Lecture 1

Differential Geometry

We begin with the study of geometry of curves and surfaces in three dimensional spaces. This topic deals with some of the most beautiful and useful results in mathematics and yet most of them can be understood if one has the background knowledge in Calculus.

A curve in $\mathbb{R}^2$ can be described as the set of points $C = \{ (x,y) \in \mathbb{R}^2 / f(x,y) = c \}$, where $c$ is a real number.

A curve in $\mathbb{R}^3$ might be defined by a pair of equations

$$f_1(x,y,z) = c_1, f_2(x,y,z) = c_2.$$  

curves of this type are called level curves.

**Definition 1:** A parametrized curve in $\mathbb{R}^n$, $n \geq 2$, is a map $\gamma : I \rightarrow \mathbb{R}^n$, for some interval $I \subset \mathbb{R}$.

**Examples:**

(i) For $y = x^2$ (parabola) A parametric representation is given by $\gamma(t) = (f_1(t), f_2(t))$, where

$$f_1(t) = t, \quad f_2(t) = t^2, \quad -\infty < t < \infty.$$

(ii) $x^2 + y^2 = 1$ (circle): $\gamma(t) = (\cos t, \sin t), \quad 0 \leq t \leq 2\pi$.

(iii) Helix: $\gamma(t) = (\cos t, \sin t, at), \quad \theta \leq t \leq a\pi$, for $a > 0$.

(iv) Logarithmic Spiral $\gamma(t) = (e^{t\cos}, e^{t\sin})$  

$-\infty < t < \infty.$
Definition: A parametrized curve \( \gamma: I \rightarrow \mathbb{R}^n \) is said to be smooth if for \( \gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t)) \), each of its components \( \gamma_i(t) \), \( i = 1, \ldots, n \), has derivatives of all orders. i.e. \( \gamma'_i(t), \gamma''_i(t), \ldots, \gamma^{(k-1)}_i(t) \) exist for all \( t \in I \).

Let \( C \) be a curve represented by \( \gamma(t) \). Let \( P = \gamma(t_o) \) and \( Q = \gamma(t_o + \Delta t) \) be two points on the curve. Then, \( P \) the vector \( \frac{\gamma(t_o + \Delta t) - \gamma(t_o)}{\Delta t} \) is parallel to the chord joining \( P \) and \( Q \). Clearly, \nabla \\
\[ \gamma'(t_o) = \lim_{\Delta t \to 0} \frac{\gamma(t_o + \Delta t) - \gamma(t_o)}{\Delta t}, \] if it exists and is not equal to the zero vector, represents the tangent vector to \( C \) at \( \gamma(t_o) \).

We assume hereafter that the curve \( C \) is smooth.

Arc Length

For \( a \in \mathbb{R}^n \), \( a = (a_1, \ldots, a_n) \), its magnitude \( \|a\| = \left( \sum_{i=1}^n a_i^2 \right)^{1/2} \).

By considering a partition of the curve \( C \), we approximate the length of each segment by the straight line joining the endpoints of the segment. This is \( \| \gamma'(t) \| \Delta t \). Summing over all the segments and taking the limit as the number of segments \( \to \infty \), we get...
the arc length to be given by
\[ L = \int_a^b \| r'(t) \| \, dt, \quad \text{if} \quad I = [a, b]. \]

The arc length of a curve \( r \) starting at \( r(t_0) \), till any point \( r(t) \), on the curve is given by
\[ s = s(t) = \int_{t_0}^t \| r'(u) \| \, du. \]

The arc length is negative or positive according as whether \( t > t_0 \) or \( t < t_0 \). Also, arc length depends on the starting point \( r(t_0) \).

If \( r : (a, b) \to \mathbb{R}^n \) is a parametrized curve, its speed at \( r(t) \) is \( \| r'(t) \| \).

\( r \) is said to be unit speed curve if \( \| r'(t) \| = 1 \) \( \forall t \).

If \( r \) is a curve with constant magnitude, then the position vector \( r(t) \) and \( r'(t) \) are orthogonal vectors. (i.e. \( r(t) \cdot r'(t) = 0 \).)

\[ \Rightarrow \text{If } r \text{ is a unit speed curve, then } r'' \text{ is zero or perpendicular to } r'. \]

**Definition 2**: A parametrized curve \( \tilde{r} : (\bar{a}, \bar{b}) \to \mathbb{R}^n \) is a reparametrization of a parametrized curve \( r : (a, b) \to \mathbb{R}^n \) if there exists a smooth bijective map \( \phi : (\bar{a}, \bar{b}) \to (a, b) \) such that \( \tilde{r}' : (a, b) \to (\bar{a}, \bar{b}) \) is also smooth and \( \tilde{r}(\xi) = r(\phi(\xi)) \), for all \( \xi \in (\bar{a}, \bar{b}) \).
Example: Let \( \gamma(t) = (\cos t, \sin t, at), \ a > 0, \ 0 \leq t \leq 2\pi \).

