Homework 1

1. Verify that \( y(t) = e^{-t/2} \) is a solution of the DE \( 2y' + y = 0 \).

2. Verify that \( x^2y + y^2 = 1 \) is a solution of the DE \( 2xydx + (x^2 + 2y)dy = 0 \). (Assume an appropriate interval of definition.)

3. Let \( y(t) = e^{-\int_0^t e^{u^2} du + c_1 e^{-t^2}} \), where \( c_1 \) is an arbitrary constant. Show that \( y(t) \) is a solution of \( y' + 2ty = 1 \).

4. Verify that \( x(t) = e^{-2t} + 3e^{6t}, \ y(t) = -e^{-2t} + 5e^{6t} \) is a solution of the system of DE on \((-\infty, \infty)\)
\[
\frac{dx}{dt} = x + 3y \quad \frac{dy}{dt} = 5x + 3y.
\]

5. Make up a DE that does not have any real solutions.

6. Consider \( t(y')^2 - 4y' - 12t^3 = 0 \). Can we write it in the form \( \frac{dy}{dt} = f(t, y) \).

7. Verify that \( y(t) = e^{-\sin t} \) is a solution of \( y' + (\cos t)y = 0 \).

8. Verify that \( y(t) = e^{2it} \) is a solution of \( y'' + 4y = 0 \).

9. Make up a DE of the form \( \frac{dy}{dt} = 2y - t + g(y) \), that has \( y(t) = e^{2t} \) as a solution.

10. Solve the ODE \( y' = y \). Describe in graphical terms the difference between the solutions of the IVPs \( y' = y, \ y(0) = y_0 \) and \( y' = y, \ y(t_0) = y_0, y_0 \neq 0 \).

11. Consider the ODE \( y' - y = 0 \). Let \( \phi(t) \) be its solution on \((-\infty, \infty)\), such that \( \phi(0) = 1 \). Verify that \( \phi(t_1 + t) \), for some real number \( t_1 \), is a solution of \( y' - y = 0, y(0) = \phi(t_1) \). Use this and show \( \phi(t_1 + t) = \phi(t_1)\phi(t) \).

12. Solve the following ODEs by separation of variables:

   (i) \( dt - t^2 dy = 0 \) \hspace{1cm} (ii) \( y \ln t \frac{dt}{dy} = \frac{(y + 1)^2}{t} \).

   (iii) \( \sec^2 t dy + \csc y dt = 0 \) \hspace{1cm} (iv) \( \frac{dP}{dt} = P - P^2 \).

   (v) \( \frac{dy}{dx} = \frac{xy + 3x - y - 3}{xy - 2x + 4y - 8} \)
13. Solve the following IVPs
   (i) \( x^2 \frac{dy}{dx} = y - xy, \quad y(-1) = -1. \)
   (ii) \( y' = \sqrt{y}, \quad y(t_0) = y_0(> 0). \) What can you say if \( y_0 < 0. \)

14. Solve the homogeneous ODEs
   (i) \( (t - y) dt + t dy = 0. \)
   (ii) \( \frac{dy}{dx} = \frac{y - x}{y + x}. \)
   (iii) \( (y^2 + ty) dt - t^2 dy = 0 \)

15. Sketch in the \( xy \)-plane, the regions for which the ODE
   \( (1 - x^2)\left(\frac{dy}{dx}\right)^2 = 1 - y^2 \) has real solutions. Find solutions of the ODE in these regions. Find singular solutions of the equation.