Differential Equations/Linear Algebra - MTH 2201/2202

Homework 1

Name: 01/15/2008

1. Given $y(t) = e^{-\frac{t}{2}}$. Then $y'(t) = -\frac{1}{2} e^{-\frac{t}{2}}$. Substituting $y(t)$ and $y'(t)$ in the given differential equation, we obtain

$$2y' + y = 2 \left( -\frac{1}{2} e^{-\frac{t}{2}} \right) + e^{-\frac{t}{2}} = 0.$$ 

2. Consider $x^2y + y^2 = 1$. Differentiating on both sides with respect to $y$, we get

$$2xy \frac{dx}{dy} + x^2 + 2y = 0 \Rightarrow 2xydx + (x^2 + 2y)dy = 0.$$ 

3. Given $y(t) = e^{-t^2} \int_0^t e^{u^2} du + c_1 e^{-t^2}$. Then,

$$y'(t) = e^{-t^2} e^{u^2} + (-2t)e^{-t^2} \int_0^t e^{u^2} du + c_1 e^{-t^2} (-2t) = 1 - 2te^{-t^2} \int_0^t e^{u^2} - c_1 e^{-t^2} (2t)$$

Substituting $y(t)$ and $y'(t)$ in the given differential equation, we obtain

$$y'(t) + 2ty = 1.$$ 

4. Given $x(t) = e^{-2t} + 3e^{6t}$ and $y(t) = -e^{-2t} + 5e^{6t}$. Then,

$$x'(t) = -2e^{-2t} + 18e^{6t}$$

and $y'(t) = 2e^{-2t} + 30e^{6t}$. Substituting $x(t)$,
$y(t), x'(t)$ and $y'(t)$, we get

\[
\frac{dx}{dt} = x + 3y, \quad \frac{dy}{dt} = 5x + 3y.
\]

5. If $y(t) = e^{it}$ then $y'(t) = ie^{it}$ and $y''(t) = -e^{it}$. Therefore,

\[
y''(t) + y(t) = 0
\]

is the differential equation with no real solutions.

6. No.

7. Given $y(t) = e^{-\sin t}$. Then $y'(t) = -\cos t \ e^{-\sin t}$. Substituting $y(t)$ and $y'(t)$ in the given differential equation, we obtain

\[
y'(t) + \cos t \ y(t) = 0.
\]

8. If $y(t) = e^{2it}$ then $y'(t) = 2i \ e^{2it}$ and $y''(t) = -4 \ e^{2it}$. Substituting $y''(t)$ and $y(t)$ in the given differential equation, we obtain

\[
y''(t) + 4 \ y(t) = 0.
\]

9. Given $y(t) = e^{2t}$. Then $y'(t) = 2 \ e^{2t}$. Substituting $y(t)$ and $y'(t)$ in the given differential equation

\[
y'(t) = 2y - t + g(y),
\]
we obtain, \( g(t) = t \).

10. The solution of differential equation \( y'(t) = y(t) \) is \( y(t) = C \ e^t \).

(i) If \( y(0) = y_0 \) then \( y(t) = y_0 e^t \).

(ii) If \( y(t_0) = y_0 \) then \( y(t) = y_0 e^{t-t_0} \).

11. \( \phi(t) \) is the solution of \( y' - y = 0 \) on \(( -\infty, \infty) \) satisfying \( \phi(0) = 1 \).

Therefore,

\[
\phi(t) = e^t. \tag{1}
\]

\( \phi(t + t_1) \) is the solution of \( y' - y = 0 \) satisfying \( y(0) = \phi(t_1) \) where \( t_1 \) is the real number. Therefore,

\[
\phi(t + t_1) = \phi(t_1) \ e^t. \tag{2}
\]

Substituting (1) in (2), we get,

\[
\phi(t + t_1) = \phi(t_1) \phi(t). \]

12. (i) \( dt - t^2 dy = 0 \Rightarrow \frac{1}{t^2} dt = dy \).

Integrating on both sides, we get

\[
y = \frac{-1}{t} + c.
\]
(ii) \( y \ln t \frac{dt}{dy} = \frac{(y+1)^2}{t} \). By separation of variables, we get
\[
t \ln t \, dt = \frac{(y + 1)^2}{y} \, dy.
\]
Integrating on both sides, we get
\[
\frac{t^2}{2} \ln t - \frac{t^2}{4} + c = \frac{y^2}{2} + 2y + \ln y.
\]

(iii) \( \sec^2 t \, dy + \csc y \, dt = 0 \). By separation of variables, we get
\[
- \sin y \, dy = \cos^2 t \, dt.
\]
Integrating on both sides
\[
\cos y + c_1 = \int \frac{\cos 2t + 1}{2} \, dt = \frac{1}{4} \sin 2t + \frac{t}{2} + c_2.
\]
Therefore,
\[
\cos y = \frac{1}{4} \sin 2t + \frac{t}{2} + c
\]
where \( c = c_2 - c_1 \).

