Definition: Let \( f : (a, b) \to \mathbb{R} \) be a piecewise continuous function. If the derivative \( f' \) is also a piecewise continuous function on \((a, b)\), \( f \) is said to be \textit{piecewise smooth} on \((a, b)\).

Theorem: If a function \( f \) is PWS on \((a, b)\) then at each point \( x_0 \) in \([a, b]\) the one sided derivatives of \( f \), from the interior at the end points, exist and are the same as the corresponding one sided limits of \( f' \):

\[
\left\{ \begin{array}{l}
\left. f'_R(x_0) \right|_{x=x_0} = f'_1(x_0+),
\left. f'_L(x_0) \right|_{x=x_0} = f'_1(x_0-).
\end{array} \right.
\]

Bessel's Inequality:

\[
\frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 \leq \frac{2}{\pi} \int_{0}^{\pi} [f(x)]^2 \, dx \quad N=1, 2, \ldots
\]

Consequence: \( \lim_{n \to \infty} a_n = 0. \)

Fourier Theorem: Let \( f : \mathbb{R} \to \mathbb{R} \) be a function such that \( f \) is

(i) PWC on \((-\pi, \pi)\)

(ii) periodic with period \(2\pi\), on the entire \( x \) axis.

The Fourier series

\[
\frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right)
\]

with coefficients

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx
\]
Converges to the mean value \( \frac{f(x^+) + f(x^-)}{2} \)

of the one-sided limits of \( f \) at each point \( x \) \((-\infty, x < \infty)\).

where both the one-sided derivatives \( f'_R(x), f'_L(x) \) exist.

**Corollary** Let \( f \) be a PWS function on \((-\pi, \pi)\).

Let \( F \) denote the periodic extension, with period \( 2\pi \), of \( f \). At each point \( x \) \((-\infty, x < \infty)\) the Fourier series for \( f \) on \((-\pi, \pi)\) converges to the mean value of the one-sided limits of \( F(x^+) \) and \( F(x^-) \), namely

\[
\frac{F(x^+) + F(x^-)}{2}
\]

**Remark:** If \( f \) and \( F \) are function in the above Corollary, then \( F(x^+) = f(x^+) \) and \( F(x^-) = f(x^-) \) on \(-\pi < x < \pi\).

\( \therefore \) Fourier series for \( f \) on \((-\pi, \pi)\) converges to the number

\[
\frac{f(x^+) + f(x^-)}{2}
\]

\( \therefore \) If \( f \) is continuous at \( x \), then to \( f(x) \).

At the end points \( x = \pm \pi \), the series converges to

\[
\frac{f(-\pi^+) + f(\pi^-)}{2}
\]

**Example**

\[
f(x) = \begin{cases} 
0 & -\pi < x \leq 0 \\
x & 0 < x < \pi .
\end{cases}
\]

\[
\frac{\pi}{4} + \sum_{h=1}^{\infty} \frac{(-1)^{h-1}}{\pi h^2} \cos hx + \frac{(-1)^{h+1}}{h} \sin nx
\]
Example: \( f(x) = \sin x, \quad 0 < x < \pi \)

\( f \) is PWS on \((0, \pi)\) and continuous on \([0, \pi]\)

\[
\sin x = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1} \quad \text{on} \quad [0, \pi].
\]

Example

\[ f(x) = 3 \sqrt{x}, \quad -\pi < x < \pi \]

\( f \) is PWC on \((-\pi, \pi)\), but \( f'(x) = \frac{1}{3} \frac{1}{x^{1/2}} \)

when \( x \to 0 \), and \( f'(0^+), f'(0^-) \) do not exist.

\( \therefore f \) is not PWS on \(-\pi < x < \pi\).

Lemma: Let \( f \) denote a function such that

(i) \( f \) is continuous on \(-\pi \leq x \leq \pi\)

(ii) \( f(-\pi) = f(\pi) \)

(iii) \( f' \) is PWC on \(-\pi < x < \pi\)

If \( a_n, b_n \) are its Fourier coefficients, then the series \( \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \) converges.

