Parametrizations of Titchmarsh's 'm(λ)'-Functions in the Limit Circle Case

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Contents

1 Introduction 3

2 Preliminaries 4

3 The Limit-circle $m(\lambda)$ -Functions and Associated Weyl Spaces 9

4 Boundary Functionals 20

5 The Characteristic Functions of K. Kodaira and the Rellich Initial Numbers 24

6 The Weyl Spaces of Hermann Weyl 28

7 Titchmarsh’s $m(\lambda)$-Functions in the Limit Circle Case 34

8 Examples 42

9 The Weyl Spaces of M.H. Stone 48

10 Bibliography 52
1 Introduction

We consider the differential expression,

$$\tau = -\frac{d^2}{dx^2} + q(x) \text{ for } x \in [a, c), \text{ with } a < c \leq \infty,$$

and assume that $q(x)$ is real-valued and continuous in $[a, c)$ and belongs to the limit circle case at $c$. Let $L_2[a, c)$ be the Hilbert Space of complex-valued square integrable functions on $[a, c)$ and let $L_0$ and $L_1$ be the operators taking $f$ into $\tau f$ defined by,

$$D(L_0) := \{ f \in L_2[a, c) \mid f \text{ and } f' \text{ are absolutely continuous and } f = 0 \text{ in the Neighborhood of } a \text{ and of } c \}$$

$$D(L_1) := \{ f \in L_2[a, c) \mid f \text{ and } f' \text{ are absolutely continuous on compact subsets of } [a, c) \text{ and } \tau f \in L_2[a, c) \} \text{.}$$

The self-adjoint extensions of $L_0$ which arise from separated boundary conditions at $a$ and $c$ have been characterized in the literature by a variety of different brands of boundary conditions at the (singular) point $c$. In contrast to the boundary conditions originally given by H. Weyl ([37, p.232, Equa. 41]), those given by M.H. Stone in his treatise of Linear Transformations in Hilbert Space ([29, p.475, Theorem 10.17]), which depend on solutions of the eigenvalue equation for $\lambda = i$, and those used by K. Kodaira in [16, p. 924, Equa. 1.7III], which depend on an arbitrary element of $D(L_1)$, E.C. Titchmarsh in his remarkable book, Eigenfunction Expansions Associated with Second Order Differential Equations, Part I, gave boundary conditions at the singular point which depend on a function of the eigenvalue parameter, $m(\lambda)$ (cf. [31, p.31, Equa. 2.7.2]). In Titchmarsh’s treatment each boundary condition at $c$ is determined by a function of $\lambda$, $m(\lambda)$, which is meromorphic and whose poles (which are real and simple) coincide with the eigenvalues of the corresponding self-adjoint extension of $L_0$.

In this dissertation I show that Titchmarsh’s limit-circle ‘$m(\lambda)$’ -functions (corresponding to a fixed boundary condition at $a$) can be represented in the form,

$$m(\lambda) = \frac{a(\lambda) \cos \gamma + b(\lambda) \sin \gamma}{c(\lambda) \cos \gamma + d(\lambda) \sin \gamma}, \quad \gamma \in [0, \pi),$$

where $a(\lambda)$, $b(\lambda)$, $c(\lambda)$, and $d(\lambda)$ are entire functions of $\lambda$. Each value of $\gamma$ corresponds to a different boundary condition at $c$ and the spectrum of the operator associated with this boundary condition is then determined as the poles of the associated $m(\lambda)$ function (or the zeros of $c(\lambda) \cos \gamma + d(\lambda) \sin \gamma$). Also, the connection between the Titchmarsh boundary conditions associated with the $m(\lambda)$-functions of (4) and the boundary conditions arising out of Hilbert-Space
approach taken by M.H. Stone will be established. This permits the conclusion that the $\gamma$-parametrization in (4) yields all possible limit-circle $m(\lambda)$-functions.

The only paper which was found to contain a representation of the type (4) was the 1950 paper of D.B. Sears and E.C. Titchmarsh, “Some Eigenfunction Formulae” [27], which was written to correct some erroneous statements made in the first edition of Titchmarsh’s book (notably Theorem 5.8, p.108\(^1\), which asserts the existence of a continuous spectrum under limit circle conditions at $\infty$). There, for the case $c = \infty$, and under additional assumptions on $q(x)$ (to guarantee the occurrence of the limit circle case and to permit application of the Liouville Transformation), a representation of the type (4) is obtained (p. 168, 3 lines before Section 3). As will be seen in this dissertation these additional assumptions on $q(x)$ are unnecessary for the representation (4).

2 Preliminaries

Letting $b \in (a, c)$ we consider first the Hilbert Space $L_2[a, b]$ of complex-valued square integrable functions on $[a, b]$ and define operators $L_{0,b}$ and $L_{1,b}$ taking $f$ into $\tau f$ by,

$$D (L_{0,b}) := \{ f \in L_2[a,b] | f \text{ and } f' \text{ are absolutely continuous and } f = 0 \text{ in a Neighborhood of } a \text{ and of } b \}^2$$

(1)

$$D (L_{1,b}) := \{ f \in L_2[a,b] | f \text{ and } f' \text{ absolutely continuous in } [a,b] \text{ and } \tau f \in L_2[a,b] \}.$$  

(2)

For $f, g \in d(L_1)$ we define,

$$W_x(f,g) := f(x)g'(x) - f'(x)g(x) \text{ for } x \in [a,c].$$  

(3)

The formula,

$$\int_a^x (\tau f) g - f (\tau g) \, dt = W_x (f, g) - W_a (f, g) \text{ for } x \in [a,c]$$

(4)

holds for all $f, g$ in $D(L_1)$ and will be called Green’s Formula. A self-adjoint extension of $L_{0,b}$ arising from separated boundary conditions at $a$ and $b$ is necessarily one of the operators $L_{b}^{\alpha,\beta}$, for some $\alpha \in [0, \pi)$ and $\beta \in [0, \pi)$, defined by,

$$D (L_{b}^{\alpha,\beta}) := U_{b}^{\alpha,\beta} := \{ f \in D(L_{1,b}) | R_{\alpha}^b (f) = 0, R_{\beta}^b (f) = 0 \}.$$  

---

\(^1\)This false theorem was recently repeated in the book of E. Hille, Lectures in Ordinary Differential Equations, cf. [13, p.531, Theorem 10.3.9].

\(^2\)Primes denote derivatives with respect to $x$. 

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where,

\[ R_a^\alpha (f) := \cos \alpha f (a) + \sin \alpha f' (a) \quad (6) \]
\[ R_b^\beta (f) := \cos \beta f (b) + \sin \beta f' (b) \quad (7) \]

Because of the analogy to the singular limit circle case it will be desirable to summarize briefly Titchmarsh’s approach to the expansion theorem associated with the self-adjoint operator \( L_{b, a}^{\alpha, \beta} \). In order to prove the expansion theorem for the operator \( L_{b, a}^{\alpha, \beta} (\alpha, \beta \in [0, \pi] \text{ fixed}) \) Titchmarsh introduces a function \( \psi_{b, \lambda} (\alpha, \lambda) \) associated with this operator as follows: For each \( \lambda \in \mathbb{C} (\mathbb{C} = \text{the complex numbers}) \) let \( \{ \phi_{a, \lambda} (x), \phi_{a, \lambda} (x) \} \) be the fundamental system of the eigenvalue equation,

\[ \tau u = \lambda u, \quad (8) \]

defined by the initial conditions,

\[ \begin{pmatrix} \phi_{a, \lambda} (a) \\ \phi'_{a, \lambda} (a) \end{pmatrix} = \begin{pmatrix} \sin \alpha & \cos \alpha \\ -\cos \alpha & \sin \alpha \end{pmatrix} \quad \text{for all } \lambda \in \mathbb{C}. \quad (9) \]

