$$

Repeating the above process, the conclusion of the theorem is obtained.

We shall next extend a typical result in Lyapunov-like theory.

Let $V : \mathbb{R}_+ \times K_c(\mathbb{R}^n) \times \mathcal{C} \to \mathbb{R}_+$. Then V is said to belong to class V_0 if

(A1) $V(t, U, \Phi)$ is continuous in $(t_{k-1}, t_k] \times K_c(\mathbb{R}^n) \times \mathcal{C}$ and for each $U \in K_c(\mathbb{R}^n)$, $\Phi \in \mathcal{C}$, $k = 1,\ 2,\ \cdots$,

$$\lim_{(t,W,\Phi)\to(t_k^+,U,\Phi)} V(t, W, \Phi) = V(t_k^+, U, \Phi)$$

exists;

(A2) $V \in C[(t_{k-1}, t_k] \times K_c(\mathbb{R}^n) \times \mathcal{C}, \mathbb{R}_+^n]$ satisfies

$$|V(t, U, \Phi) - V(t, W, \Phi| \leq LD[U, W],\ L > 0.$$

For $(t, U, \phi) \in (t_{k-1}, t_k] \times K_c(\mathbb{R}^n) \times \mathcal{C}$, we define

$$D^+V(t, U, \Phi) = \limsup_{h\to 0^+} \frac{1}{h}[V(t + h,\ U + hF(t, U_t),\ U_{t+h}) - V(t, U, U_t)].$$

To investigate stability criteria the following comparison result in terms of a Lyapunov function on product spaces is needed. (See Lakshmikantham, Leela and Sivasundaram [1]).

Theorem 5.5.3 *Suppose that*

(i) $V : \mathbb{R}_+ \times K_c(\mathbb{R}^n) \times \mathcal{C} \to \mathbb{R}_+$ *and* $V \in V_0$.

(ii) $D^+V(t, U, \Phi) \leq g(t, V(t, U, \Phi)), \quad t \neq t_k,$ *where* $g : (t_{k-1}, t_k] \times \mathbb{R}_+ \to \mathbb{R}$ *is continuous and for each* $w \in \mathbb{R}_+$, $\lim_{(t,z)\to(t_k^+,w)} g(t, z) = g(t_k^+, w)$ *exists.*

(iii) $V(t_k^+, U(t_0, \Phi_0)(t_k^+), \quad U_{t_k^+}(t_0, \Phi_0)) \leq J_k[V(t_k, U(t_0, \Phi_0)(t_k), U_{t_k}(t_0, \Phi_0)],$ $t = t_k$, *and* $J_k(w)$ *is nondecreasing in* w.

Let $r(t) = r(t, t_0, w_0)$ *be the maximal solution of the scalar impulsive differential equation (5.5.3) existing on* $t \geq t_0$. *Then*

$$V(t, U(t_0, \Phi_0)(t),\ U_t(t_0, \Phi_0)) \leq r(t),\ t \geq t_0,$$

where $U(t_0, \Phi_0)(t)$ *is any solution of the impulsive set differential equation with delay (5.5.1) existing on* $t \geq t_0$.

Proof Let $U(t_0, \Phi_0)(t)$ be any solution of (5.5.1) existing on $[t_0, \infty)$. Define $m(t) = V(t, U(t_0, \Phi_0)(t),\ U_t(t_0, \Phi_0))$, so that
$m(t_0) = V(t_0, U(t_0, \Phi_0)(t_0), \Phi_0)$, and suppose that $m(t_0) \leq w_0$.

Now for $t \in (t_{k-1}, t_k],\ k = 1,\ 2, \cdots,$

$$\begin{aligned} m(t+h) - m(t) &= V(t+h, U(t_0, \Phi_0)(t+h),\ U_{t+h}(t_0, \Phi_0)) \\ &\quad - V(t, U(t_0, \Phi_0)(t), U_t(t_0, \Phi_0)) \\ &= V(t+h, U(t_0, \Phi_0)(t+h),\ U_{t+h}(t_0, \Phi_0)) \\ &\quad - V(t+h, U(t_0, \Phi_0)(t) + hF(t, U_t),\ U_{t+h}(t_0, \Phi_0)) \\ &\quad + V(t+h, U(t_0, \Phi_0)(t) + hF(t, U_t),\ U_{t+h}(t_0, \Phi_0)) \\ &\quad - V(t, U(t_0, \Phi_0)(t),\ U_t(t_0, \Phi_0)) \\ &\leq LD[U(t_0, \Phi_0)(t) + hF(t, U_t),\ U(t_0, \Phi_0)(t+h)) \\ &\quad + V(t+h, U(t_0, \Phi_0)(t) + hF(t, U_t),\ U_{t+h}(t_0, \Phi_0)) \\ &\quad - V(t, U(t_0, \Phi_0)(t),\ U_t(t_0, \Phi_0)), \end{aligned}$$

using (A2). Thus

$$\begin{aligned} D^+m(t) &= \limsup_{h\to 0^+} \frac{1}{h}[m(t+h) - m(t)] \\ &\leq D^+V(t, U(t_0, \Phi_0)(t),\ U_t(t_0, \Phi_0)) \\ &\quad + L \limsup_{h\to 0^+} D[U(t_0, \Phi_0)(t+h),\ U(t_0, \Phi_0)(t) + hF(t, U_t)]. \end{aligned}$$

Using the properties of the Hausdorff metric D and the fact that $U(t_0, \Phi_0)(t)$ is a solution (5.5.1), it is not difficult to show that

$$\limsup_{h\to 0^+} D[U(t_0, \Phi_0)(t+h),\ U(t_0, \Phi_0)(t) + hF(t, U_t)] = 0.$$

Therefore, using (ii), we have

$$\begin{aligned} D^+m(t) &\le D^+V(t, U(t_0,\Phi_0)(t),\ U_t(t_0,\Phi_0)) \\ &\le g(t, V(t, U(t_0,\Phi_0)(t),\ U_t(t_0,\Phi_0)) \\ &= g(t, m(t)). \end{aligned}$$

For $t = t_k$, we get from (iii)

$$\begin{aligned} m(t_k^+) &= V(t_k^+, U(t_0,\Phi_0)(t_k^+),\ U_{t_k^+}(t_0,\Phi_0)) \\ &\le J_k[V(t_k, U(t_0,\Phi_0)(t_k),\ U_{t_k}(t_0,\Phi_0))] \\ &= J_k[m(t_k)]. \end{aligned}$$

Hence by Theorem 4.6.1 we get

$$m(t) \le r(t),\ t \ge t_0.$$

We will now define stability of the null solution of an impulsive set differential equation with delay.

Definition 5.5.1 *Let $U(t) = U(t_0,\Phi_0)(t)$ be any solution of (5.5.1). Then the trivial solution $U(t) \equiv \theta$ is said to be stable if for each $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(t_0,\epsilon) > 0$ such that $D_0[\Phi_0,\theta] < \delta$ implies $D[U(t),\theta] < \epsilon$, $t \ge t_0$.*

The other definitions can be formulated similarly. (See Lakshmikantham and Leela [1]).

We set, as before

$$\begin{aligned} S(\rho) &= [U \in K_c(\mathbb{R}^n)\ :\ D[U,\theta] < \rho] \\ \tilde{S}(\rho) &= [\Phi \in \mathcal{C}\ :\ D_0[\Phi,\theta] < \rho] \\ \mathcal{K} &= \{a \in C[\mathbb{R}_+,\mathbb{R}_+]\ :\ a(0) = 0 \text{ and } a(u) \text{ is strictly increasing}\}. \end{aligned}$$

We shall now give a typical result on stability criteria.

In order to obtain the trivial solution of (5.5.1) we assume that $F(t,\theta) \equiv \theta$ and $I_k(\theta) \equiv \theta$ for all k.

Theorem 5.5.4 *Assume that*

(i) $V :\ \mathbb{R}_+ \times S(\rho) \times \tilde{S}(\rho) \to \mathbb{R}_+,\ V \in V_0$ *and* $D^+V(t,U,\phi) \le g(t, V(t,U,\phi)),\ t \ne t_k,\ g :\ \mathbb{R}^2{}_+ \to \mathbb{R},\ g(t,0) \equiv 0$ *and g satisfies the assumptions given in Theorem 5.5.3;*

(ii) there exists $\rho_0 > 0$ such that $U_{t_k} \in \tilde{S}(\rho_0)$ implies that $I_k(U_{t_k}) \in \tilde{S}(\rho)$ for all k and
$V(t_k^+,\ U(t_0,\Phi_0)(t_k^+),\ U_{t_k^+}(t_0,\Phi_0)) \le J_k[V(t_k, U(t_0,\Phi_0)(t_k),\ U_{t_k}(t_0,\Phi_0))]$, *$t = t_k,\ U_{t_k} \in \tilde{S}(\rho_0)$ and $J_k\ :\ \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing and $J_k(0) = 0$ for all k;*

(iii) $b(D_0[U,\theta]) \le V(t,U,\Phi) \le a(D_0[\Phi,\theta])$, *where $a,\ b \in \mathcal{K}$.*

Then the stability properties of the trivial solution of (5.5.3)imply the corresponding stability properties of the trivial solution of (5.5.1).

