**PURPOSE:** EXAMINE OPERATION OF COMBUSTION CHAMBER IN A LIQUID H₂-O₂ ROCKET

![Diagram of combustion chamber]

**INJECTION, ATRIONIZATION, VAPORIZATION**

**RAPID COMBUSTION ZONE**

**STREAMLINE COMBUSTION ZONE**

**ENTIRE REGION IS SUBSONIC FLOW**

**CHEMICAL REACTIONS MAY OR MAY NOT CONTINUE TO TAKE PLACE.**

**NOTE:** THE PRESSURE IS DETERMINED BY THE MASS FLOW THROUGH THE NOZZLE THROAT AREA.

\[
\dot{m} = \frac{P_c A*}{C*(T_c)}
\]

IN THIS PROBLEM, WE TAKE \( P_c = 100 \text{ ATM} \).

THE OVERALL GOVERNING EQUATION IS:

\[ H_2 + MO_2 \rightarrow \alpha H_2O + \beta H_2 + \delta OH + \gamma H + \epsilon O_2 + \eta O \]

IF THE REACTANTS BEGAN AT 300K:

\[ 0 = \alpha \left[ h_{H_2O}(T_c) + \Delta H_{f, H_2O}^o \right] + \beta \left[ h_{H_2}(T_c) \right] + \delta \left[ h_{OH}(T_c) + \Delta H_{f, OH}^o \right] + \gamma \left[ h_{H}(T_c) + \Delta H_{f, H}^o \right] + \epsilon \left[ h_{O_2}(T_c) \right] + \eta \left[ h_{O}(T_c) + \Delta H_{f, O}^o \right] \]
WE ARE INTERESTED IN THE FUEL RICH CASE ($0 < \frac{1}{a} < \frac{1}{3}$)
IN THIS CASE WE WILL ASSUME $\varepsilon < 1$ AND $\gamma < 1$.

$$H_2 + \text{MO}_2 \rightarrow \alpha H_2O + \beta H_2 + \delta \text{O}_2 + \upsilon \text{H}$$

1) **FIRST APPROXIMATION**, LET $S = U = 0$

$$H_2 + \text{MO}_2 \rightarrow \alpha H_2O + \beta H_2$$

$H$: $2 = 2\alpha + 2\beta$, $\beta = 1 - \alpha$, $\beta = 1 - 2m$

$O$: $2m = \alpha$

$$0 = 2m \left[ h_{H_2O} (T_c) + \Delta H^\circ_{H_2O} \right]
+ (1 - 2m) \left[ h_{H_2} (T_c) + \Delta H^\circ_{H_2} \right]$$

$$0 = 2m \left[ 10.6 (T_c - 298) - 57,800 \right]
+ (1 - 2m) \left[ 7.5 (T_c - 298) \right]$$

**IN GENERAL:**

$$T_c = 298 + \frac{(1.15 \times 10^5) m}{7.5 + 6.2m}$$

**FOR $m = 0.25$**

$$T_c = 3480 \text{ K} \quad \text{(FIRST ESTIMATE)}$$

2) **NEXT, RECALL RELATIONS FOR** $K_p (T)'s$:

$$K_{H_2O} (T_c) = \frac{P_{H_2O}}{(P_{H_2})(P_{O_2})^{1/2}}$$

$$K_{H_2O} (T) = \frac{(P_{H_2O})^2}{(P_{H_2})(P_{O_2})}$$

$$K_{H_2} (T) = \frac{P_{H_2}^2}{P_{H_2}}$$
For equation 2.0, relate $K_{H_2O}$ to $K_{OH}$

Eliminate $P_{O_2}$

$$K_{H_2O}^2 = \frac{P_{H_2O}^2}{P_{H_2}P_{O_2}}$$
$$P_{O_2} = \frac{P_{OH}^2}{K_{OH}P_{H_2}}$$

$$K_{H_2O}^2 = \frac{P_{H_2O}^2}{P_{H_2}P_{OH}}$$

For equation 2.6, this is simply the expression for $K_{H}$

$$K_{H} = \frac{P_{H}}{P_{H_2}}$$

3) For this question recall $\sigma = \sum = \alpha + \beta$

And recall:

$$P_{H_2O} = \frac{\alpha}{\beta} P_e$$
$$P_{H_2} = \frac{\beta}{\alpha} P_e$$
$$P_{OH} = \frac{\delta}{\beta} P_e$$
$$P_{H} = \frac{\beta}{\delta} P_e$$