\[
\gamma'(t) = (-\sin t, \cos t, a)
\]

\[
\|\gamma'(t)\| = \sqrt{1 + a^2} \neq 0.
\]

\[
s(t) = \int_0^t \sqrt{1 + a^2} \, du = \sqrt{1 + a^2} \ t.
\]

i.e. \( s = \sqrt{1 + a^2} \ t \quad \Rightarrow \quad t = \frac{s}{\sqrt{1 + a^2}}.
\]

(\text{So, the mapping } \phi \text{ as referred to earlier, is,} \)

\[
\phi: [0, 2\sqrt{1 + a^2} \pi) \longrightarrow [0, 2\pi]
\]

\[
s \quad \longrightarrow \quad \frac{s}{\sqrt{1 + a^2}}.
\]

\[
\phi^{-1}: [0, 2\pi] \longrightarrow [0, 2\sqrt{1 + a^2} \pi]
\]

\[
t \quad \longrightarrow \quad \sqrt{1 + a^2} \ t.
\]

\[
\tilde{\gamma}(s) = \left( \cos \left( \frac{s}{\sqrt{1 + a^2}} \right), \sin \left( \frac{s}{\sqrt{1 + a^2}} \right), \frac{a}{\sqrt{1 + a^2}} \ s \right), \ s \in [0, 2\sqrt{1 + a^2} \pi].
\]

is a reparametrization of \( \gamma \). Infact, we have reparametrized in terms of arc length parameter.

Example 2 \( \gamma(t) = (e^t \cos t, e^t \sin t), \ t \in I \).

\[
\gamma'(t) = (e^t \cos t - e^t \sin t, e^t \sin t + e^t \cos t)
\]

\[
\|\gamma'(t)\|^2 = 2 e^{2t} \neq 0.
\]

\[
s = s(t) = \int_0^t e^u \, du = e^t - 1, \quad \Rightarrow \quad t = \ln \left( 1 + \frac{s}{\sqrt{2}} \right)
\]

\[
\tilde{\gamma}(s) = \left( \left(1 + \frac{s}{\sqrt{2}}\right) \cos \left( \ln \left( 1 + \frac{s}{\sqrt{2}} \right) \right), \left(1 + \frac{s}{\sqrt{2}}\right) \sin \left( \ln \left( 1 + \frac{s}{\sqrt{2}} \right) \right) \right).
\]

is a reparametrization.
Definition 3: A point \( \gamma(t) \) of a parametrized curve \( \gamma \) is called a regular point if \( \gamma'(t) \neq 0 \). Otherwise \( \gamma(t) \) is a singular point of \( \gamma \).

A curve \( \gamma \) is regular if all of its points are regular.

Observations:
1. Any reparametrization of a regular curve is regular.
2. If \( \gamma(t) \) is a regular curve, its arc length, starting at any point of \( \gamma \), is a smooth function of \( t \).
3. A parametrized curve \( \gamma \) has a unit speed reparametrization \( \iff \) it is regular. That is, if \( \tilde{\gamma}(u) \) is a reparametrization, \( \| \tilde{\gamma}'(u) \| = 1 \iff \gamma \) is regular.
4. For a regular curve \( \gamma \), if \( \tilde{\gamma}(u) \) is a unit speed reparametrization; i.e., \( \tilde{\gamma}'(u(t)) = \gamma(t) \times t \).

\( u \) is smooth. If \( s \) is the arc length of \( \gamma \), parameterized, then \( u = \pm s + c \), where \( c \) is a constant.
curvature

If \( \gamma \) is a unit speed curve with parameter \( s \), its curvature \( k(s) \) at the point \( \gamma(s) \) is defined to be \( \| \gamma'(s) \| \).

This definition of curvature is independent of a particular parametric form, since the only unit speed reparametrizations of \( \gamma \) are of the form \( \gamma(u) \), where \( u = \pm s + c \).

Then, by chain rule,

\[
\frac{dr}{ds} = \frac{dr}{du} \frac{du}{ds} = \pm \frac{dr}{du}
\]

\[
\frac{d^2r}{ds^2} = \frac{d}{ds}(\pm \frac{dr}{du}) = \frac{d^2r}{du^2}.
\]

Result: Let \( \gamma(t) \) be a regular curve. Then its curvature is

\[
k = \frac{||\gamma'' \times \gamma'\||}{||\gamma'\||^3}
\]

Example: curvature of the circular helix \( (a \cos \theta, a \sin \theta, b \theta) \), \(-\infty < \theta < \infty \) is \( \frac{1}{a^2 + b^2} \).

