Practice Problems - Legendre Polynomials and Bessel’s Functions

(1) The Legendre Polynomials are given by

\[ P_n(x) = \frac{1}{2^n} \sum_{k=0}^{m} \frac{(-1)^k (2n-2k)!}{k! (n-k)!} x^{n-2k}, \]

\( (n = 0, 1, \ldots) \) where \( m = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases} \)

Use this to establish the following properties of \( P_n(x) \):

(i) \( P_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2} \)

(ii) \( P'_{2n}(0) = 0 \)

(iii) \( P_{2n+1}(0) = 0 \)

(iv) \( P'_{2n+1}(0) = (2n+1)P_{2n}(0) \).

(2) Use Rodrigues’s formula to establish, for \( n = 0, 1, 2, \ldots \),

\[ P_n(1) = 1 \]

and \( P_n(-1) = (-1)^n \).

(3) Use Rodrigue’s formula to show that all the roots of \( P_n(x) \) lie in the interval \((-1, 1)\) and are distinct. (Use Rolle’s theorem)

(4) Use Rodrigue’s formula and find \( P_0(x), P_1(x), P_2(x) \).

(5) Let \( y_1(x) \) be a solution of the second order ODE \( y'' + p_1(x)y' + p_0(x)y = 0 \) then the method of variation of parameters gives the following formula for the second linearly independent solution:

\[ y_2(x) = y_1(x) \int_0^x \frac{1}{y_1(t)} \exp \left( -\int_0^t p_1(s) \, ds \right) \, dt. \]

Use this formula and establish that \( Q_n(x) = \frac{1}{2} \ln \frac{1+x}{1-x}, \quad Q_1(x) = \frac{\pi}{2} \ln \frac{1+x}{1-x} - 1. \)

(6) Show that \( x^2J''_n(x) = (n^2-n-x^2)J_n(x) + xJ_{n+1}(x), \quad n = 0, 1, 2, \ldots. \)

(7) Transform the differential equation \( x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + \left( \alpha^2 x^2 - n^2 \right) y = 0 \) into Bessel’s equation

\[ s^2 \frac{d^2y}{ds^2} + s \frac{dy}{ds} + (s^2 - n^2) y = 0 \]

for \( n = 0, 1, 2, \ldots. \) (Use the substitution \( s = \alpha x \)).