Proof Let $0 < \epsilon < \min(\rho, \rho_0)$, $t_0 \in \mathbb{R}_+$ be given. Suppose that the trivial solution of (5.5.3) is stable. Then, given $b(\epsilon) > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta_1 = \delta_1(t_0, \epsilon) > 0$ such that $0 \leq w_0 < \delta_1$ implies $w(t, t_0, w_0) < b(\epsilon)$, $t \geq t_0$, where $w(t, t_0, w_0)$ is any solution of (5.5.3). Let $w_0 = a(D_0[\Phi_0, \theta])$ and choose a $\delta = \delta(t_0, \epsilon)$ such that $a(\delta) < \delta_1$.

We claim that with this δ we have $D_0[\Phi_0, \theta] < \delta$ implies $D[U(t_0, \Phi_0)(t), \theta] < \epsilon$, $t \geq t_0$ for any solution $U(t_0, \Phi_0)(t)$ of (5.5.1). If this is not true there would exist a solution $U(t) = U(t_0, \Phi_0)(t)$ of (5.5.1) with $D_0[\Phi_0, \theta] < \delta$ and a $t^* > t_0$ satisfying $t_k < t^* \leq t_{k+1}$, for some k, $\epsilon \leq D[U(t_0, \Phi_0)(t^*), \theta]$ and $D[U(t_0, \Phi_0)(t), \theta] < \epsilon$, $t_0 \leq t \leq t_k$.

Since $0 < \epsilon < \rho_0$, condition (ii) shows that $D[U(t_0, \Phi_0)(t_k), \theta] < \epsilon$ and $D_0[U_{t_k^+}(t_0, \Phi_0), \theta] = D_0[I_k(U_{t_k}(t_0, \Phi_0)), \theta] < \rho$.

Hence we can find a t^0 such that $t_k < t^0 \leq t^*$ satisfying

$$\epsilon \leq D[U(t_0, U(t_0, \Phi_0)(t^0), \theta] < \rho.$$

Now setting $m(t) = V(t, U(t_0, \Phi_0)(t), \quad U_t(t_0, \Phi_0))$, $\quad t_0 \leq t \leq t^0$ and using hypothesis (i), (ii), and Theorem 5.5.3, we get the estimate

$$V(t, U(t_0, \Phi_0)(t), U_t(t_0, \Phi_0)) \leq r(t, t_0, a(D_0[\Phi_0, \theta])), \quad t_0 \leq t \leq t^0,$$

where $r(t, t_0, w_0)$ is the maximal solution of (5.5.3).We are then led, because of (iii), to the contradiction

$$\begin{aligned} b(\epsilon) &\leq b(D[U(t_0, \Phi_0)(t^0), \theta) \\ &\leq V(t^0, U(t_0, \Phi_0)(t^0),\ U_{t^0}(t_0, \Phi_0)) \\ &\leq r(t^0, t_0,\ a(D_0[\Phi_0, \theta])) \\ &< r(t^0, t_0, a(\delta)) < r(t^0, t_0, \delta_1) < b(\epsilon), \end{aligned}$$

which proves that the trivial solution of (5.5.1) is stable.

Example 5.5.1 *Consider the set differential equation with delay on* $\mathbb{R}$

$$D_H U = -U(t - \tau),\ U_0 = [\phi_1, \phi_2], \tag{5.5.5}$$

where $U(t) = [u_1(t), u_2(t)]$.

This can be written as

$$[u'_1, u'_2] = [-u_2(t - \tau), -u_1(t - \tau)]$$

which is equivalent to the system of ordinary differential equations with delay

$$\begin{aligned} u'_1 &= -u_2(t - \tau), \quad u_{1,0} = \phi_1, \\ u'_2 &= -u_1(t - \tau), \quad u_{2,0} = \phi_2, \end{aligned} \tag{5.5.6}$$

which reduces to

$$\begin{cases} u''_1 = u_1(t-2\tau) \\ u''_2 = u_2(t-2\tau) \end{cases} \tag{5.5.7}$$

Suppose the initial functions are given by

$$\begin{cases} \phi_1(s) = \left(\dfrac{u_{10}-u_{20}}{2}\right)e^{\lambda_1 s} + \left(\dfrac{u_{10}+u_{20}}{2}\right)e^{-\lambda_2 s}, \\ \phi_2(s) = \left(\dfrac{u_{20}-u_{10}}{2}\right)e^{\lambda_1 s} + \left(\dfrac{u_{10}+u_{20}}{2}\right)e^{-\lambda_2 s}, \end{cases} \quad -2\tau \le s \le 0. \tag{5.5.8}$$

We choose $\lambda_1, \lambda_2 > 0$ *such that*

$$\lambda_1^2 = e^{-2\lambda_1 \tau}, \quad \lambda_2^2 = e^{2\lambda_2\tau},$$

so that $e^{\lambda_1 t}, \quad e^{-\lambda_2 t}$ *satisfy (5.5.7).*

As a result , we have

$$\begin{cases} u_1(t) = c_1 e^{\lambda_1 t} + c_2 e^{-\lambda_2 t} \\ u_2(t) = c_3 e^{\lambda_1 t} + c_4 e^{-\lambda_2 t} \end{cases} \quad t \ge 0. \tag{5.5.9}$$

Using (5.5.6), (5.5.8) and (5.5.9) at $t = 0$*, we compute the values of* $c_1,\ c_2,\ c_3$ *and* c_4 *to find that* $c_1 = \dfrac{u_{10}-u_{20}}{2}$, $c_2 = \dfrac{u_{10}+u_{20}}{2}$, $c_3 = \dfrac{u_{20}-u_{10}}{2}$ *and* $c_4 = \dfrac{u_{10}+u_{20}}{2}$.

Hence, the solutions (5.5.9) are given by

$$\begin{cases} u_1(t) = \left(\dfrac{u_{10}-u_{20}}{2}\right)e^{\lambda_1 t} + \left(\dfrac{u_{10}+u_{20}}{2}\right)e^{-\lambda_2 t}, \\ u_2(t) = \left(\dfrac{u_{20}-u_{10}}{2}\right)e^{\lambda_1 t} + \left(\dfrac{u_{10}+u_{20}}{2}\right)e^{-\lambda_2 t}, \qquad t \ge 0. \end{cases} \tag{5.5.10}$$

Thus, the solution of (5.5.7) is given by (5.5.8) and (5.5.10), where $\phi_1(0) = u_{10}$ *and* $\phi_2(0) = u_{20}$.

Case 1: If $u_{20} = u_{10} = u_0$*, then (5.5.10) reduces to*

$$\begin{cases} u_1(t) = u_0 e^{-\lambda_2 t}, \\ u_2(t) = u_0 e^{-\lambda_2 t}, \qquad t \ge 0, \end{cases}$$

or

$$U(t) = [u_0, u_0]e^{-\lambda_2 t}, \ t \ge 0.$$

In this case impulses have no role to play.

Case 2: If $u_{10} = -u_{20} = -u_0$*, then (5.5.10) reduces to*

$$\begin{cases} u_1(t) = -u_0 e^{\lambda_1 t} \\ u_2(t) = u_0 e^{\lambda_1 t}, \qquad t \ge 0, \end{cases}$$

or

$$U(t) = [-u_0, u_0]e^{\lambda_1 t},\ t \geq 0.$$

Now suppose we introduce impulses to the set differential equation with delay, (5.5.5) at $t = t_k,\ k = 1,\ 2,\ \ldots$ as

$$U_{t_k^+} = a_k U_{t_k}, \quad a_k > 0. \tag{5.5.11}$$

Then the solution to the impulsive set differential equation with delay (5.5.5), (5.5.11), is given by

$$U(t) = \prod_{0<t_k<t} a_k[-u_0, u_0]e^{\lambda_1 t},\ t \geq 0. \tag{5.5.12}$$

If the a_k's satisfy

$$\lambda_1 t_{k+1} + \ln a_k \leq \lambda_1 t_k \quad \text{for all} \quad k, \tag{5.5.13}$$

then $a_k \leq e^{\lambda_1(t_k - t_{k+1})}$ and using this estimate in (5.5.12), we obtain

$$||U(t)|| \leq ||U_0||\ e^{\lambda_1 t_1}$$

where $||U(t)|| = D[U(t), \theta]$. Choosing $\delta = \frac{\epsilon}{2}e^{-\lambda_1 t_1}$, it follows that $||U(t)|| < \epsilon,\quad t \geq 0$, provided $||U_0|| < \delta$. Hence the stability of the trivial solution of (5.5.5) and (5.5.11) follows.

For asymptotic stability, we strengthen the assumption (5.5.13) to

$$\lambda_1 t_{k+1} + \ln(\alpha a_k) \leq \lambda_1 t_k \quad \text{for all} \quad k, \quad \text{where} \quad \alpha > 1. \tag{5.5.14}$$

Then $a_k \leq \frac{1}{\alpha}e^{\lambda_1(t_k - t_{k+1})}$ and using this estimate in (5.5.12), we obtain

$$\limsup_{h\to\infty} \|U(t)\| = 0.$$

Thus the trivial solution of the impulsive differential equation with delay (5.5.5) and (5.5.11) is asymptotically stable.

5.6 Set Difference Equations

It is well known that difference equations appear as the natural description of observed evolution phenomenon, because most measurements of time evolving variables are discrete and as such these equations are, in their own right, important models. More importantly, difference equations also appear in the study of discretization methods for differential equations. Consequently, initial value problems (IVPs) of difference equations should be formulated as,

$$\Delta u_n \equiv u_{n+1} - u_n = F(n, u_n),\ \ u_{n_0} = u_0,\ n \geq n_0 \geq 0$$

to represent discretization of corresponding ODEs. Nonetheless, in the literature of difference equations, we find usually the following type of formulation

$$u_{n+1} = f(n, u_n),\ \ u_{n_0} = u_0,\ \ n \geq n_0 \geq 0,$$

where, of course , $f(n, u_n) = u_n + F(n, u_n)$.