Then we have:

\[ W_a (\phi_{a, \lambda}, \theta_{a, \lambda}) = 1 \text{ for all } \lambda \in \mathbb{C} \text{ and all } x \in [a, c]. \quad (10) \]

\[ \phi_{a, \lambda} (x) = \overline{\phi_{a, \lambda} (\overline{x})} \text{ and } \theta_{a, \lambda} (x) = \overline{\theta_{a, \lambda} (\overline{x})} \quad (11) \]

For fixed \( x \in (a, c), \phi_{a, \lambda} (x), \phi'_{a, \lambda} (x), \theta_{a, \lambda} (x), \]

and \( \theta'_{a, \lambda} (x) \) are entire functions of \( \lambda. \)

\[ R_a^\alpha (f) = W_a (\phi_{a, \lambda}, f) \text{ for all } f \in D (L_{1, b}) \text{ and all } \lambda \in \mathbb{C}. \quad (13) \]

\[ R_a^\alpha (\phi_{a, \lambda}) = 0 \text{ for all } \lambda \in \mathbb{C}. \quad (14) \]

\[ W_a \left( \phi_{a, \lambda}, \phi'_{a, \lambda'} \right) = \sin (\alpha' - \alpha) \quad \text{for all } \lambda, \lambda' \in \mathbb{C}. \]

\[ W_a \left( \theta_{a, \lambda}, \theta'_{a, \lambda'} \right) = \sin (\alpha' - \alpha) \quad \text{for all } \lambda, \lambda' \in \mathbb{C}. \quad (15) \]

For \( \text{Im } \lambda \neq 0, \) a solution of \( (8) \) satisfying the boundary conditions at \( b \) is,

\[ \psi_{b, \lambda} (x) := \theta_{a, \lambda} (x) + \ell_{b, \lambda}^{\alpha, \beta} (\lambda) \phi_{a, \lambda} (x) \quad (16) \]

---

3We will indicate the dependence of \( \phi_{a, \lambda} \) and \( \theta_{\alpha, \lambda} \) on \( \alpha \) by writing \( \phi'^{\alpha}_{a, \lambda} \) and \( \theta'^{\alpha}_{a, \lambda} \) when this is desirable for clarity.

4Bars denote complex conjugates.
where $\ell_b^{\alpha,\beta}(\lambda)$ is defined (for $\text{Im} \lambda \neq 0$) by,

$$
\ell_b^{\alpha,\beta}(\lambda) := - \frac{\theta_{a,\lambda}(b) \cos \beta + \theta'_{a,\lambda}(b) \sin \beta}{\phi_{a,\lambda}(b) \cos \beta + \phi'_{a,\lambda}(b) \sin \beta}.
$$

(17)

Since $\psi_{b,\lambda}(x)$ and $\psi_{b,\lambda'}(x)$ satisfy the boundary condition (7) at $b$ it follows that

$$
W_b(\psi_{b,\lambda}, \psi_{b,\lambda'}) = 0 \text{ for all } \lambda, \lambda', \text{Im} \lambda \neq 0, \text{Im} \lambda' \neq 0. \tag{18}
$$

The function $\ell_b^{\alpha,\beta}(\lambda)$ is Titchmarsh’s $'m(\lambda)'$-function for the operator $L_b^{\alpha,\beta}$ and the expansion theorem follows as in [31, Chapter 2], with $m(\lambda) = \ell_b^{\alpha,\beta}(\lambda)$. Thus if $\lambda_n, n = 1, 2, 3, \cdots$, are the poles of $\ell_b^{\alpha,\beta}(\lambda)$, and if

$$
\nu_{b,n}^{\alpha,\beta} := \Re \lim_{\lambda \to \lambda_n} \ell_b^{\alpha,\beta}(\lambda),
$$

(19)

then the eigenfunctions of the operator $L_b^{\alpha,\beta}$ are (cf. [31, Equa. 2.5.6 and Theorem 2.7 (i) and (ii))):

$$
\psi_n^{\alpha,\beta}(x) := \sqrt{\nu_{b,n}^{\alpha,\beta}} \phi_{a,\lambda_n}^{\alpha,\beta}(x). \tag{20}
$$

Moreover, it follows from Titchmarsh’s proof of the expansion theorem that these eigenfunctions form a complete orthonormal set for $L_2[a,b]$.

Since it will be needed for later reference we now give a brief summary of Titchmarsh’s discussion of Weyl’s limit-point/limit-circle Alternative (cf. [31, p. 24-25]): For each $\lambda$, Im $\lambda \neq 0$,

$$
A_{b,\lambda}^{\alpha}(z) := - \frac{\theta_{a,\lambda}(b) z + \theta'_{a,\lambda}(b)}{\phi_{a,\lambda}(b) z + \phi'_{a,\lambda}(b)} \tag{21}
$$

is linear fractional transformation (invertible because of (10) ) which maps the real $z$-axis onto the circle $C_{b,\lambda}^{\alpha}$ whose center and radius are respectively,

$$
\varphi_{b,\lambda}^{\alpha} := \frac{W_b(\theta_{a,\lambda}, \phi_{a,\lambda})}{W_b(\theta_{a,\lambda}, \phi'_{a,\lambda})} \tag{22}
$$

$$
r_{b,\lambda}^{\alpha} := \frac{1}{W_b(\theta_{a,\lambda}, \phi_{a,\lambda})} = \frac{1}{2 |\nu| \int_a^b |\theta_{a,\lambda}(x)|^2 dx} \tag{22}
$$

where $\nu = \text{Im} \lambda$.

If $w \in \mathbb{C}$ and $z = \left(A_{b,\lambda}^{\alpha}\right)^{-1}(w)$, the following algebraic equation$^6$ holds for all

$^6$In the regular case equation (18) corresponds to Titchmarsh [31, p. 26, Lemma 2.3].

$^6$For $\lambda = i$ compare with Hellwig [11, p.230, Equa. (15) and (14)], and with [36, p.228, Equa 29].
\( \lambda, \Im \lambda \neq 0 : \)

\[
|\theta_{a,\lambda} (b) + w\phi_{a,\lambda} (b)|^2 \frac{\Im z}{\nu} = \frac{W_b (\theta_{a,\lambda} + w\phi_{a,\lambda}, \theta_{a,\lambda} + w\phi_{a,\lambda})}{2i\nu} = \int_a^b \left| \phi_{a,\lambda} (x) \right|^2 dx \left\{ |w - \phi_{b,\lambda}^\alpha|^2 - (r_{b,\lambda}^\alpha)^2 \right\}
\]

\[
= \int_a^b \left| \theta_{a,\lambda} (x) + w\phi_{a,\lambda} (x) \right|^2 dx + \frac{\Im w}{\nu} . \quad (23)
\]

From this equation the following result can be observed:

**Theorem 1. (Weyl Alternative)**

(i) For each \( \lambda, \Im \lambda \neq 0, \) we have either

1. \( \lim_{b \to c} r_{b,\lambda}^\alpha = 0 \) or
2. \( \lim_{b \to c} r_{b,\lambda}^\alpha =: r_{c,\lambda}^\alpha > 0. \)

(ii)\(^7\) If case (2) occurs for a single \( \lambda, \Im \lambda \neq 0, \) then it occurs for all \( \lambda, \Im \lambda \neq 0. \)

