PROBLEMS 1.3

1. (a) Verify each product.
\[
\begin{bmatrix}
1 & -3 \\
2 & -7
\end{bmatrix}
\begin{bmatrix}
-2 & 4 \\
3 & 2
\end{bmatrix} = \begin{bmatrix}
0 & 19 \\
2 & 12
\end{bmatrix};
\begin{bmatrix}
0 & 3 \\
1 & 2
\end{bmatrix}
\begin{bmatrix}
2 & 4 \\
-1 & 2
\end{bmatrix} = \begin{bmatrix}
4 & 4 \\
1 & 4
\end{bmatrix};
\begin{bmatrix}
2 & 1 \\
3 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
4 & 1
\end{bmatrix} = \begin{bmatrix}
19 & 4 \\
12 & 4
\end{bmatrix}.
\]

(b) Evaluate the products.
\[
\begin{bmatrix}
-4 & 6 \\
2 & 0
\end{bmatrix}
\begin{bmatrix}
4 & 1 \\
-1 & 2
\end{bmatrix} = \begin{bmatrix}
5 & 3 \\
-2 & 5
\end{bmatrix};
\begin{bmatrix}
0 & -2 \\
2 & -3
\end{bmatrix}
\begin{bmatrix}
5 & 1 \\
3 & -1
\end{bmatrix} = \begin{bmatrix}
6 & 6 \\
2 & 2
\end{bmatrix};
\]

2. Evaluate each product.
\[
\begin{bmatrix}
1 & 6 \\
7 & 3
\end{bmatrix}
\begin{bmatrix}
2 & -1 \\
-4 & 2
\end{bmatrix} = \begin{bmatrix}
3 & -2 \\
2 & 0
\end{bmatrix};
\begin{bmatrix}
2 & 1 \\
3 & 1
\end{bmatrix}
\begin{bmatrix}
2 & -1 \\
0 & 3
\end{bmatrix} = \begin{bmatrix}
4 & -2 \\
6 & 3
\end{bmatrix};
\begin{bmatrix}
-2 & -2 \\
-3 & 2
\end{bmatrix}
\begin{bmatrix}
5 & 3 \\
2 & -6
\end{bmatrix} = \begin{bmatrix}
-6 & -12 \\
-6 & 6
\end{bmatrix};
\begin{bmatrix}
2 & 6 \\
3 & 5
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
2 & 7 \\
6 & 5
\end{bmatrix}.
\]

3. Compute \(AB\), \(AC\), \(B+C\), and compare \(AB=AC\) with \(A(B+C)\) for
\[
A = \begin{bmatrix}
1 & -3 & 2 \\
3 & 0 & 2
\end{bmatrix},
B = \begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix}, \text{ and } C = \begin{bmatrix}
-2 & 2 & 2 \\
1 & 3 & 0
\end{bmatrix}.
\]

4. Show that if the third row of \(A\) equals four times its first row, then the same holds for those rows of \(AB\) for all \(B\) for which the product is defined.

5. Show that if the third column of \(A\) equals two times its first column, then the same holds for those columns of \(BA\) for all \(B\) for which the product is defined.

6. Suppose that \(D = \text{diag}(d_1, \ldots, d_p)\). Show that \(DA\) is obtained from \(D\) just by replacing the \(i\)th row of \(A\) by \(d_i\) times it for all \(i\). State and show analogous result for the columns of \(AD\) for \(r \times p\) \(A\).

7. Suppose that \(A\) is \(p \times q\) and that \(r\) is a number. While \(rA\) is of course defined,show that the matrix product \([r]A\) makes sense if and only if \(p = 1\); show that if \(p = 1\), then \([r]A = rA\). State and verify the similar results concerning \(A\).

8. Find real \(2 \times 2\) matrices \(X\) and \(Y\) — neither of which equals \(0\) — for which \(XX + YY = 0\).

9. A product different from that in Definition 1.14 is important in many areas of science and engineering; it is called the cross product (or vector product), denoted by \(\times\) and defined as
\[
a \times b = [a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1],
\]

another \(1 \times 3\) row matrix.
(a) Show that \(a \times b = -b \times a\), so that \(\times\) is not commutative.
(b) Show that \(a \times a = 0\) for all \(a\).
(c) Show by example that \(\times\) is not associative.
(d) Show that there cannot exist an "identity matrix for \(\times\)," that is, that there cannot exist a special \(1 \times 3\) matrix \(e\) for which \(e \times a = a\) for all \(a\).