In this section, we plan to discuss set difference equations parallel to set differential equations, and develop the theory of such equations. We shall only provide a few typical results so as to advance the investigation of set difference equations further, since the theory of such equations is a lot richer than the corresponding SDEs.

Let $\mathbb{N}$ denote the natural numbers and $\mathbb{N}^+ = \mathbb{N} \cup \{0\}$. We denote by $\mathbb{N}^+_{n_0}$ the set

$$\mathbb{N}^+_{n_0} = \{n_0, n_0 + 1, \cdots, n_0 + l, \cdots\},$$

with $l \in \mathbb{N}^+$ and $n_0 \in \mathbb{N}^+$.

We consider the set difference equation of the form

$$U_{n+1} = F(n, U_n), \quad U_{n_0} = U_0, \tag{5.6.1}$$

where $F : \mathbb{N}^+_{n_0} \times K_c(\mathbb{R}^q) \to K_c(\mathbb{R}^q)$ is continuous in U for each n and $U_n \in K_c(\mathbb{R}^q)$ for each $n \geq n_0$.

Since we shall be using n in difference equations, we shall employ the metric space $(K_c(\mathbb{R}^q), D)$ for $(K_c(\mathbb{R}^n), D)$ used earlier. This will avoid confusion. The possibility of obtaining the values of solutions of (5.6.1) recursively is very important and does not have a counter part in other kinds of equations. For this reason, we sometimes reduce continuous problems to approximate difference problems. For simple set difference equations we can find solutions in closed form. However, deducing information on the qualitative and quantitative behavior of solutions of (5.6.1) by the comparison principle, is very effective as usual.

We need the following comparison principle for ordinary difference equations. See Lakshmikantham and Trigiante [1] for details.

Theorem 5.6.1 *Let $n \in \mathbb{N}^+_{n_0}$, $\quad r \geq 0$ and $g(n, r)$ be a nondecreasing function in r for each n. Suppose that for each $n \geq n_0$, the inequalities*

$$y_{n+1} \leq g(n, y_n), \tag{5.6.2}$$

$$z_{n+1} \geq g(n, z_n), \tag{5.6.3}$$

hold. If $y_{n_0} \leq z_{n_0}$, then $y_n \leq z_n$ for all $n \geq n_0$.

Corollary 5.6.1 *Let $n \in \mathbb{N}^+_{n_0}$, $k_n \geq 0$ and $y_{n+1} \leq k_n y_n + p_n$. Then,*

$$y_n \leq y_{n_0} \prod_{s=n_0}^{n-1} k_s \;+\; \sum_{s=n_0}^{n-1} p_s \prod_{\tau=s+1}^{n-1} k_\tau, \quad n \geq n_0. \tag{5.6.4}$$

Corollary 5.6.2 *(Discrete Gronwall Inequality)*

Let $n \in \mathbb{N}^+_{n_0}$, $k_n \geq 0$ and

$$y_{n+1} \leq y_{n_0} + \sum_{s=n_0}^{n} [k_s y_s + p_s].$$

Then,

$$y_n \le y_{n_0} \prod_{s=n_0}^{n-1} (1+k_s) \;+\; \sum_{s=n_0}^{n-1} p_s \prod_{\tau=s+1}^{n-1} (1+k_\tau) \tag{5.6.5}$$

$$\le y_{n_0}\, exp\left(\sum_{s=n_0}^{n-1} k_s\right) \;+\; \sum_{s=n_0}^{n-1} p_s\, exp\left(\sum_{\tau=s+1}^{n-1} k_\tau\right), \quad n \ge n_0.$$

The following theorem estimates the solution of the set difference equation in terms of the solution of the scalar difference equation

$$z_{n+1} = g(n, z_n), \quad z_{n_0} = z_0, \tag{5.6.6}$$

where $g(n,r)$ is continuous in r for each n and nondecreasing in r for each n. We prove the following result.

Theorem 5.6.2 *Assume that $F(n,U)$ is continuous in U for each n and*

$$D[F(n,U),\theta] = \|F(n,U)\| \le g(n, \|U\|) \tag{5.6.7}$$

where $g(n,r)$ is given in (5.6.6). Then, $\|U_{n_0}\| \le z_{n_0}$ implies,

$$\|U_{n+1}\| \le z_{n+1}, \;\; \text{for } \; n \ge n_0. \tag{5.6.8}$$

Proof Set $y_{n+1} = \|U_{n+1}\|$, so that we get

$$y_{n+1} = \|F(n,U_n)\| \le g(n, \|U_n\|) = g(n, y_n), \quad n \ge n_0.$$

Let z_{n+1} be the solution of (5.6.6) with $z_{n_0} = y_{n_0}$. Then, Theorem 5.6.1 yields immediately,

$$y_{n+1} \le z_{n+1}, \quad n \ge n_0,$$

which implies (5.6.8), completing the proof.

The assumption (5.6.7) can be replaced by a weaker condition, namely

$$D[F(n,U),\theta] = \|F(n,U)\| \le \|U\| + w(n, \|U\|), \quad U \in K_c(\mathbb{R}^q).$$

Now, set $g(n,r) = r + w(n,r)$ and assume that $g(n,r)$ is nondecreasing in r, for each n. This version of Theorem 5.6.2 is more suitable because $w(n,r)$ need not be positive, and hence the solutions of (5.6.6) could have better properties. This observation is useful in extending the Lyapunov-like method for (5.6.1).

Let $V : \mathbb{N}_{n_0}^{+} \times K_c(\mathbb{R}^q) \to \mathbb{R}_+$ be a given function. We have the following comparison result.

Theorem 5.6.3 *Let $V(n,U)$ given above satisfy*

$$\begin{aligned} V(n+1, U_{n+1}) \;\; &\le \;\; V(n, U_n) + w(n, V(n, U_n)) \\ &\equiv \;\; g(n, V(n, U_n)), \quad n \ge n_0. \end{aligned}$$

Then, $V(n, U_{n_0}) \le z_{n_0}$ implies,

$$V(n+1, U_{n+1}) \le z_{n+1}, \quad n \ge n_0, \tag{5.6.9}$$

where $z_{n+1} = z_{n+1}(n_0, z_{n0})$ is the solution of (5.6.6).

Proof Set $y_{n+1} = V(n+1, U_{n+1})$, so that $y_{n_0} = V(n_0, U_{n_0}) \le z_{n_0}$ and

$$y_{n+1} \le y_n + w(n, y_n), \quad n \ge n_0.$$

Consequently, $g(n,r) = r + w(n,r)$. Hence, by Theorem 5.6.1, we get

$$y_{n+1} \le z_{n+1}, \quad n \ge n_0,$$

where z_{n+1} is the solution of (5.6.7). This implies the stated estimate.

Using Theorem 5.6.3., we can prove the stability results for the solutions of set difference equation (5.6.1).

Theorem 5.6.4 *Let the assumptions of Theorem 5.6.3 hold. Suppose further that*

$$b(\|U\|) \le V(n, U) \le a(\|U\|),$$

where $a, b \in \mathcal{K}$, $n \in \mathbb{N}^+_{n_0}$ *and* $U \in K_c(\mathbb{R}^q)$. *The stability properties of the trivial solution of* (5.6.6) *imply the corresponding stability properties of the trivial solution of* (5.6.1).

Proof Suppose that the trivial solution of (5.6.6) is asymptotically stable. Then, it is stable. Let, $0 < \varepsilon$, $n_0 \in \mathbb{N}$ be given. Then, $b(\varepsilon) > 0$, $n_0 \in \mathbb{N}$, there exists a $\delta_1 = \delta_1(n_0, \varepsilon)$ such that

$$0 \le z_{n_0} < \delta_1 \text{ implies } z_{n+1} < b(\varepsilon), \quad n \ge n_0.$$

Choose $\delta = \delta(n_0, \varepsilon)$ satisfying

$$a(\delta) < \delta_1.$$

Then Theorem 5.6.3 gives,

$$V(n+1, U_{n+1}) \le z_{n+1}, \quad n \ge n_0,$$

which shows that

$$b(\|U_{n+1}\|) \le V(n+1, U_{n+1}) \le z_{n+1}, \quad n \ge n_0.$$

Let $\|U_{n_0}\| < \delta$. Choose $z_{n_0} = V(n_0, U_{n_0})$ so that we have

$$z_{n_0} \le a(\|U_{n_0}\|) \le a(\delta) < \delta_1.$$

We then get,

$$b(\|U_{n+1}\|) < b(\varepsilon), \quad n \ge n_0,$$

which implies the stability of the trivial solution of (5.6.1).

For asymptotic stability, we observe that

$$b(\|U_{n+1}\|) \le V(n+1, U_{n+1}) \le z_{n+1}, \quad n \ge n_0.$$

Since $z_{n+1} \to 0$ as $n \to \infty$, we get $\|U_{n+1}\| \to 0$ as $n \to \infty$. The proof is complete.