From (22) and (23) it follows that in case (1) (the limit point case) there is only one solution of (8) for \( \Im \lambda \neq 0 \) which is in \( L^2[a, c], \) and in case (2) (the limit circle case) two linearly independent solutions of (8) in \( L^2[a, c]. \) In the limit circle case the circle \( C_{b,\lambda}^\alpha \) converges (for each \( \lambda, \Im \lambda \neq 0 \)) to a limit circle \( C_{c,\lambda}^\alpha, \)

whose center and radius are, respectively

\[
\theta_{c,\lambda}^\alpha := -\frac{W_c (\theta_{a,\lambda}, \phi_{a,\lambda})}{W_c (\theta_{a,\lambda}, \phi_{a,\lambda})}, \quad r_{c,\lambda}^\alpha := \frac{1}{|W_c (\theta_{a,\lambda}, \phi_{a,\lambda})|} = \frac{1}{2 |\nu| \cdot \int_a^b \left| \theta_{a,\lambda} (x) \right|^2 dx}, \quad (24)
\]

where

\[
W_c (f, g) := \lim_{x \to c} W_x (f, g) \text{ for } f, g \text{ in } D (L_1).
\]

**Remark 1.1** Since

\[
\ell_{b,\lambda}^{\alpha,\beta} (\lambda) = A_{b,\lambda}^{\alpha} (\cot \beta) \text{ for all } \beta \in [0, \pi), \quad (25)
\]

it is clear that for each \( \lambda, \Im \lambda \neq 0, \)

\[
f_{a,\lambda,b} (\beta) : \beta \to \ell_{b,\lambda}^{\alpha,\beta} (\lambda)
\]

\(^7\)Titchmarsh [31, p.43-44]. In the limit circle case there are two linearly independent solutions of (1.8) in \( L^2[a, c] \) for real \( \lambda \) as well as complex \( \lambda \) although this does not follow from Titchmarsh’s argument. Compare Hellwig [11, p.223, Theorem 1].
is a one-to-one mapping from \([0, \pi)\) onto the circle \(C_{b, \lambda}\).

In the present paper we apply a theorem of Hartman ([9, p. 273, Theorem 1.1 and 1.2]) to define for each \(\lambda \in \mathcal{C}\) a fundamental system \(\{\phi_{c, \lambda}(x), \theta_{c, \lambda}(x)\}\) of (8) by "initial conditions" at the singular point \(c\) which have the properties (10)-(15) (and notably the property (13)) with \(c\) in the place of \(a\), and with a Rellich-type boundary condition at \(c\) in place of (7). Making use of the two fundamental systems \(\{\phi_{a, \lambda}(x), \theta_{a, \lambda}(x)\}\) and \(\{\phi_{c, \lambda}(x), \theta_{c, \lambda}(x)\}\) we define (Definition 2 below) an \('m(\lambda)'\)-function associated with each boundary condition at \(c\). Titchmarsh's proof of the expansion theorem ([31, p. 31, Theorem 2.7 (i) and (ii)]) for the corresponding operator will apply with our definition of the \(m(\lambda)\) function. We then establish the equivalence of our definition with Titchmarsh's definition afterwards in Section 6. The advantage of our approach to the definition is that we obtain the representation (0.4) almost automatically.

In the following sections a Weyl-Space will refer to any space \(U\) of the form,

\[
U = \{ f \in D(L_1) \mid R^\alpha_c(f) = 0, \ R_c(f) = 0 \},
\]

where \(R_c(f) = 0\) is a boundary condition at \(c\) such that \(U\) is the domain of a self-adjoint extension of \(L_0\).

It should perhaps be mentioned that Titchmarsh's book has been regarded with some suspicion as far as the limit circle case is concerned. Titchmarsh's simultaneous treatment of the limit circle case and the 'discrete' limit point case in Chapter 2 ([30] and [31]) is, in particular, a little confusing since the \('m(\lambda)'\) functions in the two different cases are, of course, different objects. This confusion is depicted for example in the following words of D.B. Sears (cf. [26, p.49]):

"In the limit point case,

\[
m(\lambda) = \lim_{b \to \infty} \ell_b(\lambda)^8
\]

exists as an analytic function, regular in either half plane. Titchmarsh uses this function also in the limit circle case, and thus requires no distinction in his theorems between the two possibilities. This seems to require some further discussion. Levinson appears to abandon the limit circle case in his second paper. While the function \(k(t)\) is not unique in this case, the whole theory may in fact be put in a form in which the type of the equation is immaterial, except for a few details."

In the above cited paper Sears modifies the approach of N. Levinson in [17] and shows that the limit circle case can be treated by making use of Levinson's

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\(^8\)This is the function \(\ell_0^{p, \beta}(\lambda)\) defined in (1.17).
'spectral functions'. He must however, take his limit-circle 'm(λ)' function to be 
\[ m(λ) = \lim_{n \to \infty} \ell_{b_n}(λ) \]
where \( \{b_n\} \) is a 'suitable sequence', in order to obtain a well-defined spectral function \( k(t) \), cf. [17, p. 50, Equa. 4.4]. He then remarks later (p.57),

"Naturally the spectrum in the limit circle case depends on the sequence through which \( b \to \infty \)."

As we shall see this paper the limit-circle \( m(λ) \)-functions (and corresponding spectra) do indeed depend on the choice of the sequence \( \{b_n\} \) and we shall give two examples showing that different choices of the sequence do produce different limit functions. Moreover, we show that the class of all limit functions for all possible choices of the sequence \( \{b_n\} \) can be indexed by a real parameter \( γ \) and represented in the form (4).

I am indebted to Professor Dr. J. Walter for insisting that different choices of the sequence \( \{b_n\} \) would yield different \( m(λ) \)-functions. I am also grateful for his suggestion that I investigate example 1 of Section 7 below to see that this is the case. I thank also Professor Dr. G. Hellwig and his Assistants for many lively discussions in his Colloquium on the Spectral Theory of Singular Elliptic Differential Operators.

3 The Limit-circle 'm(λ)' -Functions and Associated Weyl Spaces

Let \( \{u(x), v(x)\} \) be real-valued fundamental system of (8) for \( λ = 0 \) satisfying 
\[ W(u, v) = 1 \]  
and put 
\[ Y(x) := \begin{pmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{pmatrix} \text{ for } x \in [a, c]. \]  
(2)

For each \( f \in D(L_1) \) we define a vector-valued function \( Sf \) for \( x \in [a, c] \) by,
\[ (Sf)(x) := (Y(x))^{-1} \begin{pmatrix} f(x) \\ f'(x) \end{pmatrix} = \begin{pmatrix} W_x(f, v) \\ -W_x(f, u) \end{pmatrix}. \]  
(3)

Because we assume \( q(x) \) belongs to the limit circle case at \( a \), the solutions \( u(x) \) and \( v(x) \) are square integrable over \( [a, c] \) and also belong to \( D(L_1) \). It follows

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9A treatment of the limit circle case employing the notion of the spectral function was later included in the book of Coddington and Levinson, cf [4, p.242-46]. This approach was also included in the second edition of Titchmarsh’s book, cf. [31, Chapter VI]. Levinson’s spectral function is Titchmarsh’s function ‘\( k(λ) \)’, cf. [31, p.138, Equa. 6.7.5].
from (2.4) that both components of $Sf$ can be continuously extended to be defined at the singular point $c$.\textsuperscript{10}

$$\left( Sf \right)(c) := \lim_{x \to c} \left( Sf \right)(x) = \begin{pmatrix} W_c(f, v) \\ -W_c(f, u) \end{pmatrix} \quad \text{for all } f \in D(L_1). \quad (4)$$