Definition: Let \( S(x) \) denote the sum of an infinite series of function \( f_n(x) \), where the series is convergent for all \( x \)
in some interval \([a, b]\).

i.e.

\[ S(x) = \sum_{n=1}^{\infty} f_n(x) = \lim_{N \to \infty} S_N(x). \]

where \( S_N(x) = \sum_{n=1}^{N} f_n(x) \).

Pointwise convergence.
Definition (Uniform Convergence)

If for every $\varepsilon > 0$, there exist an $N_\varepsilon$, independent of $x$, such that

$$|s_n(x) - s(x)| < \varepsilon$$

whenever $N > N_\varepsilon$, $a \leq x \leq b$.

Weierstrass M-Test

Suppose that $\sum_{n=1}^{\infty} M_n$ is a convergent series of real numbers where $M_n$ are such that

$$|f_n(x)| \leq M_n, \quad a \leq x \leq b,$$

for each $n$.

Then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[a, b]$.

Properties: (of a uniformly convergent series)

1. If $f_n$ are continuous functions and if $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent, then the sum $s(x)$ is also a continuous function.

2. The series can be integrated term-wise over $[a, b]$.

   i.e.,

   $$\int_{a}^{b} s(x) \, dx = \sum_{n=1}^{\infty} \int_{a}^{b} f_n(x) \, dx$$

3. If $f_n$ and their derivatives $f_n'$ are continuous, and if $\sum_{n=1}^{\infty} f_n(x)$ converges, and $\sum_{n=1}^{\infty} f_n'(x)$ is uniformly convergent, then

   $$s'(x) = \sum_{n=1}^{\infty} f_n'(x).$$
Theorem: (Uniform Convergence of Fourier Series)

Let $f$ be a function with the following properties:

(i) $f$ is continuous on $[-\pi, \pi]$

(ii) $f(-\pi) = f(\pi)$

(iii) $f'$ is P.W.C. on $(-\pi, \pi)$

Then, the F.S. \( \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \),

for $f$, with F.C.

\( a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \),

\( b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \),

converges absolutely and uniformly to $f(x)$ on $[-\pi, \pi]$.

Example

$\ f(x) = x, -\pi < x < \pi$

has a (point-wise) convergent F.S. given by

\[ x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx, \quad -\pi < x < \pi. \]

Differentiating series we get

\[ 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cos nx \]

which does not converge.
Theorem: Let \( f \) denote a function such that

(i) \( f \) is continuous on \([-\pi, \pi]\)

(ii) \( f(-\pi) = f(\pi) \)

(iii) its derivative \( f' \) is PWG on \((-\pi, \pi)\).

Then, the F.S. representation

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx, \quad -\pi \leq x \leq \pi
\]

where

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx
\]

is differentiable at each point \( x \) in \((-\pi, \pi)\) at which the second order derivative \( f'' \) exists.

\[
f'(x) = \sum_{n=1}^{\infty} n \left( -a_n \sin nx + b_n \cos nx \right).
\]

Theorem: Let \( f \) be a function that is PWG on \((-\pi, \pi)\). Regardless of whether the series

\[
\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx
\]

converges or not, the following equation is valid on \(-\pi \leq x \leq \pi\):

\[
\int_{-\pi}^{\pi} f(x) \, dx = \frac{a_0}{2} (x+\pi) + \sum_{n=1}^{\infty} \frac{1}{n} \left[ a_n \sin nx - b_n \left( \cos nx + (-1)^n \right) \right]
\]

Example

\[
\cosh ax = \frac{\sinh a\pi}{a\pi} \left[ 1 + 2e^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2} \cos nx \right]
\]

\[
a \sinh ax = \frac{2a \sinh a\pi}{\pi} \left[ \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2} \left( -n \sin nx \right) \right]
\]

\[
a \cosh ax = \frac{2a \sinh a\pi}{a^2+n^2} \left[ \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2} \left( -n \sin nx \right) \right]
\]

\[
a \sinh ax = \frac{2a \sinh a\pi}{\pi} \left[ \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2} \left( -n \sin nx \right) \right]
\]