As an example, take $g(n,r) = a_n r$ where $a_n \in \mathbb{R}$. Then the solution of

$$z_{n+1} = a_n z_n, \ \ z_{n_0} = z_0,$$

is given by

$$z_n = z_0 \prod_{i=n_0}^{n-1} a_i.$$

We have the following two cases.

(a) If

$$|\prod_{i=n_0}^{n-1} a_i| \leq M(n_0),$$

then $|z_n| \leq |z_0| M(n_0)$ and therefore enough to take $\delta(\varepsilon, n_0) = \dfrac{\varepsilon}{M(n_0)}$, to get stability.

(b) If

$$\lim_{n\to\infty} |\prod_{i=n_0}^{n-1} a_i| \ = \ 0,$$

then asymptotic stability results.

Consequently, Theorem 5.6.3. yields the corresponding stability properties of the trivial solution of (5.6.1).

Corresponding to the familiar example in SDE, namely $D_H U = (-1)U$, let us consider the example in set difference equation

$$\Delta U_n \equiv U_{n+1} - U_n = (-1)U_n, \quad U_0 = U^0 \in K_c(\mathbb{R}).$$

Of course, we need to assume that the Hukuhara difference $U_{n+1} - U_n$ exists for all n. Letting $U_n = [u_n, v_n], U_0 = U^0 = [u_0, v_0]$, using the interval methods as before, we solve the ordinary difference equations,

$$u_{n+1} = u_n - v_n, \quad v_{n+1} = v_n - u_n,$$

which yield

$$u_{n+1} = 2^n[u_0 - v_0], \ \ v_{n+1} = 2^n[v_0 - u_0],$$

and therefore

$$U_{n+1} = [u_0 - v_0, v_0 - u_0]2^n.$$

Thus, diam $\|U_{n+1}\| \to \infty$ as $n \to \infty$.

If, on the other hand, we consider set difference equation

$$U_{n+1} = (-1)U_n, U_0 = U^0 \in K_c(\mathbb{R}),$$

we get , using the interval methods,

$$u_{n+1} = -v_n, \quad v_{n+1} = -u_n,$$

which give us

$$U_{2n} = [u_0, v_0], \;\; U_{2n+1} = [-v_0, -u_0] = (-1)U^0.$$

Hence diam $\|U_n\| =$ diam $\|U^0\|$, which is finite.

As a last example, if we analyze

$$U_{n+1} = \frac{1}{2}U_n, \;\; U_0 = U^0 = [u_0, v_0],$$

we find that $U_n = \left(\frac{1}{2}\right)^n U_0$ and consequently, diam$\|U_n\| \to 0$ as $n \to \infty$.

5.7 Set Differential Equations with Causal Operators

We shall devote this section to extend certain basic results to set differential equations (SDEs) with causal or nonanticipative maps of Volterra type, since such equations provide a unified treatment of the basic theory of SDEs, SDEs with delay and set integro differential equations, which in turn include ordinary dynamic systems of the corresponding types.

Let $E = C[[t_0, T], K_c(\mathbb{R}^n)]$ with norm

$$D_0[U, \theta] = \sup_{t_0 \le t \le T} D[U(t), \theta].$$

Definition 5.7.1 *Suppose that $Q \in C[E, E]$, then Q is said to be a causal map or a nonanticipative map if $U(s) = V(s),\; t_0 \le s \le t \le T$, where $U, V \in E$, then $(QU)(s) = (QV)(s),\; t_0 \le s \le t$.*

We define the IVP for SDE with causal map, using the Hukuhara derivative as follows:

$$D_H U(t) = (QU)(t), \quad U(t_0) = U_0 \in K_c(\mathbb{R}^n). \tag{5.7.1}$$

Before we proceed to prove an existence and uniqueness result for (5.7.1), we need the following comparison results.

Theorem 5.7.1 *Assume that $m \in C[J, \mathbb{R}_+]$, $g \in C[J \times \mathbb{R}_+, \mathbb{R}_+]$ and for $t \in J = [t_0, T]$,*

$$D_- m(t) \le g(t, |m|_0(t)),$$

where $|m|_0(t) = \sup_{0 \le s \le t} |m(s)|$. Suppose that $r(t) = r(t, t_0, w_0)$ is the maximal solution of the scalar differential equation

$$w' = g(t, w), \quad w(t_0) = w_0 \ge 0, \tag{5.7.2}$$

existing on J. Then $m(t_0) \le w_0$ implies $m(t) \le r(t), \;\; t \in J$.

Proof To prove the stated inequality, it is enough to prove that

$$m(t) < w(t, t_0, w_0, \epsilon),\; t \ge t_0,\; t \in J, \tag{5.7.3}$$

where $w(t, t_0, w_0, \epsilon)$ is any solution of

$$w' = g(t, w) + \epsilon, \quad w(t_0) = w_0 + \epsilon,\ \epsilon > 0,$$

since $\lim_{\epsilon \to 0^+} w(t, t_0, w_0, \epsilon) = r(t, t_0, w_0)$.

If (5.7.3) is not true, there exists a $t_1 > t_0$ such that $m(t_1) = w(t_1, t_0, w_0, \epsilon)$ and $m(t) < w(t, t_0, w_0, \epsilon),\ t_0 \leq t < t_1$, in view of the fact $m(t_0) < w_0 + \epsilon$.

Hence,

$$D_- m(t_1) \geq w'(t_1, t_0, w_0, \epsilon) = g(t_1, w(t_1, t_0, w_0, \epsilon)) + \epsilon. \tag{5.7.4}$$

Now $g(t, w) \geq 0$ implies that $w(t, t_0, w_0, \epsilon)$ is nondecreasing in t, and this gives

$$|m|_0(t_1) = w(t_1, t_0, w_0, \epsilon) = m(t_1),$$

which in turn yields

$$D_- m(t_1) \leq g(t_1, |m|_0(t_1)) = g(t_1, w(t_1, t_0, w_0, \epsilon))$$

which is a contradiction to (5.7.4). Hence the theorem follows.

Next we consider an estimate of any two solutions of (5.7.1) in terms of the maximal solution of (5.7.2) utilizing Theorem 5.7.1.

We define $D_0[U, V](t) = \max_{t_0 \leq s \leq t} D[U(s), V(s)]$.

Theorem 5.7.2 *Let $Q \in C[E, E]$ be a causal map such that for $t \in J$,*

$$D[(QU)(t), (QV)(t)] \leq g(t, D_0[U, V](t)), \tag{5.7.5}$$

where $g \in C[J \times \mathbb{R}_+, \mathbb{R}_+]$. Suppose further that the maximal solution $r(t, t_0, w_0)$ of the differential equation (5.7.2) exists on J. Then, if $U(t),\ V(t)$ are any two solutions of (5.7.1) through $U(t_0) = U_0,\ V(t_0) = V_0,\ U_0, V_0 \in K_c(\mathbb{R}^n)$ on J respectively, we have

$$D[U(t), V(t)] \leq r(t, t_0, w_0),\ t \in J, \tag{5.7.6}$$

provided $D[U_0, V_0] \leq w_0$.

Proof Set $m(t) = D[U(t), V(t)]$. Then $m(t_0) = D[U_0, V_0] \leq w_0$. Now for small $h > 0,\ t \in J$, consider $m(t+h) = D[U(t+h), V(t+h)]$. Using the property (1.3.8) of the Hausdorff metric D, we get successively, the following relations:

$$\begin{aligned}
m(t+h) &\leq D[U(t+h), U(t) + h(QU)(t)] + D[U(t) + h(QU)(t), V(t+h)] \\
&\leq D[U(t+h), U(t) + h(QU)(t)] \\
&\quad + D[U(t) + h(QU)(t), V(t) + h(QV)(t)] \\
&\quad + D[V(t) + h(QV)(t), V(t+h)] \\
&\leq D[U(t+h), U(t) + h(QU)(t)] \\
&\quad + D[U(t) + h(QU)(t), U(t) + h(QV)(t)] \\
&\quad + D[U(t) + h(QV)(t), V(t) + h(QV)(t)] \\
&\quad + D[V(t) + h(QV)(t), V(t+h)].
\end{aligned}$$

Next using the property (1.3.9) of the Hausdorff metric D and the fact that the Hukuhara differences $U(t+h)-U(t)$ and $V(t+h)-V(t)$ exist for small $h>0$, we arrive at,

$$\begin{aligned} m(t+h) \leq\ & D[U(t)+Z(t,h),U(t)+h(QU)(t)] \\ & +D[h(QU)(t),h(QV)(t)]+D[U(t),V(t)] \\ & +D[V(t)+h(QV)(t),V(t)+Y(t,h)], \end{aligned}$$

where $U(t+h) = U(t) + Z(t,h)$ and $V(t+h) = V(t) + Y(t,h)$. Again the property (1.3.9) gives

$$\begin{aligned} m(t+h) \leq\ & D[Z(t,h),h(QU)(t)]+D[h(QU)(t),h(QV)(t)] \\ & +D[U(t),V(t)]+D[h(QV)(t),Y(t,h)]. \end{aligned}$$

Since the Hukuhara differences exist, we can replace $Z(t,h)$ and $Y(t,h)$ with $U(t+h)-U(t)$ and $V(t+h)-V(t)$ respectively. This gives, on subtracting $m(t)$ and dividing both sides with $h>0$,

$$\begin{aligned} \frac{m(t+h)-m(t)}{h} \leq\ & D\left[\frac{U(t+h)-U(t)}{h},(QU)(t)\right] \\ & +D[(QU)(t),(QV)(t)] \\ & +D\left[(QV)(t),\frac{V(t+h)-V(t)}{h}\right]. \end{aligned}$$

Now, taking limit supremum as $h\to 0^+$ and using the fact that $U(t)$ and $V(t)$ are solutions of (5.7.1) along with assumption (5.7.5), we obtain

$$\begin{aligned} D^+m(t) &\leq D[(QU)(t),(QV)(t)] \\ &\leq g(t,D_0[U,V](t)) \\ &= g(t,|m|_0(t)), \quad t\in J. \end{aligned}$$

Now, Theorem 5.7.1 guarantees the stated conclusion and the proof is complete.