10. A product different from that in Definition 1.14 is important in many areas of science and engineering; it is defined between any two real \(1 \times 3\) row matrices. If \(A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}\) and \(B = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}\), then the Kronecker product (or tensor product) \(A \otimes B\) is defined as the \(pr \times qs\) matrix containing all possible products of one entry of \(A\) with one entry of \(B\), arranged in a special way: denoting \(\langle A \rangle_{ij}\) by \(a_{ij}\), the \(i\)th row of \(A \otimes B\) is created by writing down \(a_{13}B\) followed by \(a_{12}B\), and so on; and continues through the \(p\)th set of \(r\) rows. For example,
\[
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
\otimes
\begin{bmatrix}
5 & 6 & 7 \\
8 & 9 & 10
\end{bmatrix} =
\begin{bmatrix}
5 & 6 & 7 & 10 & 12 & 14 \\
8 & 9 & 10 & 16 & 18 & 20 \\
15 & 18 & 20 & 24 & 28 & 30 \\
24 & 27 & 30 & 32 & 36 & 40
\end{bmatrix}.
\]

(a) Show by example that \(\otimes\) is not commutative.
(b) Show by example that \(\otimes\) is not associative.
(c) Show that \([1] = \text{an "identity matrix" in the sense that } [1] \otimes A = A \text{ for all } A\).
11. A product different from that in Definition 1.14 that is useful primarily in connection with discrete Fourier analysis (see Problems 58 and 59) is defined entry-by-entry between any two \( p \times q \) matrices \( A \) and \( B \) by defining
\[
\langle A \odot B \rangle_{ij} = \langle A \rangle_{ij} \langle B \rangle_{ij}
\]
(a) Show that \( \odot \) is commutative.
(b) Show that \( \odot \) is associative.
(c) Find the \( p \times q \) "identity matrix" \( I \) so that \( B \odot A = A \odot B = A \) for all \( p \times q \) \( A \).

12. Find a matrix \( A \) and a matrix \( B \) so that \( AB \) and \( BA \) are both defined, but that \( AB \) and \( BA \) have the same shape, but so that \( AB \neq BA \).

13. Find two \( 2 \times 2 \) matrices \( A \) and \( B \) with \( AB \neq BA \) but with \( AB = BA \).

14. Suppose that \( A \) is a \( 2 \times 2 \) matrix that commutes with every \( 2 \times 2 \) matrix. Show that \( A \) must be a scalar multiple of \( I_2 \).

15. Compute \( A^2 \) and \( A^3 \) for
\[
A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}
\]

16. Verify that \( A(BC) = (AB)C \) if
\[
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}
\]

17. (a) Explicitly write out \( I_3 \).
(b) Show that \( I_3 A = A \) for all \( 2 \times 3 \) \( A \).

18. (a) Explicitly write out \( I_3 \).
(b) Show that \( B I_3 = B \) for all \( 2 \times 3 \) \( B \).

19. Use MATLAB or other software to discover experimentally what happens to \( A^k \) for large positive integers \( n \) if \( A \) is:
(a) \[
\begin{bmatrix} 0.6 & 0.5 \\ -0.2 & 1.2 \end{bmatrix}
\]
(b) \[
\begin{bmatrix} 0.6 & 0.5 \\ -0.1 & 1.2 \end{bmatrix}
\]
(c) \[
\begin{bmatrix} 0.6 & 0.5 \\ -0.1 & 1.2 \end{bmatrix}
\]
(d) \[
\begin{bmatrix} 0.9 & 1.0 \\ 0 & 0.9 \end{bmatrix}
\]
(e) \[
\begin{bmatrix} 0.99 & 1.0 \\ 0 & 0.99 \end{bmatrix}
\]

20. Determine, in terms of the real number \( r \), what happens to \( A^n \) for large positive integers \( n \) if
\[
A = \begin{bmatrix} r & 1 \\ 0 & r \end{bmatrix}
\]

21. Prove Theorem 1.19(b)—other than the one version proved in the text—on the distributive law.

22. Prove Theorem 1.19(e) on \( I \) as a multiplicative identity.

23. Prove Theorem 1.19(c) on scalar multiples of matrix products.
24. Prove Theorem 1.19(c) on matrix multiplication with 0.
25. Compute \( A^T, B^T, (AB)^T, (A^T B)^T \), and \( B^T A^T \) and verify that \((AB)^T = B^T A^T\).
\[
A = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}
\]

