(1) In the following problems, (i) prove that $T$ is a linear transformation (ii) find bases for $N(T)$ and $R(T)$ (iii) find nullity $T$ and rank $T$ and verify the dimension theorem (iv) use the results discussed in the lecture to determine whether $T$ is one to one or onto.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2, a_3) = (a_1 + a_2, 0, 2a_1 - a_2)$.
(b) $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T(f(x)) = xf(x) + f'(x)$.
(c) $T : M_{n \times n}(F) \to F$ defined by $T(A) = tr(A)$.

(2) Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear, $T(1, 0) = (1, 4)$, and $T(1, 1) = (2, 5)$. What is $T(2, 3)$? Is $T$ one to one?

(3) Prove that there is a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1, 1) = (1, 0, 2)$ and $T(2, 3) = (1, -1, 4)$. What is $T(8, 11)$?

(4) Let $T : V \to W$ be a linear transformation. Prove that $T$ is one to one if and only if $T$ carries linearly independent subsets of $V$ onto linearly independent subsets of $W$.

(5) Let $T : V \to W$ be a linear transformation between finite dimensional vector spaces $V$ and $W$. Prove that if $\dim(V) > \dim(W)$, then $T$ cannot be one to one.

(6) Give an example of distinct linear transformations $T$ and $U$ such that $N(T) = N(U)$ and $R(T) = R(U)$.

(7) Let $T : V \to V$ be linear. Prove that the subspaces $R(T), N(T)$ are all $T$- invariant.

(8) Suppose that $V = R(T) \oplus W$ where $W$ is a $T$- invariant subspace of $V$. Prove that $W \subseteq N(T)$. 