For two elements $f, g \in D(L_1)$ we define the ‘Wronskian’ of $Sf$ and $Sg$ by,

$$\widetilde{W}_x(Sf, Sg) := (Sf)_1(x)(Sg)_2(x) - (Sf)_2(x)(Sg)_1(x) \quad \text{for } x \in [a, c],$$

where $(Sf)_1(x)$ and $(Sf)_2(x)$ are the first and second components of $Sf$. Because of (1) we have the algebraic identity,

$$W_x(Sf, Sg) := \widetilde{W}_x(Sf, Sg) \quad \text{for } x \in [a, c),$$

and because of the existence of the limits in (4) we can take the limit as $x \to c$ on both sides to obtain,

$$W_c(f, g) = \widetilde{W}_c(Sf, Sg) \quad \text{for all } f, g \in D(L_1). \quad (5)$$

We now define a two-parameter family of ‘Weyl’-Spaces arising from separated boundary conditions as follows:

$$U_{2}^{\alpha, \gamma} := \{ f \in D(L_1) \mid R_\alpha^c(f) = 0 \text{ and } R_\gamma^c(f) = 0 \}, \quad (6)$$

where $\alpha, \gamma \in [0, \pi)$, and where $R_\alpha^c(f)$ is defined by,

$$R_\alpha^c(f) := (Sf)_1(c) \cos \gamma + (Sf)_2(c) \sin \gamma \quad \text{for all } f \in D(L_1). \quad (7)$$

The operator taking $f$ into $\tau f$ which has $U_{2}^{\alpha, \gamma}$ as its domain will be denoted $L^{\alpha, \gamma}$. The symmetry of $L^{\alpha, \gamma}$ follows easily since

$$R_\beta^c(f) = R_\beta^c(g) = 0 \implies \widetilde{W}_c(Sf, Sg) = 0$$

and therefore $W_c(f, g) = 0$ by (5). Also $L^{\alpha, \gamma}$ is a ‘real’ operator, that is

$$L^{\alpha, \gamma}(\mathcal{F}) \quad \text{for all } f \in U_{2}^{\alpha, \gamma},$$

since $U_{2}^{\alpha, \gamma}$ is closed under complex conjugation. The self-adjointness of $L^{\alpha, \gamma}$ will be deducible from the completeness of its eigenfunctions.

In order to treat the spectral theory of the operator $L^{\alpha, \gamma}$ (for fixed $\alpha, \gamma$) in a manner analogous to the discussion of the operator $L_0^{\alpha, \beta}$ in section 1, we now define a fundamental system of (8) for all $\lambda \in \mathbb{C}$ by initial conditions at the

\textsuperscript{10}This is not possible in the limit point case for this reason the following discussion does not apply when the limit point case occurs at $c$.

\textsuperscript{11}Compare K. Kodaira [16, p.924, Equa. (1.5) with $1=0$].

\textsuperscript{12}Boundary conditions at the singular point were first written in this form by F. Rellich in [22, p. 354, Equa. (9)]. Compare also K. Jörgens [15, Corollary to Theorem 4, p. 9.10] and J. Weidmann [36, p. 272].
singular point $c$. This can be done as follows: We write the equation (8) in the form,
\[
\frac{dz}{dx} = \begin{pmatrix} 0 & 1 \\ -\lambda - q & 0 \end{pmatrix} z \text{ with } z := \begin{pmatrix} u \\ u' \end{pmatrix},
\]
and introduce the change of variables\(^{13}\),
\[
y := (Y(x))^{-1} z,
\]
obtaining
\[
\frac{dy}{dx} = \lambda B(x) y, \quad \text{where}
\]
\[
B(x) := \begin{pmatrix} uv & v^2 \\ -u^2 & -uv \end{pmatrix} \text{ for } x \in [a,c).
\]

**Remark 2.1** The mapping $S$ on $D(L_{1})$ defines for each $\lambda \in \mathbb{C}$ ($\lambda = 0$ included) a one-to-one correspondence between the solution spaces of (8) and (10).

**Theorem 2.** Let $q(x)$ be continuous in $[a,c)$, $a < c \leq \infty$, and belong to the limit circle case at $c$. We have:

(i) If $y_{\lambda}(x)$ is a solution of (10) for some $\lambda \in \mathbb{C}$, then $y_{\lambda}(c) := \lim_{x \to c} y_{\lambda}(x)$ exists.

Moreover, for all $x_0 \in [a,c)$ and all $x \in [a,c)$ we have,
\[
|y_{\lambda}(x_0)| \exp \left\{ -|\lambda| \cdot \left| \int_{x_0}^{x} u^2 + v^2 \, ds \right| \right\} 
\leq |y_{\lambda}(x)| 
\leq |y_{\lambda}(x_0)| \exp \left\{ |\lambda| \cdot \left| \int_{x_0}^{x} u^2 + v^2 \, ds \right| \right\} \quad (12)
\]
and
\[
|y_{\lambda}(x) - y_{\lambda}(c)| \leq |y_{\lambda}(x_0)| \cdot |\lambda| \cdot M(x) \exp \{ |\lambda| M(x_0) \} \quad (13)
\]
where $M(x) := \int_{x}^{c} u^2 + v^2 \, ds$,

and where the absolute value of the complex vector $y_{\lambda}(x)$ is defined as usual, that is,
\[
|y_{\lambda}(x)| := \sqrt{|y_{1,\lambda}(x)|^2 + |y_{2,\lambda}(x)|^2}.
\]

(ii) To each vector $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ in $\mathbb{C} \times \mathbb{C}$ there exists (for every $\lambda \in \mathbb{C}$) a unique solution $y_{\lambda}(x)$ of (10) such that
\[
y_{\lambda}(c) := \lim_{x \to c} y_{\lambda}(x) = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.
\]
\[^{13}\text{Compare Hartman [9, p.330, Equa 2.28 with } q = \lambda - q_0].\]
Proof. For \( \lambda \neq 0 \) the assumptions of [9, p. 273, Theorem 1.1 and 1.2], are fulfilled since,
\[
\int_a^c |\lambda B(x)| \, dx = |\lambda| \int_a^c u^2 + v^2 \, dx,
\]
which is finite because \( u \) and \( v \) are in \( L_2[a,c] \). (The inequality (12) comes from Hartman [9, Equa. (1.9), p.274], and the inequality (13) comes from [9, Equa. (1.14), p.275].) For \( \lambda = 0 \) the solutions of (10) are constant vectors and we have equality in (12) and (13).

Using Theorem 2 we now make the following definition:

Definition 1. Let \( \gamma \in [0, \pi) \). For each \( \lambda \in \mathbb{C} \), let \( y^{(1)}_\lambda (x) \) and \( y^{(2)}_\lambda (x) \) be the unique solutions of (10) defined by,
\[
\begin{pmatrix}
  y^{(1)}_{1,\lambda}(c) & y^{(2)}_{1,\lambda}(c) \\
  y^{(1)}_{2,\lambda}(c) & y^{(2)}_{2,\lambda}(c)
\end{pmatrix} =
\begin{pmatrix}
  \sin \gamma & \cos \gamma \\
  -\cos \gamma & \sin \gamma
\end{pmatrix}
\]
and let \( \phi^{(1)}_{c,\lambda}(x) \) and \( \theta^{(2)}_{c,\lambda}(x) \) denote the solutions of (8) defined by,
\[
\begin{pmatrix}
  \phi^{(1)}_{c,\lambda}(x) & \theta^{(2)}_{c,\lambda}(x) \\
  \phi^{(1)}_{c,\lambda}'(x) & \theta^{(2)}_{c,\lambda}'(x)
\end{pmatrix} = Y(x)
\begin{pmatrix}
  y^{(1)}_{1,\lambda}(c) & y^{(2)}_{1,\lambda}(c) \\
  y^{(1)}_{2,\lambda}(c) & y^{(2)}_{2,\lambda}(c)
\end{pmatrix}
\]

We have the following facts analogous to (10)-(15):
\[
W_x (\phi^{(1)}_{c,\lambda},\theta^{(2)}_{c,\lambda}) = \overline{W_x (y^{(1)}_\lambda, y^{(2)}_\lambda)} = 1
\]
for all \( \lambda \in \mathbb{C} \) and all \( x \in [a,c] \).
\[
\phi^{(1)}_{c,\lambda}(x) = \overline{\phi^{(1)}_{c,\lambda}(x)} \quad \text{and} \quad \theta^{(2)}_{c,\lambda}(x) = \overline{\theta^{(2)}_{c,\lambda}(x)}
\]
(17)

For fixed \( x \in (a,c) \), \( \phi^{(1)}_{c,\lambda}(x) \), \( \phi^{(1)}_{c,\lambda}'(x) \), \( \theta^{(2)}_{c,\lambda}(x) \), and \( \theta^{(2)}_{c,\lambda}'(x) \) are entire functions of \( \lambda \).\(^{14}\)

\[
R^{(1)}_c (f) = W_x (\phi^{(1)}_{c,\lambda}, f) \quad \text{for all} \quad f \in D(L_1)
\]
and all \( \lambda \in \mathbb{C} \).