Corollary 5.7.1 *Let $Q\in C[E,E]$ be a causal map and*

$$D[(QU)(t),\theta]\leq g(t,D_0[U,\theta](t)),$$

where $g\in C[J\times\mathbb{R}_+,\mathbb{R}_+]$. Also, suppose that $r(t,t_0,w_0)$ is the maximal solution of the scalar differential equation (5.7.2). If $U(t,t_0,U_0)$ is any solution of (5.7.1) through (t_0,U_0) with $U_0\in K_c(\mathbb{R}^n)$, then

$$D[U_0,\theta]\leq w_0 \textit{ implies } D[U(t),\theta]\leq r(t,t_0,w_0),\ t\in J.$$

Let us begin by proving a local existence result using successive approximations.

Theorem 5.7.3 *Assume that*

(a) $Q \in C[B, E]$ *where* $B = B(U_0, b) = \{U \in E : D_0[U, U_0] \leq b\}$, *is a causal map and* $D_0[(QU), \theta](t) \leq M_1$, *on* B;

(b) $g \in C[J \times [0, 2b], \mathbb{R}_+]$, $g(t, w) \leq M_2$ *on* $J \times [0, 2b]$, $g(t, 0) \equiv 0$, $g(t, w)$ *is nondecreasing in* w *for each* $t \in J$ *and* $w(t) = 0$ *is the only solution of*

$$w' = g(t, w), \quad w(t_0) = 0 \text{ on } J; \tag{5.7.7}$$

(c) $D[(QU)(t), (QV)(t)] \leq g(t, D_0[U, V](t))$ *on* B.

Then, the successive approximations defined by

$$U_{n+1}(t) = U_0 + \int_{t_0}^{t} (QU_n)(s)\, ds, \quad n = 0, 1, 2, \cdots, \tag{5.7.8}$$

exist on $J_0 = [t_0, t_0 + \eta)$ *where* $\eta = \min[T - t_0, \frac{b}{M}]$ *and* $M = \max(M_1, M_2)$, *and converge uniformly to the unique solution* $U(t)$ *of (5.7.1).*

Proof For $t \in J_0$, we have, by induction, using property (1.3.9) and (1.7.11) of the Hausdorff metric D,

$$\begin{aligned} D[U_{n+1}(t), U_0] &= D\left[U_0 + \int_{t_0}^{t} (QU_n)(s)\, ds, U_0\right] \\ &= D\left[\int_{t_0}^{t} (QU_n)(s)\, ds, \theta\right] \\ &\leq \int_{t_0}^{t} D[(QU_n)(s), \theta]\, ds \\ &\leq \int_{t_0}^{t} D_0[QU_n, \theta](t)\, ds \leq M_1(t - t_0) \leq M(t - t_0) \leq b, \end{aligned}$$

which shows the successive approximations are well defined on J_0.

Next, we define successive approximations for the problem (5.7.7) as follows:

$$\begin{aligned} w_0(t) &= M(t - t_0) \\ w_{n+1}(t) &= \int_{t_0}^{t} g(s, w_n(s))\, ds, \quad t \in J_0, \ n = 0, 1, 2, \cdots. \end{aligned}$$

Then,

$$w_1(t) = \int_{t_0}^{t} g(s, w_0(s))\, ds \leq M_2(t - t_0) \leq M(t - t_0) = w_0(t).$$

Assume for some $k > 1$, $t \in J_0$, that

$$w_k(t) \leq w_{k-1}(t).$$

Then, using the monotonicity of g, we get

$$w_{k+1}(t) = \int_{t_0}^{t} g(s, w_k(s))\ ds \leq \int_{t_0}^{t} g(s, w_{k-1}(s))\ ds = w_k(t).$$

Hence, the sequence $\{w_k(t)\}$ is monotone decreasing.

Since $w_k'(t) = g(t, w_{k-1}(t)) \leq M_2,\ t \in J_0$, we conclude by the Ascoli-Arzela theorem and the monotonicity of the sequence $\{w_k(t)\}$ that

$$\lim_{t \to \infty} w_n(t) = w(t)$$

uniformly on J_0. Since $w(t)$ satisfies equation (5.7.7), we get $w(t) \equiv 0$ on J_0 from condition (b) on J_0. Observing that for each $t \in J_0, t_0 \leq s \leq t$,

$$\begin{aligned} D[U_1(s), U_0] &= D\left[U_0 + \int_{t_0}^{s} (QU_0)(\xi)\ \ d\xi, U_0\right] \\ &= D\left[\int_{t_0}^{s} (QU_0)(\xi)\ \ d\xi, \theta\right] \\ &\leq \int_{t_0}^{s} D\left[(QU_0)(\xi), \theta\right]\ \ d\xi \\ &\leq D_0[(QU_0), \theta](\xi)\ (s - t_0) \leq D_0[(QU_0), \theta](\xi)\ (t - t_0) \\ &\leq M_1(t - t_0) \leq M(t - t_0) = w_0(t), \end{aligned}$$

which implies that $D_0[U_1, U_0](t) \leq w_0(t)$.

We assume for some $k > 1$,

$$D_0[U_k, U_{k-1}](t) \leq w_{k-1}(t), \quad t \in J_0.$$

Consider, for any $t \in J_0, \quad t_0 \leq s \leq t$,

$$\begin{aligned} D[U_{k+1}(s), U_k(s)] &\leq \int_{t_0}^{s} D[(QU_k)(\xi), (QU_{k-1})(\xi)]\ d\xi \\ &\leq \int_{t_0}^{s} g(\xi, D_0[U_k, U_{k-1}](\xi))\ d\xi \\ &\leq \int_{t_0}^{s} g(\xi, w_{k-1}(\xi))\ d\xi \\ &\leq \int_{t_0}^{t} g(\xi, w_{k-1}(\xi))\ d\xi = w_k(t), \end{aligned}$$

which further gives,

$$D_0[U_{k+1}, U_k](t) \leq w_k(t), \quad t \in J_0.$$

Thus we conclude that

$$D_0[U_{n+1}, U_n](t) \leq w_n(t), \tag{5.7.9}$$

for $t \in J_0$ and for all $n = 0, 1, 2, \cdots$.

We claim that $\{U_n(t)\}$ is a Cauchy sequence. To show this, let $n \le m$. Setting $v(t) = D[U_n(t), U_m(t)]$ and using (5.7.8), we get

$$\begin{aligned}
D^+v(t) &\le D[D_H U_n(t), D_H U_m(t)] \\
&= D[(QU_{n-1})(t), (QU_{m-1})(t)] \\
&\le D[(QU_{n-1})(t), (QU_n)(t)] + D[(QU_n)(t), (QU_m)(t)] \\
&\quad +D[(QU_m)(t), (QU_{m-1})(t)] \\
&\le g(t, D_0[U_{n-1}, U_n](t)) + g(t, D_0[U_n, U_m](t)) \\
&\quad +g(t, D_0[U_{m-1}, U_m](t)) \\
&\le g(t, w_{n-1}(t)) + g(t, |v|_0(t)) + g(t, w_{n-1}(t)) \\
&= g(t, |v|_0(t)) + 2g(t, w_{n-1}(t)).
\end{aligned}$$

The above inequalities yield, on using the Theorem 5.7.1, the estimate

$$v(t) \le r_n(t), \quad t \in J_0,$$

where $r_n(t)$ is the maximal solution of

$$\begin{aligned}
r_n'(t) &= g(t, r_n) + 2g(t, w_{n-1}(t)), \\
r_n(t_0) &= 0,
\end{aligned}$$

for each n. Since as $n \to \infty$, $2g(t, w_{n-1}(t)) \to 0$ uniformly on J_0, it follows by Lemma 1.3.1 in Lakshmikantham and Leela [1] that $r_n(t) \to 0$, as $n \to \infty$ uniformly on J_0. This implies from (5.7.9) that $U_n(t)$ converges uniformly to $U(t)$ on J_0 and clearly $U(t)$ is a solution of (5.7.1).

To prove uniqueness, let $V(t)$ be another solution of (5.7.1) on J_0. Set $m(t) = D[U(t), V(t)]$. Then $m(t_0) = 0$ and

$$D^+m(t) \le g(t, |m|_0(t)), \quad t \in J_0.$$

Since $m(t_0) = 0$, it follows from Theorem 5.7.1 that

$$m(t) \le r(t, t_0, 0), \quad t \in J_0,$$

where $r(t, t_0, 0)$ is the maximal solution of (5.7.7). The assumption (b) now shows that $U(t) = V(t)$, $t \in J_0$, proving uniqueness.

Assuming local existence, we next discuss a global existence result.

