Homework 1: Matrices

1. How many $3 \times 3$ matrices $A$ can you find such that $A$ has constant entries and

\[
A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y \\ x - y \\ 0 \end{bmatrix}
\]

2. Let $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 5 \\ -2 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 8 & -3 & -5 \\ 0 & 1 & 2 \\ 4 & -7 & 6 \end{bmatrix}$, $C = \begin{bmatrix} 0 & -2 & 3 \\ 1 & 7 & 4 \\ 3 & 5 & 9 \end{bmatrix}$.

(a) Find a matrix $X$ that satisfies the equation, $B + (A + X)^T = C$.

(b) Find a matrix that satisfies the equation $B + (\text{tr}(A)X)^T = C$.

(c) (i) Find the first row vector of $AB$ (ii) second column vector of $AB$.

(d) If $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, write out the equations $Ax = b$.

3. Show, by giving an example, that for square matrices, $(A + B)(A - B)$ need not be equal to $A^2 - B^2$.

4. Let

\[
A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix}
\]

(a) A matrix $S$ is said to be a square root of a matrix $M$ if $SS = M$. Find two square roots of $A$.

(b) How many different square roots of $B$ can you find?.

(c) Do you think that every matrix has a square root? Explain your reasoning.

5. Find the inverse of $A$. (i) $A = \begin{bmatrix} 1 & 5 \\ 2 & 20 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$
6. Find the reduced row echelon form $R$ of the matrix $A$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

7. $A = \begin{bmatrix} c & c & c \\ 1 & c & c \\ 1 & 1 & c \end{bmatrix}$, find all values of $c$, if any, for which $A$ is invertible.

8. $A = \begin{bmatrix} 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & 0 \\ k_4 & 0 & 0 & 0 \end{bmatrix}$, state conditions on the constants under which $A$ will be invertible, and find $A^{-1}$.