26. Evaluate:
(a) \[
\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 3 \\ 1 & 0 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 3 \\ 1 & 0 & 6 \end{bmatrix}
\]
(b) \[
\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 3 \\ 1 & 0 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 3 \\ 1 & 0 & 6 \end{bmatrix}
\]
(c) \[
\begin{bmatrix} -1 & 3 & 6 \\ 1 & 2 & 3 \\ -1 & 3 & 6 \end{bmatrix}\begin{bmatrix} -1 & 3 & 6 \\ 1 & 2 & 3 \\ -1 & 3 & 6 \end{bmatrix}
\]
(d) \[
\begin{bmatrix} -1 & 3 & 6 \\ 1 & 2 & 3 \\ -1 & 3 & 6 \end{bmatrix}\begin{bmatrix} -1 & 3 & 6 \\ 1 & 2 & 3 \\ -1 & 3 & 6 \end{bmatrix}
\]
27. Suppose that \( A \) is \( p \times p \) and \( x \) is \( p \times 1 \). Show that \( x^T A x = x^T (A^T A) x \). If \( x \)
28. Prove Theorem 1.12(a) on \((A^T)^T\) and \((A^T)^H\).
29. Prove Theorem 1.12(b) on \((A \pm B)^T\) and \((A \pm B)^H\).
30. Prove Theorem 1.12(c) on \((eA^T)^T\) and \((eA)^H\).
31. Prove the improved part of Theorem 1.12(d), that \((AB)^T = B^T A^T\).
32. Prove that \((ABC)^T = C^T B^T A^T\).
33. Prove the extension of Theorem 1.12(c) to apply to a product of an arbitrary number \( k \) of matrices \( A_1 A_2 \cdots A_k \).
34. Show that \( I_p \) and the \( n \times p \) zero matrix \( 0 \) are symmetric and are hermitian.
35. (a) Show that every diagonal matrix is symmetric.
(b) Show that the diagonal matrix \( D \) is hermitian if and only if all diagonal entries in \( D \) are real.
36. Explicitly write down the general \( 3 \times 3 \) symmetric matrix.
37. Explicitly write down the general \( 2 \times 2 \) hermitian matrix.
38. Show that \( AB \) need not be symmetric even though \( A \) and \( B \) are symmetric.

A = \[
\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & -2 & 1 \end{bmatrix}
\]
39. Show by example that \( AB \) can possibly be symmetric if \( A \) is symmetric and \( B \) is symmetric.
40. (a) Prove that \( A \) is symmetric if and only if \( A^T \) is symmetric.
(b) Prove that \( A \) is hermitian if and only if \( A^H \) is hermitian.
41. (a) Prove that if \( A \) is symmetric, then \( A^2 \) is symmetric.
(b) Prove that if \( A \) is hermitian, then \( A^2 \) is hermitian.
(c) What about the converse of (a)?
42. (e) Prove that if \( A \) is Hermitian, then \( A^3 \) is Hermitian.
(b) Prove that if \( A \) is hermitian, then \( A^n \) is hermitian for all positive integers \( n \).
(c) What about the converse of (a)?
43. Verify that $B^T A B$ is symmetric, in accordance with (1.25), if
\[
A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & -2 \\ -1 & 3 & 0 \end{bmatrix}.
\]

44. Prove the hermitian analogue of (1.25): $B^T A B$ is hermitian if $A$ is hermitian.

45. A matrix $A$ is said to be skew-symmetric if $A^T = -A$.

(a) Show that a skew-symmetric matrix must be square and that its entries on the main diagonal must be zeros.

(b) Show that, given any square matrix $A$, the matrix $A - A^T$ is skew-symmetric while the matrix $A + A^T$ is symmetric.

(c) Write $A = (A + A^T)/2 + (A - A^T)/2$ show that every square matrix can be uniquely written as the sum of a symmetric matrix and a skew-symmetric matrix.

46. If $x \in \mathbb{R}^n$, show that $x^T x$ is real (even if $x$ has complex entries).

47. Generalize Problem 45 to the case of skew-hermitian matrices $A$ for which $A^T = -A$.

48. By considering its real and pure-imaginary parts, prove that every hermitian matrix is the sum of a real symmetric matrix and $i$ times a real skew-symmetric matrix.

49. A product different from that in Definition 1.14 is important in many areas of science and engineering, especially in problems involving the processing of discrete signals; the convolution $x * y$ is defined between any two $1 \times p$ row matrices
\[
x = [x_1, x_2, \ldots, x_p] \quad \text{and} \quad y = [y_1, y_2, \ldots, y_p],
\]
and is itself a $p \times p$ row matrix:
\[
(x * y)_p = x_1 y_1 + x_2 y_2 + \ldots + x_p y_p
\]
\[
= x_1 y_p + x_2 y_{p-1} + \ldots + x_p y_1.
\]
For $p = 2$, this gives simply
\[
\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} x_1 y_1 + x_2 y_2 \\ x_1 y_2 + x_2 y_1 \end{bmatrix}.
\]

(a) Find the explicit formula for $[x_1, x_2, x_3] * [y_1, y_2, y_3]$.