Theorem 5.7.4 *Let $Q \in C[E, E]$ be a causal map such that*

$$D[(QU)(t), \theta] \le g(t, D_0[U, \theta](t)), \tag{5.7.10}$$

where $g \in C[\mathbb{R}_+^2, \mathbb{R}_+]$, $g(t, w)$ is nondecreasing in w for each $t \in \mathbb{R}_+$ and the maximal solution $r(t) = r(t, t_0, w_0)$ of (5.7.2) exists on $[t_0, \infty)$. Suppose further that Q is smooth enough to guarantee the local existence of solutions of (5.7.1) for any $(t_0, U_0) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n)$. Then, the largest interval of existence for any solution $U(t, t_0, U_0)$ of (5.7.1) is $[t_0, \infty)$, whenever $D[U_0, \theta] \le w_0$.

Proof Suppose that $U(t) = U(t, t_0, U_0)$ is any solution of (5.7.1) existing on $[t_0, \beta),\ t_0 < \beta < \infty$, with $D[U_0, \theta] \leq w_0$, and the value of β cannot be increased. Define $m(t) = D[U(t), \theta]$ and note that $m(t_0) \leq w_0$. Then it follows that,

$$D^+ m(t) \leq D[D_H U(t), \theta] \leq D[(QU)(t), \theta] \leq g(t, D_0[U, \theta](t)).$$

Using Comparison Theorem 5.7.1, we obtain

$$m(t) \leq r(t), \quad t_0 \leq t < \beta. \tag{5.7.11}$$

For any $t_1,\ t_2$ such that $t_0 < t_1 < t_2 < \beta$, using (5.7.10) and the properties of Hausdorff metric D,

$$\begin{aligned} D[U(t_1), U(t_2)] &= D\left[\int_{t_0}^{t_1} (QU)(s)\ ds, \int_{t_0}^{t_2} (QU)(s)\ ds\right] \\ &\leq \int_{t_1}^{t_2} D[(QU)(s), \theta]\ ds \\ &\leq \int_{t_1}^{t_2} g(s, D_0[U, \theta](s))\ ds. \end{aligned}$$

Employing the estimate (5.7.11) and the monotonicity of $g(t, w)$, we find,

$$D[U(t_1), U(t_2)] \leq \int_{t_1}^{t_2} g(s, r(s))\ ds = r(t_2) - r(t_1).$$

Since $\lim_{t \to \beta^-} r(t, t_0, w_0)$ exists, taking the limit as $t_1, t_2 \to \beta^-$, we get that $\{U(t_n)\}$ is a Cauchy sequence and therefore $\lim_{t \to \beta^-} U(t, t_0, U_0) = U_\beta$ exists.

We then consider the IVP

$$D_H U(t) = (QU)(t), \quad U(\beta) = U_\beta.$$

As we have assumed the local existence, we note that $U(t, t_0, U_0)$ can be continued beyond β, contradicting our assumption that β cannot be increased. Thus every solution $U(t, t_0, U_0)$ of (5.7.1) such that $D[U_0, \theta] \leq w_0$ exists globally on $[t_0, \infty)$ and hence the proof.

To prove a comparison result in terms of Lyapunov-like functions, we need the following known result from Lakshmikantham and Rama Mohana Rao [1].

Lemma 5.7.1 *Let* $g_0,\ g \in C[\mathbb{R}_+^2, \mathbb{R}]$ *be such that*

$$g_0(t, w) \leq g(t, w), \quad (t, w) \in \mathbb{R}_+^2. \tag{5.7.12}$$

Then the right maximal solution $r(t, t_0, w_0)$ *of (5.7.2) and the left maximal solution* $\eta(t, T_0, v_0)$ *of*

$$v' = g_0(t, v), \quad v(T_0) = v_0 \geq 0, \tag{5.7.13}$$

satisfy the relation

$$r(t, t_0, w_0) \leq \eta(t, T_0, v_0),\ t \in [t_0, T_0],$$

whenever $r(T_0, t_0, w_0) \leq v_0$.

Theorem 5.7.5 *Assume that*

(i) $L \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), \mathbb{R}_+]$, $L(t,U)$ *is locally Lipschitzian in* U, *i.e.*

$$|L(t,U) - L(t,V)| \leq KD[U,V], \quad U,V \in K_c(\mathbb{R}^n), \quad t \in \mathbb{R}_+;$$

(ii) $g_0,\ g \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}]$,

$$g_0(t,w) \leq g(t,w),$$

$\eta(t,T_0,v_0)$ *is the left maximal solution of*

$$v' = g_0(t,v), \quad v(T_0) = v_0 \geq 0,$$

existing on $t_0 \leq t \leq T_0$ *and* $r(t,t_0,w_0)$ *is the maximal solution of (5.7.2) existing on* $[t_0,\infty)$;

(iii) $D_-L(t,U(t)) \leq g(t, L(t,U(t)))$ *on* Ω, *where*

$$\Omega = [U \in E : L(s,U(s)) \leq \eta(s,t,L(t,U(t))), \quad t_0 \leq s \leq t],$$

and

$$D_-L(t,U(t)) \;=\; liminf_{h\to 0^-} \frac{1}{h}\left[L(t+h, U(t) + h(QU)(t)) - L(t,U(t))\right].$$

Then we have,

$$L(t,U(t,t_0,U_0)) \leq r(t,t_0,w_0), \quad t \geq t_0. \tag{5.7.14}$$

whenever $L(t_0,U_0) \leq w_0$.

Proof To prove (5.7.14), we set $m(t) = L(t,U(t,t_0,U_0)),\ t \geq t_0$ so that $m(t_0) = L(t_0,U_0) \leq w_0$. Let $w(t,\epsilon)$ be any solution of

$$w' = g(t,w) + \epsilon, \quad w(t_0) = w_0 + \epsilon,$$

for sufficiently small $\epsilon > 0$. Then, since $r(t,t_0,w_0) = \lim_{\epsilon\to 0^+} w(t,\epsilon)$, it is enough to prove that

$$m(t) < w(t,\epsilon) \text{ for } t \geq t_0.$$

If this is not true, there exists a $t_1 > t_0$ such that $m(t_1) = w(t_1,\epsilon)$, $m(t) < w(t,\epsilon),\ t_0 \leq t < t_1$. This implies that

$$D_-m(t_1) \geq w'(t,\epsilon) = g(t_1, m(t_1)) + \epsilon. \tag{5.7.15}$$

Now consider the left maximal solution $\eta(s,t_1,m(t_1))$ of (5.7.13) with $v(t_1) = m(t_1)$ on the interval $t_0 \leq s \leq t_1$. By Lemma 5.7.1, we have

$$r(s,t_0,w_0) \leq \eta(s,t_1,m(t_1)),\ s \in [t_0,t_1].$$

Since

$$r(t_1,t_0,w_0) = \lim_{\epsilon\to 0^+} w(t_1,\epsilon) = m(t_1) = \eta(t_1,t_1,m(t_1))$$

and $m(s) \le w(s,\epsilon)$ for $t_0 \le s \le t_1$, it follows that

$$m(s) \le r(s,t_0,w_0) \le \eta(s,t_1,m(t_1)), \quad t_0 \le s \le t_1.$$

This inequality implies that hypothesis (iii) holds for $U(s,t_0,U_0)$ on $t_0 \le s \le t_1$, and hence, standard computation yields,

$$D_- m(t_1) \le g(t_1, m(t_1))$$

which contradicts the relation (5.7.15). Thus $m(t) \le r(t,t_0,w_0),\ t \ge t_0$ and the proof is complete.

The above comparison result in terms of Lyapunov like functions is a useful tool to establish some appropriate stability and boundedness results for set differential equations with causal maps (5.7.1) analogous to original Lyapunov results and their extensions. In order to match the behavior of solutions of (5.7.1) with the corresponding ordinary differential equation with causal map, we need to use the existence of Hukuhara difference $U_0 - V_0 = W_0$ in the initial condition as in Section 3.3 and study the stability or boundedness criteria of $U(t,t_0,U_0-V_0) = U(t,t_0,W_0)$ of (5.7.1).

We present a simple example in $K_c(\mathbb{R})$.

Consider

$$D_H U(t) = -\int_0^t U(s)\, ds, \quad U(0) = U_0 \in K_c(\mathbb{R}).$$

Then using interval methods, we get

$$\begin{aligned} u_1' &= -\int_0^t u_2(s)\, ds, \\ u_2' &= -\int_0^t u_1(s)\, ds, \end{aligned}$$

where $U = [u_1, u_2]$ and $U_0 = [u_{10}, u_{20}]$. Clearly this yields

$$\begin{aligned} u_1^{(4)} &= u_1, \quad u_1(0) = u_{10}, \\ u_2^{(4)} &= u_2, \quad u_2(0) = u_{20}, \end{aligned}$$

whose solutions are given by

$$\begin{aligned} u_1(t) &= \left(\frac{u_{10}-u_{20}}{2}\right)\left(\frac{e^t+e^{-t}}{2}\right) + \left(\frac{u_{10}+u_{20}}{2}\right)\cos t \\ u_2(t) &= \left(\frac{u_{20}-u_{10}}{2}\right)\left(\frac{e^t+e^{-t}}{2}\right) + \left(\frac{u_{10}+u_{20}}{2}\right)\cos t. \end{aligned}$$

That is, $t \geq 0$,

$$\begin{aligned} U(t,0,U_0) &= \left[-\frac{1}{2}(u_{20}-u_{10}), \frac{1}{2}(u_{20}-u_{10})\right]\left(\frac{e^t+e^{-t}}{2}\right) \\ &\quad + \left[\frac{1}{2}\left(\frac{u_{10}+u_{20}}{2}\right), \frac{1}{2}\left(\frac{u_{10}+u_{20}}{2}\right)\right]\cos t,\ \ t \geq 0. \end{aligned}$$

Thus choosing

$$V_0 = \left[-\frac{1}{2}(u_{20}-u_{10}), \frac{1}{2}(u_{20}-u_{10})\right],$$

we obtain

$$U(t,0,W_0) = \left[\frac{1}{2}\left(\frac{u_{10}+u_{20}}{2}\right), \frac{1}{2}\left(\frac{u_{10}+u_{20}}{2}\right)\right]\cos t,\ \ t \geq 0.$$

which implies the stability of the trivial solution of (5.7.1).