(iv) \( \frac{dp}{dt} = p - p^2 \). By separation of variables, we get
\[
\frac{dp}{p(p - 1)} = dt.
\]
Integrating on both sides
\[
\ln p + \ln(1 - p) = t + c \Rightarrow p(1 - p) = ce^t.
\]
(v) \[
\frac{dy}{dx} = \frac{xy + 3x - y - 3}{xy - 2x + 4y - 8} \Rightarrow \frac{dy}{dx} = \frac{(y + 3)(x - 1)}{(y - 2)(x + 4)}.
\]

By separation of variables, we get

\[
\frac{(y - 2)}{(y + 3)} dy = \frac{(x - 1)}{(x + 4)} dx.
\]

Rearranging the terms, we have

\[
\left( \frac{y + 3}{y + 3} - \frac{5}{y + 3} \right) dy = \left( \frac{x + 4}{x + 4} - \frac{5}{x + 4} \right) dx.
\]

Integrating on both sides, we get

\[
y - 5 \ln |y + 3| = x - 5 \ln |x + 4| + c.
\]

\[
\Rightarrow y - x = 5 \left| \frac{y + 3}{x + 4} \right| + c.
\]

13. (i)

\[
(t - y)dt + tdy = 0. \quad (1)
\]

Let

\[
y = vt \quad \text{and} \quad \frac{dy}{dt} = v + t \frac{dv}{dt}. \quad (2)
\]

By separation of variables, we get

\[
\frac{dy}{dt} = \frac{y - t}{t}.
\]

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Substituting (2) in (1), we get

\[ dv = -\frac{dt}{t} \Rightarrow v = -\ln t + c_1 \Rightarrow y = -t \ln t + c_1 t. \]

(ii)

\[ \frac{dy}{dx} = \frac{y - x}{y + x}. \]  \hspace{2cm} (1)

Let

\[ y = vx, \quad \frac{dy}{dx} = v + x \frac{dv}{dx} \]  \hspace{2cm} (2)

Substituting (2) in (1), we get

\[ \frac{v + 1}{v^2 + 1} dv = -\frac{dx}{x} \Rightarrow \frac{1}{2} \frac{2v}{v^2 + 1} dv + \frac{1}{v^2 + 1} dv = -\frac{dx}{x}. \]

Integrating both sides,

\[ \frac{1}{2} \ln(v^2 + 1) + \tan^{-1}(v) = -\ln x + c_1. \]

Since \( v = \frac{y}{x} \),

\[ \Rightarrow \frac{1}{2} \ln \frac{y^2 + x^2}{x^2} + \tan^{-1} \frac{y}{x} = -\ln x + c_1. \]

(iii)

\[ (y^2 + ty) dt - t^2 dy = 0. \]  \hspace{2cm} (1)
Let
\[ y = vt, \quad \frac{dy}{dt} = v + t \frac{dv}{dt}. \]  
\[ (2) \]

Substituting (2) in (1), we get
\[ v^2 = t \frac{dv}{dt} \Rightarrow \frac{dt}{t} = \frac{dv}{v^2}. \]

Integrating on both sides, we get
\[ -\frac{1}{v} = \ln t + c_1 \Rightarrow y = -\frac{t}{\ln t + c_1}. \]

14. (i)

\[ x^2 \frac{dy}{dx} = y - xy, \quad y(-1) = -1 \]
\[ \Rightarrow x^2 \frac{dy}{dx} = y(1 - x) \]

By separation of variables,
\[ \frac{1}{y} \frac{dy}{x^2} = \frac{1 - x}{dx}. \]

Integrating on both sides,
\[ \ln y = -\frac{1}{x} - \ln x + c_1 \]
\[ \Rightarrow y(x) = e^{-\frac{1}{x}} \frac{1}{x} c \]
where $c = e^c$. From the initial condition, we have $c = \frac{1}{e}$. Therefore,

$$y(x) = \frac{1}{x} e^{-(\frac{1}{x}+1)}.$$

(ii) By separation of variables, we have

$$\frac{dy}{\sqrt{y}} = dx.$$

Integrating on both sides and rearranging the terms, we get

$$y(x) = \left( \frac{x + c}{2} \right)^2.$$

From the initial condition $y(x_0) = y_0$, we obtain

$$y(x) = \left( \frac{x + 2\sqrt{y_0} - x_0}{2} \right)^2$$

where $c = 2\sqrt{y_0} - x_0$. 

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