From equation 2.0:

$$\frac{K_{H_2O}^2}{K_{OH}} = \frac{(\frac{\alpha}{\beta} P_e)^2}{(\frac{\beta}{\alpha} P_e)(\frac{\delta}{\beta} P_e)^2} = \frac{\alpha^2}{\beta^2} \frac{\sigma}{P_e}$$

$$S^2 = \frac{\alpha^2}{\beta} \frac{\sigma}{P_e} \frac{K_{OH}}{K_{H_2O}^2}$$
For, $\sigma = \alpha + \beta = 1$

$$\delta^2 = \frac{4m^2}{(1-2m) \rho c} \frac{K_{POH}}{K_{H_2O}}$$

Look up:

$$K_{POH} (T \approx 3500) \text{ (or interpolate)}$$

$$= 1.70$$

$$K_{H_2O} = 4.93$$

So,

$$\delta^2 \rho c = \frac{(4)(0.25)^2}{1-0.5} \frac{1.7}{(4.93)^2} = 0.035$$

$$\delta \rho c^{1/2} = 0.187$$

$$\rho c = 100 \text{ atm}, \quad \delta = 0.0187 \text{ as a second estimate}$$

Same procedure to calculate $u$

$$K_{H} = \frac{\rho_H^2}{\rho_{H_2O}} = \left( \frac{\frac{1}{\sigma} \rho_c}{\rho_{H_2O}} \right)^2 = \frac{\rho_c}{\rho_{H_2O}} \frac{\sigma^2}{\beta^2}$$

$$u^2 = \beta K_{H} \left( \frac{1}{\rho_c} \right)$$

$$K_H = 0.359$$

$$u \rho_c^{1/2} = 0.424$$

$$u = 0.0424 \text{ as a second estimate}$$
SO, NOW WE HAVE

\[ H_2 + 0.25O_2 \rightarrow \alpha H_2O + \beta H_2 + \delta O + \gamma H \]

\[ \delta = \alpha + \beta + \delta + \gamma \]

\[ H: 2 = 2\alpha + 2\beta + \delta + \gamma \]

\[ O: 0.5 = \alpha + \delta \]

\[ \alpha = 0.5 - \delta = 0.5 - 0.0187 = 0.4813 \]

\[ 2 = 2(0.4813) + 2\beta + 0.0187 + 0.0424 \]

\[ \beta = 0.488 \]

This gives:

\[ H_2 + 0.25O_2 \rightarrow 0.4813H_2O + 0.488H_2 + 0.0187O + 0.0424H \]

Find \( T_c \) for this second estimate of the composition.

\[ 0 = 0.4813 \left[ 10.6(T_c-298)-57.800 \right] \]

\[ + 0.488 \left[ 7.5(T_c-298) \right] \]

\[ + 0.0187 \left[ 7.8(T_c-298) + 10.06 \right] \]

\[ + 0.0424 \left[ 5(T_c-298) + 52.09 \right] \]

Solving for \( T_c \)

\[ T_c = 3345 \text{ K} \]

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We could go back now and find the composition at this temp. From that composition find a new temp, and eventually converge to final values. This is exactly what a thermo-kinetic calculator does. See sample results in section 2.9 of H&P.

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This is close enough to 3480 K to stop iterating.
EFFECT OF SOLID PARTICLES ON SOLID ROCKET PERFORMANCE

CHEMICAL BALANCE:

\[ 2\text{Al} + \frac{3}{2}\text{O}_2 \rightarrow \text{Al}_2\text{O}_3(s) \]

TWO BASIC IMPACTS WE WOULD LIKE TO UNDERSTAND

1) Al HELPS TO RAISE THE COMBUSTION TEMPERATURE
2) \( \text{Al}_2\text{O}_3 \) PARTICLES IN EXHAUST ARE A PENALTY

LOOK AT NOZZLE FLOW:

\[ \begin{align*}
\text{U}_g & \rightarrow \text{U}_g + d\text{U}_g \\
\text{U}_s & \rightarrow \text{U}_s + d\text{U}_s
\end{align*} \]

U\(_g\): GAS VELOCITY
U\(_s\): VELOCITY OF SOLID \( \text{Al}_2\text{O}_3 \)

MOMENTUM BALANCE:

\[ \dot{m}_g \, d\text{U}_g + \dot{m}_s \, d\text{U}_s = -dA \, dp \]

\[ \begin{align*}
\dot{m}_g &= \rho_g \, U_g \, dA \\
\dot{m}_s &= \rho_s \, U_s \, dA
\end{align*} \]

\[ \rho_g \, U_g \, d\text{U}_g + \rho_s \, U_s \, d\text{U}_s + dp = 0 \]
LET \( \frac{PsUs}{psUs + pg Ug} = \frac{x}{1 - x} \)

\[
\frac{PsUs}{pg Ug} = \frac{x}{1 - x}
\]