We next give an example which illustrates that one can get asymptotic stability as well in SDE with causal maps.

Consider the following differential equation,

$$D_H U = -aU - b\int_0^t U(s)\,ds, \quad U(0) = U_0 \in K_c(\mathbb{R}^n), \tag{5.7.16}$$

$a, b > 0$.

As before, we take $U(t) = [u_1(t), u_2(t)]$, $U_0 = [u_{10}, u_{20}]$, which reduces to

$$\begin{aligned} u_1' &= -au_2 - b\int_0^t u_2(s)\,ds, \\ u_2' &= -au_1 - b\int_0^t u_1(s)\,ds, \\ u_1^{(4)} &= a^2 u_1'' + 2ab u_1' + b^2 u_1, \\ u_2^{(4)} &= a^2 u_2'' + 2ab u_2' + b^2 u_2, \end{aligned}$$

from which we obtain by choosing $a = 1$ and $b = 2$. Using the initial conditions,

$$\begin{aligned} u_1(t) &= \frac{1}{6}(u_{10}-u_{20})\,e^{-t} + \frac{1}{3}(u_{10}-u_{20})\,e^{2t} \\ &\quad + e^{-\frac{1}{2}t}\left[\frac{1}{2}(u_{10}+u_{20})\cos\left(\frac{\sqrt{7}}{2}t\right) - \frac{1}{2\sqrt{7}}(u_{10}+u_{20})\sin\left(\frac{\sqrt{7}}{2}t\right)\right] \\ u_2(t) &= \frac{1}{6}(u_{20}-u_{10})\,e^{-t} + \frac{1}{3}(u_{20}-u_{10})\,e^{2t} \\ &\quad + e^{-\frac{1}{2}t}\left[\frac{1}{2}(u_{20}+u_{10})\cos\left(\frac{\sqrt{7}}{2}t\right) - \frac{1}{2\sqrt{7}}(u_{20}+u_{10})\sin\left(\frac{\sqrt{7}}{2}t\right)\right]. \end{aligned}$$

Thus it follows that,

$$\begin{aligned} U(t) &= (u_{20}-u_{10})\left[-\frac{1}{6},\frac{1}{6}\right]e^{-t}+(u_{20}-u_{10})\left[-\frac{1}{3},\frac{1}{3}\right]e^{2t} \\ &\quad +(u_{10}+u_{20})\left[\frac{1}{2},\frac{1}{2}\right]e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{7}}{2}t\right) \\ &\quad -(u_{10}+u_{20})\left[\frac{1}{2\sqrt{7}},\frac{1}{2\sqrt{7}}\right]e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{7}}{2}t\right). \end{aligned}$$

Now choosing $u_{10}=u_{20}$, we eliminate the undesirable terms and therefore, we get asymptotic stability of the zero solution of (5.7.16).

5.8 Lyapunov-like Functions in $K_c(\mathbb{R}_+^d)$

Recall that in Section 2.3, a partial order in $(K_c(\mathbb{R}^n),D)$ is introduced and employing it, the existence of extremal solutions and a suitable comparison result were discussed. In this section, using the comparison result, we shall consider Lyapunov-like functions whose values are in $(K_c(\mathbb{R}_+^d),D)$. For this purpose, we shall also utilize the set differential systems developed in Section 3.7, and consequently, we consider the set differential system, namely,

$$D_HU=F(t,U),\ U(t_0)=U_0\in K_c(\mathbb{R}^n)^N, \tag{5.8.1}$$

where $F\in C[\mathbb{R}_+\times K_c(\mathbb{R}^n)^N,K_c(\mathbb{R}^n)^N],\ K_c(\mathbb{R}^n)^N=K_c(\mathbb{R}^n)\times K_c(\mathbb{R}^n)\times\cdots\times K_c(\mathbb{R}^n),\ N$ times. Since we need a partial order in $(K_c(\mathbb{R}^d),D)$, we shall assume that we have introduced it in a similar way. We shall begin by proving a comparison result in terms of Lyapunov-like functions whose values are in $K_c(\mathbb{R}_+^d)$. We also need a map,

$$\rho: K_c(\mathbb{R}^n)^N\to K_c(\mathbb{R}_+^d). \tag{5.8.2}$$

Theorem 5.8.1 *Assume that*

(i) $L\in C[\mathbb{R}_+\times K_c(\mathbb{R}^n)^N,K_c(\mathbb{R}_+^d)]$ *and whenever Hukuhara difference of* $A,B\in K_c(\mathbb{R}^n)^N$ *exists,*

$$L(t,A)\le L(t,B)+K\rho[A-B], \tag{5.8.3}$$

where $K\ge 0$ *is a local Lipschitz constant;*

(ii) $G\in C[\mathbb{R}_+\times K_c(\mathbb{R}_+^d),K_c(\mathbb{R}^d)]$, *the Hukuhara difference*

$$L(t+h,U(t+h))-L(t,U(t)) \tag{5.8.4}$$

exists, and for $t\in\mathbb{R}_+,\ U\in K_c(\mathbb{R}^n)$,

$$\begin{aligned} D^+L(t,U) &\equiv \limsup_{h\to 0^+}\frac{1}{h}[L(t+h,U+hF(t,U))-L(t,U)] \\ &\le G(t,L(t,U)). \end{aligned} \tag{5.8.5}$$

Then, if $U(t) = U(t, t_0, U_0)$ is any solution of (5.8.1) existing for $t \geq t_0$, such that $L(t_0, U_0) \leq R_0$, we have

$$L(t, U(t)) \leq R(t, t_0, R_0),\ t \geq t_0, \tag{5.8.6}$$

where $R(t, t_0, R_0)$ is the maximal solution of SDE

$$D_H R = G(t, R),\ R(t_0) = R_0 \in K_c(\mathbb{R}^d_+), \tag{5.8.7}$$

existing for $t \geq t_0$.

Proof Define $m(t) = L(t, U(t))$, where $U(t) = U(t, t_0, U_0)$ is any solution of (5.8.1) existing on $[t_0, \infty)$. Clearly, $m(t) \in K_c(\mathbb{R}^d_+)$ for each $t \in [t_0, \infty)$ and $m(t_0) \leq R_0$. Now for small $h > 0$,

$$\begin{aligned} m(t+h) &= L(t+h, U(t+h)) = L(t+h, U(t) + hF(t, U(t)) + o(h)) \\ &= L(t, U(t)) + Z(t, U(t); h) \end{aligned}$$

because of (5.8.4). It therefore follows, using (5.8.3), that

$$\begin{aligned} m(t+h) - m(t) &= Z(t, U(t); h) \\ &= L(t+h, U(t) + hF(t, U(t)) + o(h)) - L(t, U(t)) \\ &\leq L(t+h, U(t) + hF(t, U(t))) - L(t, U(t)) + K\rho(o(h)). \end{aligned}$$

Hence we arrive at

$$\begin{aligned} D^+ m(t) &\equiv \limsup_{h \to 0^+} \frac{1}{h}[m(t+h) - m(t)] \\ &\leq \limsup_{h \to 0^+} \frac{K\rho(o(h))}{h} \\ &\quad + \limsup_{h \to 0^+} \frac{[L(t+h, U(t) + hF(t, U(t)) - L(t, U(t))]}{h} \\ &\leq D^+ V(t, U(t)) \leq G(t, L(t, U(t))) = G(t, m(t)),\ t \geq t_0, \end{aligned}$$

in view of (5.8.5). This implies by Theorem 2.4.5,

$$L(t, U(t)) = m(t) \leq R(t, t_0, R_0), \quad t \geq t_0,$$

$R(t, t_0, R_0)$ being the maximal solution of (5.8.7). The proof is complete.

Employing the estimate (5.8.6), we can now discuss the desired qualitative properties of solutions $U(t)$ of set differential system (5.8.1). For this purpose, we need suitable notions of stability for SDEs (5.8.1) and (5.8.7).

Definition 5.8.1 *The trivial solution of (5.8.1) is said to be equi-stable if, given $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that*

$$D_0[W_0, \theta] < \delta \textit{ implies } D_0[U(t, t_0, W_0), \theta] < \epsilon,\ t \geq t_0,$$

where W_0 is chosen in such a way that $U_0 = V_0 + W_0$, that is, the Hukuhara difference $U_0 - V_0$ exists and is equal to W_0.

Definition 5.8.2 *The trivial solution of (5.8.7) is said to be equi-stable if given* $\epsilon > 0$ *and* $t_0 \in \mathbb{R}_+$, *there exists a* $\delta = \delta(t_0, \epsilon) > 0$ *such that*

$$\|Q_0\| < \delta \text{ implies } \|R(t, t_0, Q_0)\| < \epsilon, \ t \geq t_0,$$

where the Hukuhara difference $R_0 - S_0 = Q_0$ *is assumed to exist and* $\|R\| = \sup[\|r\| : r \in R \in K_c(\mathbb{R}^d_+)]$.