(b) Show that $*$ is commutative for general $p$.

(c) Show that $*$ is associative for general $p$.

(d) Find a $1 \times p \times p$ "identity matrix" such that $e * x = x$ and $x * e = x$ for all $1 \times p$ row vectors.

50. An important technique in applied mathematics is to approximate a given function $f(x)$ by a Fourier series: a sum of terms of the form $sin(\lambda x)$ or cos $\lambda x$, or, equivalently, of the complex form $e^{i\lambda x} = \cos \lambda x + i \sin \lambda x$. Discrete Fourier analysis chooses the coefficients $c_k$ of $e^{i\lambda x}$ by requiring $f(x)$; that all $c_k$ except possibly some specific $N$ coefficients are zero, and (2) that the series and the original function $f$ are equal at $N$ equally-spaced values of $x$. With

$N = 3$, for example we might approximate $f(x)$ for $0 \leq x \leq 2\pi$ by $f(x) = a_0 + a_1 \cos x + a_2 \sin x$, with the coefficients $a_k$ determined by requiring $f(x) = f(x)$; $x = \pi, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \ldots, \frac{\pi}{2}$, and $x = \frac{\pi}{2} = \frac{\pi}{2}$. Let $f_k$ denote $f(x_k)$

(a) Show that these requirements on the $c_k$ can be expressed as
\[
f_1 = c_0 + c_1 + c_2,
\]
\[
f_2 = c_0 + c_1 + c_2^2
\]
\[
f_3 = c_0 + c_1 + c_2^3
\]
\[where z = e^{i2\pi/N} = \cos 2\pi/N + i \sin 2\pi/N\]
\[= -\frac{1}{2} - \frac{i}{2}.
\]
(b) Show that the equations above can be expressed as
\[
Z = \begin{bmatrix}
f_1 & f_2 & f_3 \end{bmatrix}
\]
\[
= [c_0, c_1, c_2]^T,
\]
\[\text{and}
\]
\[
Z = \begin{bmatrix} 1 & 1 & 1 \\ 1 & z & z^2 \\ 1 & z^2 & z^4 \end{bmatrix}.
\]
(c) Use the values of $z$ above explicitly to find $Z$.

(d) Let $\mathbb{Z}$ denote the vector of products defined in Problems 11 and 49. Show $Z(x * y)^T = Zx^T \mathbb{Z}y^T$. Firstlet $x = \begin{bmatrix} 3 & -1 & 2 \end{bmatrix}$ and $y = \begin{bmatrix} -2 & 1 & 0 \end{bmatrix}$ and then for all $1 \times 3$ row matrices $x$ and $y$.

51. Problem 50 examines discrete Fourier analysis with $N = 3$. Suppose generally that we wish to approximate $f(x)$ for $0 \leq x \leq 2\pi$ by a finite Fourier series $s(x)$ using $N$ terms:
\[
s(x) = c_0 + c_1 e^{i\phi} + c_2 e^{i2\phi} + \cdots + c_{N-1} e^{i(N-1)\phi}
\]
and that the $c_k$ are determined by requiring $s(x_j) = f(x_j)$ where we take $j = 0, 1, 2, \ldots, N$. Show that the problem of finding the $c_k$ from the given values $f(x_j)$ is the same as solving $Zc = f$,
\[
c_k = [c_0, c_1, \ldots, c_{N-1}]^T,
\]
\[f = [f_1, f_2, \ldots, f_N]^T,
\]
and $Z$ is the $N \times N$ matrix with $(Z)_{jk} = e^{i(j-1)(k-1)\phi}$ and $z = e^{i2\pi/N}$. The product $Zc$ is called the discrete Fourier transform of the row-matrix $c$ and is extremely important in a mathematics.

52. Let $Z$ be the $N \times N$ matrix that generates the discrete Fourier transform in Problem 51, and let $\mathbb{Z} = \mathbb{Z}d$ denote the products defined in Problems 11 and 49. Show that $Z(x * y)^T = Zx^T \mathbb{Z}y^T$ for all $1 \times N$ row matrices $x$ and $y$.

1.4 MATRIX INVERSES

The preceding sections extended to matrices many ideas associated with arithmetic on numbers; one idea omitted was the matrix analogue of inversion or reciprocal, computing $1/a$ from the nonzero real number $a$. 

where $\mathbb{Z} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & z & z^2 \\ 1 & z^2 & z^4 \end{bmatrix}$.