1. Let $f : D \to \mathbb{R}$, with $D = \{ x | x \text{ is irrational} \}$ and $f(x) = \frac{2x}{x-3}$. Evaluate $\lim_{x \to \infty} f(x)$.
Solution: We will prove that $\lim_{x \to \infty} f(x) = 2$. Let $\varepsilon > 0$ be given. We need to find $M > 0$ such that $\left| \frac{2x}{x-3} - 2 \right| < \varepsilon$ whenever $x \geq M$ and $x$ is irrational. But, $\left| \frac{2x}{x-3} - 2 \right| = \frac{3}{x-3}$ if $x > 3$, and $\frac{3}{x-3} < \varepsilon$ if $x > \frac{3}{\varepsilon} + 3$. Also, $\frac{3}{\varepsilon} + 3 > 3$ thus, pick $M$ to be any real number greater than $\frac{3}{\varepsilon} + 3$. Then, for any irrational $x$ such that $x \geq M$ we have $\left| \frac{2x}{x-3} - 2 \right| < \varepsilon$.

2. Give an example of a function $f$ that is unbounded on $\mathbb{R}$ but $\lim_{x \to \infty} f(x)$ is finite.
Solution: $f(x) = \frac{1}{x}$ defined on $(0, \infty)$.

3. Verify that $\frac{x-1}{x} < \frac{1}{x} \leq 1$, for $x > 0$. Evaluate $\lim_{x \to \infty} \frac{|x|}{x}$ and $\lim_{x \to -\infty} \frac{|x|}{x}$.
Solution: Since $|x|$ lies between the curves $y = x - 1$, and $y = x$, we have $x - 1 < |x| < x$. Thus, if $x > 0$, we have $\frac{x-1}{x} < \frac{|x|}{x} \leq 1$. If $x < 0$ then we have $\frac{x-1}{x} > \frac{|x|}{x} \geq 1$. Apply Sandwich theorem to evaluate the limits.

4. Consider the function $f(x) = \begin{cases} 2x - 1 & \text{if } x \text{ is rational} \\ 5 - x & \text{if } x \text{ is irrational} \end{cases}$
Prove that $(i) \lim_{x \to 2} f(x) = 3$ (ii) $\lim_{x \to -2} f(x)$ does not exist if $a \neq 2$.
This is a worked out example 3.2.9, on page 128-129, in the text.

5. Consider the Dirichlet’s function $f : (0, 1) \to \mathbb{R}$, defined by
$$ f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{in lowest terms with } p, q \in \mathbb{N} \\ 0 & \text{if } x \text{ is irrational} \end{cases} $$
Show that $\lim_{x \to a} f(x) = 0$ for every $a \in (0, 1)$.
This is a worked out example 3.2.10, on page 129, in the text.

6. Suppose that $f : D \to \mathbb{R}$, $a$ is an accumulation point of $D$, $\lim_{x \to a} f(x) = 0$, and $f(x) \neq 0$ for any $x \in D$ in some neighborhood of $a$. Prove that $\lim_{x \to a} \frac{1}{f(x)} = \infty$.
Use the definition of limits to show this. (See Theorem 2.3.6.)