\(^{14}\)The functions \( y^{(1)}_\lambda(x) \) and \( y^{(2)}_\lambda(x) \) are continuous in \( x \) and \( \lambda \) on \( (a,c) \times \mathbb{C} \). This can be proved using easy estimates, making use of (2.12) and the analyticity in \( \lambda \) for fixed \( x \). Also for fixed \( \lambda \) the functions \( \phi^{(1)}_{c,\lambda}(x) \) and \( \theta^{(2)}_{c,\lambda}(x) \) are infinitely differentiable for \( x \in (a,c) \). This follows from [5, Corollary 4, p.1283].
\[ R_c^*(\phi'_{c,\lambda}) = 0 \text{ for all } \lambda \in C. \]  
\(15\)

\[ W_c(\phi_{c,\lambda}, \phi'_{c,\lambda}) = \sin (\gamma' - \gamma) \text{ for all } \lambda, \lambda' \in C. \]
\[ W_c(\theta_{c,\lambda}, \theta'_{c,\lambda}) = \sin (\gamma' - \gamma) \text{ for all } \lambda, \lambda' \in C. \]  

The only statement which is not self-evident is (18). Even in the case \(c < 1\), the analyticity of the solutions \(y_{c,1}(x)\) and \(y_{c,2}(x)\) in \(\lambda\) cannot be concluded from usual theorems asserting analyticity in parameters ([5, p. 1284, Corollary 5]) because the elements of the matrix \(B(x)\) may diverge as \(x \to c\). The analyticity of the functions \(c, \theta_{c,\lambda}(x), \phi_{c,\lambda}(x), \text{ and } \theta'_{c,\lambda}(x)\) in \(\lambda\) can, however, be deduced from the analyticity in \(\lambda\) of the functions \(\phi_{a,\lambda}(x), \phi'_{a,\lambda}(x), \theta_{a,\lambda}(x), \text{ and } \theta'_{a,\lambda}(x)\) as follows: We define four functions of \(\lambda\) by,

\[
\begin{align*}
    w^{(a,\gamma)}(\lambda) &:= W_x(\phi_{a,\lambda}, \phi_{c,\lambda}) \\
    p_1^{(a,\gamma)}(\lambda) &:= W_x(\phi_{c,\lambda}, \theta_{a,\lambda}) \\
    p_2^{(a,\gamma)}(\lambda) &:= W_x(\phi_{a,\lambda}, \theta_{c,\lambda}) \\
    v^{(a,\gamma)}(\lambda) &:= W_x(\theta_{c,\lambda}, \theta_{a,\lambda})
\end{align*}
\]  

(22)

where the right hand side is, of course, independent of \(x\). Evaluating the above Wronskian at \(x = c\) and applying (5) and (14) we have,

\[
\begin{align*}
    w^{(a,\gamma)}(\lambda) &= - (S\phi_{a,\lambda})_1(c) \cos \gamma - (S\phi_{a,\lambda})_2(c) \sin \gamma \\
    &= - R_c^*(\phi_{a,\lambda}) \\
    p_1^{(a,\gamma)}(\lambda) &= (S\theta_{a,\lambda})_1(c) \cos \gamma + (S\theta_{a,\lambda})_2(c) \sin \gamma \\
    &= R_c^*(\theta_{a,\lambda}) \\
    p_2^{(a,\gamma)}(\lambda) &= (S\phi_{a,\lambda})_1(c) \sin \gamma - (S\phi_{a,\lambda})_2(c) \cos \gamma \\
    v^{(a,\gamma)}(\lambda) &= -(S\theta_{a,\lambda})_1(c) \sin \gamma + (S\theta_{a,\lambda})_2(c) \cos \gamma.
\end{align*}
\]  

(23)

The following algebraic facts are readily verified:

\[ p_1(\lambda) p_2(\lambda) - w(\lambda) v(\lambda) = 1 \text{ for all } \lambda \in C. \]  

(24)
\[
\begin{pmatrix}
\phi_{c,\lambda}^\gamma (x) \\
\theta_{c,\lambda}^\gamma (x)
\end{pmatrix} =
\begin{pmatrix}
p_1 (\lambda) & w (\lambda) \\
v (\lambda) & p_2 (\lambda)
\end{pmatrix}
\begin{pmatrix}
\phi_{a,\lambda}^\alpha (x) \\
\theta_{a,\lambda}^\alpha (x)
\end{pmatrix}
\]
(25)

for all \( \lambda \in \mathbb{C} \).

\[
\begin{pmatrix}
\phi_{c,0}^\gamma (x) \\
\theta_{c,0}^\gamma (x)
\end{pmatrix} =
\begin{pmatrix}
sin \gamma & -\cos \gamma \\
\cos \gamma & \sin \gamma
\end{pmatrix}
\begin{pmatrix}
u (x) \\
v (x)
\end{pmatrix}
\]
(26)

\[
\begin{pmatrix}
\phi_{c,0}^\alpha (x) \\
\theta_{c,0}^\alpha (x)
\end{pmatrix} =
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
u (x) \\
v (x)
\end{pmatrix}
\]
(27)

where

\[
A := \cos \alpha \, v (a) + \sin \alpha \, v' (a)
\]

\[
B := -\cos \alpha \, u (a) - \sin \alpha \, u' (a)
\]

\[
C := \cos \alpha \, v' (a) - \sin \alpha \, v (a)
\]

\[
D := -\cos \alpha \, u' (a) + \sin \alpha \, u (a)
\]

Observations:

1. For fixed \( x \in (a, c) \), the analyticity of \( (S \phi_{a,\lambda}) (x) \) and \( (S \theta_{a,\lambda}) (x) \) in \( \lambda \) follows immediately from the analyticity in \( \lambda \) of \( \phi_{a,\lambda} (x) \), \( \phi'_{a,\lambda} (x) \), \( \theta_{a,\lambda} (x) \) and \( \theta'_{a,\lambda} (x) \) because of (9).

2. From (13) it is clear that

\[
\lim_{x \to c} (S \phi_{a,\lambda}) (x) = (S \phi_{a,\lambda}) (c) \quad \text{for all } \lambda \in \mathbb{C}, \quad \text{and}
\]

\[
\lim_{x \to c} (S \theta_{a,\lambda}) (x) = (S \theta_{a,\lambda}) (c) \quad \text{for all } \lambda \in \mathbb{C},
\]

and because of the estimate in (13) it follows that the convergence is uniform on compact \( \lambda \)-sets. Thus by the Weierstrass Theorem the limit functions \( (S \phi_{a,\lambda}) (c) \) and \( (S \theta_{a,\lambda}) (c) \) are entire functions of \( \lambda \).