Plane Curves

Suppose that \( \gamma(s) \) is a unit speed curve in \( \mathbb{R}^2 \).

\[
\mathbf{T} = \frac{d\gamma}{ds} \ \ \text{unit tangent vector}.
\]

\( \mathbf{N}_s \) : Signed unit normal obtained by rotating \( \mathbf{T} \) by \( \pi/2 \), in the anticlockwise direction.

\[
\mathbf{T}'' = \gamma'' \perp \mathbf{T} \ \ \text{hence } \mathbf{T} \parallel \mathbf{N}_s.
\]

\[
: \gamma' = k_0 \mathbf{N}_s.
\]

\( k_0 \) \( \parallel \) signed curvature of \( \gamma \). \( k = 1k_0 \).
If $\vec{a}$ is any fixed vector, and $\phi(s)$ is the angle through which $\vec{a}$ should be rotated to be aligned with $T$. Then,

$$k = \frac{d\phi}{ds}.$$

**Space curves:**

Let $\gamma(s)$ be unit speed curve in $\mathbb{R}^3$.

$$T = \gamma'(s) \quad \text{unit tangent vector}$$

If the curvature $k(s)$ is non-zero, the principle normal of $\gamma$ at $\gamma(s)$ is

$$n(s) = \frac{1}{k(s)} T'(s).$$

Since $||T'(s)|| = k$, $n$ is a unit vector.

The vector $\vec{b} = T \times n$ is binormal vector.

$\{T, n, b\}$ orthonormal set.

$$\vec{b}' = T' \times n + T \times n' = T \times n'$$

$\Rightarrow \vec{b}'$ is $\perp$ to $T.$

Also, $\vec{b}'$ is $\perp$ to $\vec{b}$. (Why?)

$\Rightarrow \vec{b}'$ is parallel to $n$.

$$\vec{b} = -\tau \vec{n}$$

Torsion (defined only if the curvature is non-zero)

$$\tau = \frac{(\gamma' \times \gamma'') \cdot \gamma'''}{||\gamma' \times \gamma''||^2}.$$
Results:

1. If \( \gamma \) be a regular curve with nonvanishing curvature, then the image of \( \gamma \) is contained in a plane \( \iff \gamma' = 0 \).

2. \[
\begin{align*}
\pi' &= k \pi \\
\eta' &= -k \pi + \gamma \, b \\
b' &= -\tau \eta \\
\end{align*}
\]
   Frenet-Serret Formulas.

Global Properties of Curves:

Definition: Let \( a \in \mathbb{R}^+ \). A simple closed curve in \( \mathbb{R}^2 \) with period \( a \) is a (regular) curve \( \gamma: \mathbb{R} \to \mathbb{R}^2 \) such that \( \gamma(t) = \gamma(t + a) \iff t - t^* = ka \), for some integer \( k \).

Jordan Curve Theorem: Any simple closed curve in the plane has an interior and an exterior. i.e. the set of points that are not on the curve is the disjoint union of two subsets of \( \mathbb{R}^2 \), \( \text{int}(\gamma) \) and \( \text{ext}(\gamma) \),

\[
\begin{array}{ccc}
\uparrow & \text{bounded} & \text{unbounded} \\
\text{Connected}. & & \\
\end{array}
\]

3. If \( \gamma(t) = (x(t), y(t)) \) is a positively oriented simple closed curve (signed unit normal is towards the interior) with period \( a \), then
\[
\text{Area } (\text{int } \gamma) = \frac{1}{2} \int_0^a (x'y' - y'x') \, dx.
\]
4. **Isoperimetric Inequality.**

Let \( \gamma \) be a simple closed curve, let \( l(\gamma) \) be its length and \( A(\text{int} \\gamma) \) be the area of the interior. Then,

\[
A(\text{int} \\gamma) \leq \frac{1}{4\pi} [l(\gamma)]^2
\]

with equality holding \( \Leftrightarrow \gamma \) is a circle.

5. A **vertex** of a curve \( \gamma(t) \) in \( \mathbb{R}^2 \) is a point where its signed curvature \( k_s \) has a stationary point, i.e., \( \frac{dk_s}{dt} = 0 \).

A simple, closed curve is **convex** if its interior is convex.

**Four-Vertex Theorem:**

Every simple closed curve in \( \mathbb{R}^2 \) has at least four vertices.