Let \( B_1 = \{ u_1, \ldots, u_k \} \) be a basis for \( W_1 \) and \( B_2 = \{ v_1, \ldots, v_k \} \) be a basis for \( W_2 \).

Let \( S = B_1 \cup B_2 = \{ u_1, \ldots, u_k, v_1, \ldots, v_k \} \).

Pick \( w \in W_1 + W_2 \). Then, \( w = u_1 + v_1, \ldots, u_k + v_k \in W_1 + W_2 \).

\[ w = \sum_{i=1}^{k} a_i u_i + \sum_{i=1}^{k} b_i v_i \]

\( \Rightarrow w \) is expressed as some linear combination of elements of \( S \).

\( \Rightarrow S \) is a generating set of \( W_1 + W_2 \).

Since any generating set must contain a basis, and \( S \) is a finite set, we have that \( W_1 + W_2 \) has a finite basis. \( \Rightarrow W_1 + W_2 \) is finite-dimensional.

To show \( \dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) \)

Let \( B^* = \{ w_1, \ldots, w_p \} \) be a basis for \( W_1 \cap W_2 \).

Extend \( B^* \) to bases \( B_1^* \) of \( W_1 \) and \( B_2^* \) of \( W_2 \).

That is, \( B_1^* = \{ w_1, \ldots, w_p, u_{p+1}, u_{p+2}, \ldots, u_k \} \) and \( B_2^* = \{ w_1, \ldots, w_p, v_{p+1}, v_{p+2}, \ldots, v_k \} \).

\( B_1 \rightarrow \) basis of \( W_1 \), \( B_2 \rightarrow \) basis of \( W_2 \).

\[ B = B_1 \cup B_2^* = \{ w_1, \ldots, w_p, u_{p+1}, \ldots, u_k, v_{p+1}, \ldots, v_k \} \]

Claim: \( B \) is LI.

Clearly, \( w_1, \ldots, w_p, u_{p+1} \ldots \) are LI.

If possible, let \( v_j \), for some \( j = p+1, \ldots, k \), be LD on the remaining elements.
\[ v_j = \sum_{i=1}^{p} a_i w_i + \sum_{i=p+1}^{k} b_i u_i \]

\[ \Rightarrow v_j - \sum_{i=1}^{p} a_i w_i = \sum_{i=p+1}^{k} b_i u_i \]

\[ \text{in } W_2 \]

\[ \text{in } W_1 \]

\[ \Rightarrow \text{the elements on LHS, RHS are in } W_1 \cap W_2. \]

\[ \Rightarrow v_j = \sum_{i=1}^{p} a_i w_i \]

\[ \Rightarrow v_j \text{ is LD on } W_1, \ldots, W_p. \]

This is a contradiction.

Hence, none of \( v_j, j=p+1, \ldots, k \) can be dependent on \( w_1, \ldots, w_p, u_{p+1}, \ldots, u_k \).

\[ \Rightarrow B \text{ is LI}. \]

Claim: \( B \) is a generating set of \( W_1 + W_2 \).

\[ w \in W_1 + W_2, \quad w = w_1 + w_2 \]

\[ = \left( \sum_{i=1}^{p} a_i w_i + \sum_{i=p+1}^{k} b_i u_i \right) + \left( \sum_{i=1}^{p} a_i w_i + \sum_{i=p+1}^{k} b_i u_i \right) + \sum_{i=p+1}^{k} \delta_i v_i \]

\[ = \sum ( \gamma_i + \gamma_i ) w_i + \sum \beta_i u_i \]

\[ + \sum \delta_i v_i \]

Hence the claim.

Hence the result.