SO,

\[
pg Ug \left( \frac{dUg}{\theta} + \frac{PsUs}{pg Ug} \frac{dUs}{\theta} \right) + dP = 0
\]

\[
pg Ug \left( \frac{dUg}{\theta} + \frac{x}{1 - x} \frac{dUs}{\theta} \right) + dP = 0
\]

NEXT PERFORM AN ENERGY BALANCE:

\[
\dot{m}_g \Delta h_{hg} + \dot{m}_s \Delta h_{hs} = 0
\]

\[
\dot{h_s} = \text{TOTAL ENTHALPY} = C_p dT + U dU
\]

\[
\dot{h_s} = C_p T + \frac{1}{2} U^2
\]

\[
pg Ug \left( C_p \frac{dT_g}{\theta} + U_g dU_g \right) + Ps Us \left( C_s \frac{dT_s}{\theta} + U_s dU_s \right) = 0
\]

Divide both sides by \( Ps Us + pg Ug \)

\[
\frac{pg Ug}{Ps Us + pg Ug} = (1 - x)
\]

SO,

\[
(1 - x) \left( C_p \frac{dT_g}{\theta} + U_g dU_g \right) + (x) \left( C_s \frac{dT_s}{\theta} + U_s dU_s \right) = 0
\]
From momentum balance, we have

\[ U_g \frac{dU_g}{d\bar{x}} = -\frac{dp}{E_g} - \frac{P_s}{E_g} U_s \frac{dU_s}{d\bar{x}} \]

\[ (1-x) \left( C_p \frac{dT_g}{d\bar{x}} + \left( -\frac{dp}{E_g} - \frac{P_s}{E_g} U_s \frac{dU_s}{d\bar{x}} \right) \right) + \]

\[ (x) \left( C_s \frac{dT_s}{d\bar{x}} + U_s \frac{dU_s}{d\bar{x}} \right) = 0 \]

\[ \frac{P_s}{E_g} U_s \frac{dU_s}{d\bar{x}} = U_g \frac{x}{1-x} \frac{dU_s}{d\bar{x}} \]

Simplify:

\[ C_p \frac{dT_g}{d\bar{x}} - \frac{dp}{E_g} - U_g \frac{x}{1-x} \frac{dU_s}{d\bar{x}} + \frac{x}{1-x} \left( C_s \frac{dT_s}{d\bar{x}} + U_s \frac{dU_s}{d\bar{x}} \right) = 0 \]

\[ \frac{dp}{E_g} = C_p \frac{dT_g}{d\bar{x}} + \frac{x}{1-x} \left[ C_s \frac{dT_s}{d\bar{x}} + (U_s - U_g) \frac{dU_s}{d\bar{x}} \right] \]

If \( x \rightarrow 0 \) (no solid particles)

\[ \frac{dp}{E_g} = C_p \frac{dT_g}{d\bar{x}} \]

\[ P_g = E_g R T_g \]

\[ C_p = \frac{P_g}{E_g R T_g} \]

\[ \frac{dp}{P_g} = \frac{C_p}{R} \frac{dT_g}{T_g} \]
\[
\frac{dp}{p \frac{C_p - C_v}{C_p}} = \frac{dp}{\rho} \left( 1 - \frac{1}{\delta} \right) = \frac{dT}{T}
\]

\[
\int \frac{dp}{p \left( \frac{\gamma - 1}{\delta} \right)} = \int \frac{dT}{T}
\]

\[
\frac{\delta - 1}{\delta} \ln \left( \frac{p_2}{p_1} \right) = \ln \left( \frac{T_2}{T_1} \right)
\]

\[
\left( \frac{p_2}{p_1} \right)^{\frac{\gamma - 1}{\delta}} = \frac{T_2}{T_1}
\]

\[
\left( \frac{p_2}{p_1} \right) = \left( \frac{T_2}{T_1} \right)^{\delta/(\delta - 1)} \quad (X = 0, \text{ no particles})
\]

We can use the results to examine two limiting cases of interest:

a) Very small particles

Assumptions: \( U_s = U_g \quad T_s = T_g \quad \tau \ll \frac{L}{U_g} \)

b) Very large particles

Assumptions: \( U_s \ll U_g \quad T_s \approx T_g \quad \tau \gg \frac{L}{U_g} \)