3. The analyticity of the functions \( w (\lambda), p_1 (\lambda), p_2 (\lambda) \), and \( v (\lambda) \) follows immediately from the analyticity of \( (S \phi_{a,\lambda}) (c) \) and \( (S \theta_{a,\lambda}) (c) \) in \( \lambda \) because of the algebraic relations in (23).

4. For fixed \( x \in (a, c) \) the analyticity of \( \phi_{c,\lambda} (x), \phi'_{c,\lambda} (x), \theta_{c,\lambda} (x) \) and \( \theta'_{c,\lambda} (x) \) in \( \lambda \) follows from the analyticity of the functions \( w (\lambda), p_1 (\lambda), p_2 (\lambda), \) and \( v (\lambda) \), and the analyticity in \( \lambda \) of \( \phi_{a,\lambda} (x), \phi'_{a,\lambda} (x), \theta_{a,\lambda} (x) \) and \( \theta'_{a,\lambda} (x) \) because of the algebraic relations in (25).
Remark 2.2. For fixed $x \in (a, c)$ the analyticity of $\phi_{a, \lambda}(x)$ and $\phi'_{a, \lambda}(x)$ in $\lambda$ can be proved by showing that the Cauchy-Riemann equations hold (cf. F. John [14, p.219]). A similar argument can be used to show that the solutions $y^{(1)}_\lambda(x)$ and $y^{(2)}_\lambda(x)$ of (14) are analytic for $\lambda \neq 0$, and then Theorem 2 can be used to show that $\lambda = 0$ is a removable singularity. This gives an alternate proof for analyticity in $\lambda$ of $\phi_{c, \lambda}(x)$, $\phi'_{c, \lambda}(x)$, $\theta_{c, \lambda}(x)$ and $\theta'_{c, \lambda}(x)$.

Remark 2.3. The fact that $w(\gamma), p_1(\lambda), p_2(\lambda)$, and $v(\lambda)$ have only real (and simple) zeros can be proved in the same manner as in the regular case, cf. Titchmarsh [31, p.11-12]. We note only that in the present (singular) case Titchmarsh’s proof of the simplicity of the zeros of $w^{\alpha, \gamma}(\lambda)$ must be slightly modified. One has to make use (for $\cos \gamma \neq 0$) of the equation

$$-W_c(S\phi_{a, \lambda}, S\phi_{a, \lambda}') = \frac{1}{\cos \gamma} \left[ w^{\alpha, \gamma}(\lambda') (S\phi_{a, \lambda})_1 (c) - w^{\alpha, \gamma}(\lambda) (S\phi_{a, \lambda})'_2 (c) \right]$$

instead of [31, p.12, line 13],

$$W_b(\phi_{a, \lambda}, \phi_{a, \lambda}') = \frac{1}{\cos \beta} \left[ w^{\alpha, \beta}_b(\lambda') \phi_{a, \lambda}(b) - w^{\alpha, \beta}_b(\lambda) \phi_{a, \lambda'}(b) \right],$$

which applies only for $b < c$.

Remark 2.4. Titchmarsh’s proof ([31, p.12, two lines before Theorem 1.9]) that the zeros of the function $w^{\alpha, \beta}_b(\lambda)$ (definition [31, p.7, Equa. 1.6.1]) are bounded below does not carry over to the singular case because there is no analogue of Lemma (7) in the singular case.

As a corollary of Theorem 2 and the above discussion we have the following result which gives a formula for the computation of the zeros of $w^{\alpha, \gamma}(\lambda)$:

Corollary 2.1

(i) If $\lambda = 0$ is not a zero of $w^{\alpha, \gamma}(\lambda)$, the zeros of $w^{\alpha, \gamma}(\lambda)$ are the solutions of the following real equation in $\lambda$:

$$\lambda = \frac{\int_a^c (u(x) \sin \gamma - v(x) \cos \gamma) \cdot \phi_{a, \lambda}^2(x) dx}{\int_a^c [u(x) \sin \gamma - v(x) \cos \gamma] \cdot \phi_{a, \lambda}^2(x) dx}$$  \hspace{1cm} (28)$$

(ii) If $\lambda = 0$ is a zero of $w^{\alpha, \gamma}(\lambda)$, then the remaining zeros of $w^{\alpha, \gamma}(\lambda)$ are the real solutions of the equation,

$$\int_a^c [u(x) \sin \gamma - v(x) \cos \gamma] \cdot \phi_{a, \lambda}^2(x) dx = 0.$$  \hspace{1cm} (29)$$
Proof. Applying Green’s Formula to $\phi_{a,\lambda}^\alpha (x)$ and $\phi_{c,0}^\gamma (x)$ and using (19), (23), (2.13), and (26) we obtain,

$$
(\lambda - 0) \int_a^c \phi_{a,\lambda}^\alpha (x) \cdot [u (x) \sin \gamma - v (x) \cos \gamma] \, dx
$$

$$
= - W_a (\phi_{a,0}^\alpha, \phi_{c,0}^\gamma) + w^{\alpha,\gamma} (\lambda),
$$

(30)

from which statements (i) and (ii) follow. □

Remark 2.5 Similar formulas for the zeros of $p_1 (\lambda)$, $p_2 (\lambda)$, and $v (\lambda)$ can be obtained in the same way.

In order to prove the expansion theorem for the operator $L^{\alpha,\gamma} (\alpha, \gamma \in [0, \pi]$ fixed) ala Titchmarsh, we now introduce an $'m (\lambda)'$-function associated with this operator as follows:

Definition 2. For each $\alpha, \gamma \in [0, \pi)$ we define for all $\lambda$ not zeros of $w^{\alpha,\gamma} (\lambda)$:

$$
m^{\alpha,\gamma} (\lambda) : = \frac{p_1^{\alpha,\gamma} (\lambda)}{w^{\alpha,\gamma} (\lambda)} \text{ and}^{16}
$$

(31)

$$
\psi^{\alpha,\gamma}_\lambda (x) := \theta_{a,\lambda} (x) + m^{\alpha,\gamma} (\lambda) \phi_{a,\lambda} (x).
$$

(32)

From (25) it follows immediately that

$$
\psi^{\alpha,\gamma}_\lambda (x) = \frac{1}{w^{\alpha,\gamma} (\lambda)} \phi_{c,\lambda}^\gamma (x) \text{ for all } \lambda \text{ not zeros of } w^{\alpha,\gamma} (\lambda).
$$

(33)

Thus from (20) it is clear that $\psi^{\alpha,\gamma}_\lambda (x)$ satisfies the boundary condition at $c$ and from (21) we have the following analogue of (2.18) at $c$:

$$
W_c (\psi^{\alpha,\gamma}_\lambda, \psi^{\alpha,\gamma}_\lambda') = 0 \text{ for all } \lambda, \lambda' \text{ with } \text{Im} \lambda \neq 0 \text{ and } \text{Im} \lambda' \neq 0.
$$

(34)

The expansion theorem for the operator $L^{\alpha,\gamma}$ can now be proved as in [31, p. 28-41], if one identifies Titchmarsh’s $'m (\lambda)'$ and $'\psi (x, \lambda)'$ functions with the functions $m^{\alpha,\gamma} (\lambda)$ and $\psi^{\alpha,\gamma}_\lambda (x)$ defined in (31) and (32). We note only that equation (34) corresponds to Titchmarsh’s Lemma 2.3, p.26, on which the rest of Titchmarsh’s discussion hinges. Following Titchmarsh one applies Green’s Formula to $\psi^{\alpha,\gamma}_\lambda (x)$ and $\psi^{\alpha,\gamma}_\lambda' (x)$ for $\text{Im} \lambda \neq 0 \text{ and } \text{Im} \lambda' \neq 0$ and makes use of (34) to obtain as a first result (cf. [31, p.28, Equa. 2.5.1]):

$$
\int_a^c \psi^{\alpha,\gamma}_\lambda (x) \psi^{\alpha,\gamma}_\lambda' (x) \, dx = \frac{m^{\alpha,\gamma} (\lambda) - m^{\alpha,\gamma} (\lambda')}{\lambda' - \lambda}.
$$