One can construct the other definitions of various stability and boundedness concepts based on the foregoing definitions. We shall now state a typical result on stability.

Theorem 5.8.2 *Assume that conditions (i), (ii) of Theorem 5.8.1 are satisfied. Suppose further that*

$$b(D_0[U, \theta]) \leq \|L(t, U)\| \leq a(D_0[U, \theta]), \ a, b \in \mathcal{K}. \tag{5.8.8}$$

Then the stability properties of the trivial solution of (5.8.7) imply the corresponding stability properties of the trivial solution of (5.8.1) respectively.

One can construct the proof of the theorem following the standard proofs employed already, once we have the estimates (5.8.6) and (5.8.8).

Since the comparison system in the present situation is also a SDE and not a scalar differential equation as before, it would be necessary to choose (when connecting the initial values of the two SDEs) $a(D_0[W_0, \theta]) = \|Q_0\|$ so that $\|L(t_0, W_0)\| \leq \|Q_0\|$ holds, which is required when we utilize the estimate (5.8.6).

The use of comparison SDE in $K_c(\mathbb{R}^d_+)$ provides a very general set up, which includes several possibilities. For example, if $R, G \in K_c(\mathbb{R}^d_+)$ in (5.8.7) are single valued maps, then as observed earlier, (5.8.7) reduces to ordinary differential system, and, consequently, there results the method of Vector Lyapunov-like functions.

Similarly, if $d = 1$, then the Lyapunov-like functions and the comparison system reduce to interval valued maps, which, of course, include as a special case, the usual Lyapunov-like method. There is a need to further explore this framework to get interesting results.

5.9 Set Differential Equations in $(K_c(E), D)$,

In Section 1.8, we provided the necessary background for the metric space $(K_c(E), D)$, where E is a Banach space. As we have seen, SDEs generated by the differential inclusions can be used as a tool to prove existence results of differential inclusions. The results of Section 4.4 offer such results in $(K_c(\mathbb{R}^n), D)$. In this Section, we shall only indicate some basic results similar to the ones proved in Chapter 2.

Let $\Gamma : J \times K_c(E) \to K(E) =\{$all nonempty compact subsets of $E\}$, and assume Γ is continuous on $J \times K_c(E)$, where $J = [t_0, t_0 + a]$. Consider the differential inclusion

$$x' \in \Gamma(t, x),\ x(t_0) = x_0 \in E. \tag{5.9.1}$$

Then we can define a mapping $F : J \times K_c(E) \to K_c(E),\ F(t, A) = \bar{co}\Gamma(t, A)$ where $A \in K_c(E)$. Since Γ is assumed to be continuous, we have F is continuous. We can now define SDE

$$D_H U = F(t, U),\ U(t_0) = U_0 \in K_c(E), \tag{5.9.2}$$

where, as before, $D_H U$ is the Hukuhara derivative.

One can prove several results parallel to the ones we have investigated for SDEs in the metric space $(K_c(\mathbb{R}^n), D)$. In fact, Tolstonogov [11] presents a systematic study of differential inclusions in a Banach space as well as SDEs generated by the differential inclusions. This reference has several general results for SDEs.

We shall therefore list a couple of results which can be proved almost parallel to the results considered already in $(K_c(\mathbb{R}^n), D)$. We begin with the following comparison result.

Theorem 5.9.1 *Assume that* $F \in C[J \times K_c(E), K(E)]$ *and for* $t \in J,\ U, V \in K_c(E)$,

$$D[F(t, U), F(t, V)] \le g(t, D[U, V]) \tag{5.9.3}$$

where $g \in C[J \times \mathbb{R}_+, \mathbb{R}_+]$. *Suppose further that the maximal solution* $r(t) = r(t, t_0, w_0)$ *of the scalar differential equation*

$$w' = g(t, w),\ w(t_0) = w_0 \ge 0, \tag{5.9.4}$$

exists on J. *Then, if* $U(t) = U(t, t_0, U_0),\ V(t) = V(t, t_0, V_0)$ *are any two solutions of (5.9.2) existing on* J, *we have the estimate*

$$D[U(t), V(t)] \le r(t, t_0, w_0),\ t \in J,$$

provided $D[U_0, V_0] \le w_0$.

The proof follows exactly as in the proof of Theorem 2.2.1 with suitable modifications and hence omitted. The following Corollary is immediate and is useful later.

Corollary 5.9.1 *Assume that* $F \in C[J \times K_c(E), K_c(E)]$ *and*

$$D[F(t, U), \theta] \le g(t, D[U, \theta]), \tag{5.9.5}$$

for $t \in J,\ U \in K_c(E)$, *where* g *satisfies the same assumptions as in Theorem 5.9.1. Then, if* $U(t) = U(t, t_0, U_0)$ *is any solution of (5.9.2) existing on* $J,\ D[U_0, \theta] \le w_0$ *implies*

$$D[U(t), \theta] \le r(t, t_0, w_0),\ t \in J,$$

$r(t, t_0, w_0)$ *being the maximal solution of (5.9.4) existing on J.*

We recall that $D[U,\theta] = \|U\| = \sup[\|u\| : u \in U],\ U \in K_c(E)$.

Next, we state an existence and uniqueness result under assumptions more general than the Lipschitz type condition, which provides the idea inherent in the comparison principle.

Theorem 5.9.2 *Assume that*

(i) $F \in C[R_0, K_c(E)]$ *where* $R_0 = J \times B(U_0, b)$, $B(U_0,b) = \{U \in K_c(E) : D[U,U_0] \le b\}$ *and* $D[F(t,U),\theta] \le M_0$ *on* R_0;

(ii) $g \in C[J \times [0,2b], \mathbb{R}_+]$, $g(t,w) \le M_1$ *on* $J \times [0,2b]$, $g(t,0) \equiv 0$, $g(t,w)$ *is monotone nondecreasing in* w *for each* $t \in J$, *and* $w(t) \equiv 0$ *is the only solution of*

$$w' = g(t,w),\ w(t_0) = 0 \ \text{on } J;$$

(iii) $D[F(t,U), F(t,V)] \le g(t, D[U,V])$ *for* $t \in J$, $U, V \in R_0$.

Then the successive approximations defined by

$$U_{n+1}(t) = U_0 + \int_{t_0}^{t} F(s, U_n(s))\, ds, \quad n = 0,1,2,\cdots, \tag{5.9.6}$$

exist on $J_0 = [t_0, t_0+\eta]$ *where* $\eta = \min(a, \frac{b}{M})$, $M = \max[M_0, M_1]$, *as a continuous function and converge to the unique solution* $U(t) = U(t,t_0,U_0)$ *of (5.9.2) on* J_0.

The proof of Theorem 5.9.2 proceeds similarly to the proof of Theorem 2.3.1, and, therefore, we omit the proof. We thus find that several results proved in $(K_c(\mathbb{R}^n), D)$ can be adapted to $(K_c(E), D)$, with additional conditions whenever necessary to match the Banach Space set up. For example, to prove Peano's theorem, we need to impose a suitable condition in terms of the measure of noncompactness and also employ the corresponding Ascoli-Arzela's theorem.

As we indicated earlier, there are several results in Tolstonogov [1], in this framework which connect differential inclusion in a Banach Space when the multifunction involved is not convex, as well as not continuous, and even when it is not compact, convex. Then the proofs of corresponding results become more complicated but can be constructed.

5.10 Notes and Comments

Section 5.2 begins by introducing impulses to SDEs and obtains basic results. This material is from Vasundhara Devi [1]. Section 5.3 extends the monotone iterative technique to impulsive SDEs and this is adapted from Vasundhara Devi and Vatsala [1]. In Section 5.4, delay is incorporated in the SDEs, and the fundamental results described are from Vasundhara Devi and Vatsala [1]. For ordinary differential equations with delay, refer to Lakshmikantham and Leela [3] and Hale [1]. For practical stability for ODE see Lakshmikantham, Leela and Martynyuk [1]. Monotone iterative technique for SDEs with delay was developed in McRae and Vasundhara Devi [2]. An interesting combination of impulses and delay was infused into the SDEs, and basic results were obtained in McRae and Vasundhara Devi [1] which form Section 5.5.

Section 5.6 deals with set difference equations, the material of which is compiled from Gnana Bhaskar and Shaw [1]. For the basic theory of difference equations see Lakshmikantham and Trigiante [1]. The results on differential equations with causal maps investigated in Section 5.7 are adapted from the papers of Drici, Mc Rae and Vasundhara Devi [1,2]. For a good reference for differential equations with causal operators, see Corduneanu [1]. See also Lakshmikantham and Rama Mohana Rao [1].

Also, for results in abstract spaces see Lakshmikantham and Leela [2]. Section 5.8 investigates the concept of Lyapunov-like functions, whose values are in metric spaces, which includes single, vector, matrix and cone valued Lyapunov functions as a special case. For details, see Lakshmikantham and Vasundhara Devi [1]. Finally, in Section 5.9 we indicate how one can extend most of the results obtained in the metric space $(K_c(\mathbb{R}^n), D)$ to the metric space $(K_c(E), D)$, where E is a Banach space. For details and more general results see Tolstonogov [1].

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