\( \tau \): Time scale

\( L \): Exhaust nozzle length
Also, look at coupling relations

\[ M_{\text{particle}} \frac{dU_s}{dt} = -\text{Drag} \]

\[ \frac{dU_s}{dt} = -\text{Drag} \]

\[ M_{\text{part}} = \frac{4}{3} \pi R_s^3 \rho_s \frac{dU_s}{dt} = -\text{Drag} \]

\[ \text{DRAG} = -6\pi \mu g R_s (U_g - U_s) \]

So,

\[ \frac{4}{3} \pi R_s^3 \rho_s \frac{dU_s}{dt} = 6\pi \mu g R_s (U_g - U_s) \]

\[ \frac{2}{9} R_s^2 \rho_s \frac{dU_s}{dt} = \frac{1}{g} (U_g - U_s) \]

\[ \frac{2}{9} R_s^2 \rho_s \frac{dU_s}{dt} = U_g - U_s \]

\[ \tau = \frac{2}{9} \frac{R_s^3 \rho_s}{\mu g} \text{ CALLED RELAXATION TIME, THE HEAT TRANSFER HAS SOME TIME CONSTANT.} \]

Look at order of magnitude estimate

\[ L = 1 \text{m} \]

\[ U_g = 300 \text{ m/s} \]

\[ \rho_s = 3000 \text{ Kg/m}^3 \]

\[ g = 3 \times 10^{-5} \text{ Kg/m/s} \]

\[ \frac{2}{9} \frac{(3000)R_s^2}{3 \times 10^{-5}} \ll \frac{1}{3000} \]

\[ R_s \ll 4 \times 10^{-6} \text{m = 4 µm} \]

Usually this is the case
Here we want to examine what the loss in specific impulse is when particles are present in the exhaust flow. For this comparison $T_0 = 3300$K in both cases.

$$\bar{T} = \frac{(1 - \chi) C_{pq} + \chi C_s}{(1 - \chi) C_{up} + \chi C_s}$$

$$\bar{M} = \frac{M_q}{1 - \chi}$$

$$2\text{Al} + \frac{3}{2} \text{O}_2 \rightarrow \text{Al}_2 \text{O}_3 (s) \quad C_s = 1260 \quad \frac{J}{K_g K}$$

$$\gamma = 1.17 \quad P_c / P_e = 100 \quad M_q = 18 \text{ g/mole}$$

For 20% Al loading in propellant, $\chi = 37\%$

Find $C_{pq} = \frac{5 R}{\gamma - 1} M = \frac{1.17 (8314)}{0.17/18} = 3180 \frac{J}{K_g K}$

$$C_{pq} = \frac{C_{pq}}{\bar{T}} = \frac{3180}{1.17} = 2717 \frac{J/K_g K}{\bar{T}}$$

$$\bar{T} = \frac{(1 - 0.37)(3180) + 0.37(1260)}{(1 - 0.37)(2717) + 0.37(1260)}$$

$$\bar{T} = 1.134$$

$$\bar{M} = \frac{M_q}{1 - \chi} = \frac{18}{1 - 0.37} = 28.57 \text{ g/mol}$$
THE EXIT SPEED FOR $pV_w = 100$ AND $T_c = 3300K$

$$u_e = \sqrt{\frac{2(5.82)}{8-1} \frac{2}{1.134} (1 - (0.01)^{5/8})}$$

$$u_e = \sqrt{2 \left( \frac{1.134}{0.134} \right) \left( \frac{8314}{28.57} \right) (3300)(0.4197)}$$

$$u_e = 2612 \text{ m/s}, \quad Isp = 267 \text{ sec}$$

without the aluminum,

$$u_e = \sqrt{2 \left( \frac{1.17}{0.17} \right) \left( \frac{8314}{18} \right) (3300)(0.4978)}$$

$$u_e = 3199, \quad Isp = 326 \text{ sec.}$$

the loss is:

$$\left( \frac{326 - 267}{326} \right) 100 = 18% \text{ loss}$$

Does this make sense? At first glance it might seem like we get higher $Isp$ without aluminum.

But remember, the only reason we are getting $T_c = 3300K$ is because we added aluminum in the first place. Without aluminum, we would not get close to $T_c = 3300K$. Best $T_c$ without aluminum would give $Isp$ lower than 267 sec. (Typical numbers might be around 200-220 sec.) It is most certainly a good idea to use some powdered Al in solid rocket motors.