(35)

16Then numerator and denominator cannot simultaneously be zero since this would imply the linear dependence of $\phi_{a,\lambda}^\alpha (x)$ and $\theta_{a,\lambda}^\alpha (x)$, cf. Equation (22).
Titchmarsh’s technique can then be applied to obtain the eigenfunctions from (35) and the expansion theorem for the operator $L^{\alpha, \gamma}$ follows from [31, Theorem 2.7 (i) and (ii), p.31]. Thus if $\lambda_n, n = 1, 2, 3, \cdots$, are zeros of $w^{\alpha, \gamma}(\lambda)$ and if
\[
\tau_n^{\alpha, \gamma} := \text{Res}_{\lambda=\lambda_n} m^{\alpha, \gamma}(\lambda) = \frac{\tau^{\alpha, \gamma}(\lambda_n)}{w(\lambda_n)} > 0
\] (36)
then the functions,
\[
\psi_n^{\alpha, \gamma}(x) := \sqrt{\tau_n^{\alpha, \gamma}} \phi_{n, \lambda}^{\alpha}(x),
\] (37)
are the eigenfunctions of $L^{\alpha, \gamma}$ and form a complete orthonormal set in $L_2[a, c)$. Since the eigenvalues of the operator $L^{\alpha, \gamma}$ are the zeros of $w^{\alpha, \gamma}(\lambda)$, they can be computed from (28), or (29), as the case may be.

**Remark 2.6** The self-adjointness of the operator $L^{\alpha, \gamma}$ can be inferred from Titchmarsh’s equations (2.6.2), (2.6.3), together with the equation,
\[
R^\alpha_c(\Phi_\lambda, f) = 0, \tag{38}
\]
all of which hold for arbitrary $f \in L_2[a, c)$ and arbitrary $\lambda$, $\text{Im} \lambda \neq 0$. Alternatively, the self-adjointness of $L^{\alpha, \gamma}$ can be regarded as a consequence of the fact that it is a (closed) symmetric operator whose eigenfunctions constitute (because of Titchmarsh’s proof) a complete orthonormal set for $L_2[a, c)$, cf. Hellwig [11, p.184, Theorem 6].

**Remark 2.7** The existence of infinitely many zeros of $w^{\alpha, \gamma}(\lambda)$ can be regarded as a consequence of the expansion theorem. For if $w^{\alpha, \gamma}(\lambda)$ were to have only finitely many zeros, there would be only finitely many eigenfunctions $\psi_n^{\alpha, \gamma}(x)$, which then could not span the Hilbert Space $L_2[a, c)$.

From (31) and (23) we have the following representation of the limit-circle $'m(\lambda)'$-functions,
\[
m^{\alpha, \gamma}(\lambda) = \frac{(S\theta_{a, \lambda})_1(c) \cos \gamma + (S\theta_{a, \lambda})_2(c) \sin \gamma}{(S\phi_{a, \lambda})_1(c) \cos \gamma + (S\phi_{a, \lambda})_2(c) \sin \gamma}, \tag{39}
\]
with $\gamma \in [0, \pi)$, which is of the type (4), since the functions $(S\theta_{a, \lambda})(c)$ and $(S\phi_{a, \lambda})(c)$ are analytic in $\lambda$. From (23) we also have the following corollary:

**Corollary 2.2** Fix $\alpha \in [0, \pi)$. Then for each real $\lambda_0$ there exists exactly one $\gamma \in [0, \pi)$ such that $\lambda_0$ is in the spectrum of $L^{\alpha, \gamma}$. Thus if $\{\lambda_n^{\alpha, \gamma}, n = 1, 2, 3, \cdots\}$ is the spectrum of $L^{\alpha, \gamma}$ it follows that
\[
\bigcup_{\gamma \in [0, \pi]} \bigcup_{n=1}^{\infty} \lambda_n^{\alpha, \gamma} = \mathbb{R} \text{ for each fixed } \alpha. \tag{40}
\]

**17**This equation states that for arbitrary $f \in L_2[a, c)$, $\Phi_{\lambda, f}(x) := (\lambda - L^{\alpha, \gamma})^{-1} f(x)$ satisfies the $\gamma$-boundary condition at $c$. In Titchmarsh’s terminology the analogous statement would be, $\lim_{x \to c} W_2(\psi_{\lambda}, \Phi_{\lambda, f}(\lambda', f)) = 0$ for all $\lambda, \lambda'$ with $\text{Im} \lambda \neq 0$ and $\text{Im} \lambda' \neq 0$, a fact which Titchmarsh does not mention but which follows from Lemma 2.3.
Proof. : The spectrum of $L^{\alpha,\gamma}$ coincides with the set of zeros of $w^{\alpha,\gamma}(\lambda)$. Given $\lambda_0 \in \mathbb{R}$ we note from (23) that $\lambda_0$ is a zero of $w^{\alpha,\gamma}(\lambda)$ if and only if

$$\cot \gamma = -\frac{(S\phi_{a,\lambda_0}^\alpha)_2(c)}{(S\phi_{a,\lambda_0}^\alpha)_1(c)}. \quad (41)$$

Since $(S\phi_{a,\lambda_0}^\alpha)_1(c)$ and $(S\phi_{a,\lambda_0}^\alpha)_2(c)$ are not both zero (otherwise $W_c(\phi_{a,\lambda_0}^\alpha, \theta_{a,\lambda_0}^\alpha) = 0$ by (5)) (41) defines a unique value of $\gamma \in [0, \pi)$. (We assume the convention $\arccot(\pm\infty) = 0$, that is, $\gamma = 0$ when the right hand side in (41) is infinite.)

Remark 2.8 The expansion theorem for the operator $L^{\alpha,\gamma}$ can also be obtained by making use of ‘left-handed’ $m(\lambda)$-functions which are defined by

$$m^{\alpha,\gamma}_2(\gamma) := -\frac{P^{\alpha,\gamma}_2(\lambda)}{w^{\alpha,\gamma}(\lambda)}. \quad \text{(31)''}$$

Putting

$$\psi^{\alpha,\gamma}_{2,\lambda}(x) := \theta_{c,\lambda}^\gamma(x) + m^{\alpha,\gamma}_2(\lambda) \phi_{a,\lambda}(x), \quad \text{(32)''}$$

it is clear (cf. Equa. (25)) that

$$\psi^{\alpha,\gamma}_{2,\lambda}(x) = \frac{1}{w^{\alpha,\gamma}(\lambda)} \phi_{a,\lambda}(x) \text{ for all } \lambda \text{ not } 0 \quad \text{(33)''}$$

zeros of $w^{\alpha,\gamma}(\lambda)$, and therefore satisfies the boundary condition at $a$. Analogous to (34) we then have,

$$W_0 \left( \psi^{\alpha,\gamma}_{2,\lambda}, \psi^{\alpha,\gamma}_{2,\lambda'} \right) = 0 \text{ for all } \lambda, \lambda' \text{ not zeros } \quad \text{(34)''}$$

of $w^{\alpha,\gamma}(\lambda)$. Titchmarsh’s discussion of the expansion theorem then requires only slight modification to show that the eigenfunctions of $L^{\alpha,\gamma}$ are

$$\psi^{\alpha,\gamma}_{2,\lambda,n}(x) := \sqrt{-r^{\alpha,\gamma}_{2,\alpha,n}} \phi_{c,\lambda,n}^\gamma(x) \quad \text{(37)''}$$

