Practice Problems

(1) Determine if the vector \((-2,0,3)\), can be expressed as a linear combination of the vectors \((1,3,0)\) and \((2,4,-1)\).

**Solution:** \((-2,3,0) = 4(1,3,0) + (-3)(2,4,-1)\).

(2) Determine if the polynomial \(2x^3 - 2x^2 + 12x - 6\) can be expressed as a linear combination of \(x^3 - 2x^2 - 5x - 3\) and \(3x^3 - 5x^2 - 4x - 9\).

**Solution:** \(2x^3 - 2x^2 + 12x - 6 = (-4)(x^3 - 2x^2 - 5x - 3) + 2(3x^3 - 5x^2 - 4x - 9)\). (Of course, you have to "find" a solution of a suitable system of equations to come up with this answer!

(3) Determine if the polynomial \(3x^3 - 2x^2 + 7x + 8\) can be expressed as a linear combination of \(x^3 - 2x^2 - 5x - 3\) and \(3x^3 - 5x^2 - 4x - 9\).

**Solution:** Start with \(3x^3 - 5x^2 - 4x - 9 = a(x^3 - 2x^2 - 5x - 3) + b(3x^3 - 5x^2 - 4x - 9)\) and try to find a solution of the system of equations. Show that there are no such \(a\) and \(b\).

(4) Determine if the matrix \(\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}\) is in the span of

\[
S = \{A = \begin{pmatrix} 1 & 0 \\ -1 & 4 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\}.
\]

**Solution:** Note that \(3A + 4B - 2C = \begin{pmatrix} 1 & 2 \\ -3 & 16 \end{pmatrix}\). Thus, the given matrix is not in the span of \(A, B, C\).

(5) Prove that \(\text{span}\{x\} = \{ax | a \in F\}\).

**Solution:** Follows from the definition.

(6) Show that a subset \(W\) of a vector space \(V\) is a subspace of \(V\) if and only if span \(W = W\).

**Solution:** If \(\text{span} W = W\) then \(W\) contains all linear combinations of its elements. That is, if \(x, y \in W\) then (i) \(x + y,\) (ii) \(0 \in W,\) and (iii) \(αx ∈ W\) for \(α \in F\). Hence \(W\) is a subspace of \(V\). Conversely, if \(W\) is a subspace of \(V\) then for any subset \(\{u_1, ..., u_n\}\) in \(W, α_1u_1 + ... + α_nu_n \in W\) for every \(α_1, ..., α_n \in F\). (why?) Thus, \(\text{span} W \subset W\). Hence the result.

(7) Show that if \(S_1, S_2\) are subsets of a vector space \(V\) such that \(S_1 \subseteq S_2,\) then \(\text{span}(S_1) \subseteq \text{span}(S_2)\). If \(S_1 \subseteq S_2\) and \(\text{span}(S_1) = V,\) deduce that \(\text{span}(S_2) = V\).

**Solution:** If \(u \in \text{span} S_1\). Then, \(u = \sum_{i=1}^{m} α_iu_i\) for \(u_i \in S_1,\) \(α_i \in F\). But, if \(u_i \in S_1\) then \(u_i \in S_2\). Thus The RHS is also in \(\text{span} S_2\). That implies, \(U \in \text{span} S_2\). To prove the second part, start with \(v \in V\). Since, \(S_1 \subset S_2,\) from the first part it follows \(\text{span}S_1 \subset \text{span}S_2\). This means \(V \subseteq \text{span} S_2\). Hence, \(V = \text{span} S_2\).

(8) Show that \(S_1\) and \(S_2\) are arbitrary subsets of a vector space \(V\) then \(\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)\).

**Solution:** Let \(u \in \text{span}(S_1 \cup S_2),\) then

\[
u = \sum_{i=1}^{m} α_iu_i + \sum_{i=1}^{n} β_iv_i
\]

for \(u_i \in S_1, i = 1, ..., m\) and \(v_i \in S_2, i = 1, ..., n,\) and \(α_i, β_i \in F\). Then, \(u \in \text{span}S_1 + \text{span}S_2\).
Let $S_1$ and $S_2$ be subsets of a vector space $V$. Prove that $(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$. Give an example in which $\text{span}(S_1 \cap S_2)$ and $\text{span}(S_1) \cap \text{span}(S_2)$ are equal and one in which they are not equal.

**Solution:** Start with an element in $S_1 \cap S_2$ and use the fact that $S \subseteq \text{span}(S)$, for a subset $S$ of $V$. When $S_1 = S_2$ clearly, $\text{span}(S_1 \cap S_2) = \text{span}(S_1) \cap \text{span}(S_2)$. (For an explicit example, start with two sets each containing a single (different) element of $\mathbb{R}^2$. Check the result.

Let $S_1 = \{(1,0),(0,1)\}$ and $S_2 = \{(1,1)\}$. $\text{span}(S_1) = \mathbb{R}^2$. $\text{span}(S_2) = \{(x,y)|y = x\}$; clearly $\text{span}(S_1) \cap \text{span}(S_2) = \text{span}(S_2)$. Since, $S_1 \cap S_2 = \emptyset$, $\text{span}(S_1 \cap S_2) = \{0\}$.