where $\lambda_n$ are the zeros of $w^{\alpha,\gamma}(\lambda)$ and where,

\[19\] Because of (2.18) and footnote, the functions $\phi_{c,\lambda}^\gamma(x)$ and $\theta_{c,\lambda}^\gamma(x)$ are suitable for use in [5, p. 1364, Theorem 18]. If, in this theorem, $T$ is the self-adjoint operator $L^{\alpha,\gamma}$ and $\sigma_1(t,\lambda)$ and $\sigma_2(t,\lambda)$ are functions $\phi_{c,\lambda}^\gamma(t)$ and $\theta_{c,\lambda}^\gamma(t)$ respectively, then the left-handed $m(\lambda)$-function of (31)’ (or, as the case may be, of (39)’ below) is the function $\theta_{11}^\gamma(\lambda) = \theta_{11}^\gamma(\lambda)$ defined in this theorem. If, on the other hand, $\sigma_1(t,\lambda)$ and $\sigma_2(t,\lambda)$ are chosen to be functions $\phi_{a,\lambda}^\alpha(t)$ and $\theta_{a,\lambda}^\alpha(t)$ respectively, then the function $\theta_{11}^\gamma(\lambda) = \theta_{11}^\gamma(\lambda)$ is the right-handed $m(\lambda)$-function of (31). Representations of the type (0.4) for the function $\theta_{11}^\gamma(\lambda) = \theta_{11}^\gamma(\lambda)$ do not seem to occur in the book of Dunford and Schwartz. In this book the spectral theory and expansion theorem for eigenvalue problems involving the limit circle case at both end points is accomplished in [5, Theorem 2, p.1331 (n=2)], by appealing to the theory of Compact Operators. This approach seems to have the disadvantage of not yielding information on the dependence of the eigenvalues on the choice of boundary conditions. Compare, for example, the above mentioned theorem with our Corollary 2.1.
Evaluating the wronskians in (22) at \( x = 0 \) we have the following representation of the ‘left-handed’ \( m(\lambda) \)-functions:

\[
m^{\alpha, \gamma}(\lambda) = -\frac{\theta'_{c,\lambda}(a) \cos \alpha + \theta_{c,\lambda}(a) \sin \alpha}{\phi'_{c,\lambda}(a) \cos \alpha + \phi_{c,\lambda}(a) \sin \alpha}, \quad \alpha \in [0, \pi). \quad (39)'
\]

Note: The eigenfunctions \( \psi_{2,n}^{\alpha, \gamma}(x) \) in (37)' may differ in sign from those in (37) but we have in any case equality by taking the square, and equating the squares yields the equation,

\[
p_1(\lambda_n) p_2(\lambda_n) = 1,
\]

a fact which can be verified by looking at equation (24).

**Remark 2.9** For the interval \((a, c)\) the above theory can be applied if the limit circle case occurs at \( a \) as well as at \( c \). One has only to apply Theorem 2 to redefine the fundamental system \( \{\phi_{a,\lambda}(x), \theta_{a,\lambda}(x)\} \) by initial conditions at the singular point \( a \) alá Definition 1, and to replace the boundary condition (2.6) by,

\[
R_a^\alpha(f) := (Sf)_1(a) \cos \alpha + (Sf)_2(a) \sin \alpha, \quad (1.6)'
\]

where \( Sf \) is defined as in (3). (The definitions of \( D(L_0) \) and \( D(L_1) \) also require obvious modification.) The fundamental system \( \{\phi_{a,\lambda}(x), \theta_{a,\lambda}(x)\} \) then has the properties (10)-(15) as before with the above boundary condition. The functions in (22) can be defined as before and equations (24)-(27) remain valid, with the exception that (27) must be replaced by,

\[
\left( \begin{array}{c}
\phi_{a,0}^\alpha(x) \\
\theta_{a,0}^\alpha(x)
\end{array} \right) = \left( \begin{array}{cc}
\sin \alpha & -\cos \alpha \\
\cos \alpha & \sin \alpha
\end{array} \right) \left( \begin{array}{c}
u(x) \\
v(x)
\end{array} \right). 
\quad (27)'
\]

The ‘right-handed’ \( m (\lambda) \)-functions are as in (39) and the formula (39)' for the ‘left-handed’ \( m (\lambda) \)-functions changes to,

\[
m^{\alpha, \gamma}_2(\lambda) = \frac{(S\theta_{c,\lambda})_1(a) \cos \alpha + (S\theta'_{c,\lambda})_2(a) \sin \alpha}{(S\phi_{c,\lambda})_1(a) \cos \alpha + (S\phi'_{c,\lambda})_2(a) \sin \alpha}, \quad (39)''
\]

where \( \alpha \in [0, \pi) \).

**Remark 2.10** From the above theory and from Remark 2.8 and Remark 2.9 it is clear that there is no need to break the interval into two parts (Compare
Titchmarsh [31, p.42, Section 2.18]) in order to obtain \( \lambda \)'-functions for singular problems which involve the limit circle case at both endpoints. In fact as long as the limit circle case occurs at at least one end of the interval, Theorem 2 can be applied at that end of the interval to define a fundamental system at that end point and then the \( \lambda \)'-functions (right-handed or left-handed as the case may be) can be defined in terms of this fundamental system.

**Remark 2.11** It will be observed that the assumptions made on \( q(x) \) (cf. Equation (1.1)) include the possibility that
\[
\lim_{x \to c} q(x) \text{ exists.}
\]

Thus for \( c < \infty \), the spectral theory of the regular Sturm-Liouville problem on the closed interval \([a;c]\) is contained in the above discussion.

**Remark 2.12** If for \( b \in (a;c) \), functions \( \phi_{b,\lambda}^\gamma(x) \) and \( \theta_{b,\lambda}^\gamma(x) \) are defined as in (14) and (15) by putting \( b \) in place of \( c \), then the above discussion with \( \phi_{b,\lambda}^\gamma(x) \) and \( \theta_{b,\lambda}^\gamma(x) \) in place of \( \phi_{c,\lambda}^\gamma(x) \) and \( \theta_{c,\lambda}^\gamma(x) \) yields the spectral theory of the Sturm-Liouville operators, \( A_{b}^{\alpha,\gamma} \), \( \alpha, \gamma \in [0, \pi] \), defined by,
\[
D(A_{b}^{\alpha,\gamma}) := \{ f \in D(L_{1,b}) | R_{a}^{\alpha}(f) = 0 \text{ and } R_{b}^{\alpha}(f) = 0 \},
\]
where \( R_{a}^{\alpha}(f) \) is defined by,
\[
R_{a}^{\alpha}(f) := (Sf)_{1}(b) \cos \gamma + (Sf)_{2}(b) \sin \gamma
\]
for all \( f \in D(L_{1,b}) \). If \( w_{b}^{\alpha,\gamma}(\lambda) \) denotes the function defined in (22) and \( w_{c}^{\alpha,\gamma}(\lambda) \) the function with \( \phi_{b,\lambda}^\gamma(x) \) and \( \theta_{b,\lambda}^\gamma(x) \) in place of \( \phi_{c,\lambda}^\gamma(x) \) and \( \theta_{c,\lambda}^\gamma(x) \) it will be noted from (23) and (13) that
\[
\lim_{b \to c} w_{b}^{\alpha,\gamma}(\lambda) = w_{c}^{\alpha,\gamma}(\lambda)
\]
in the \( \lambda \)-plane. It follows (by application of Rouche’s Theorem) that each zero of \( w_{c}^{\alpha,\gamma}(\lambda) \) is a limit of zeros of \( w_{b}^{\alpha,\gamma}(\lambda) \) as \( b \to c \). This means that each point of the spectrum of \( L_{c}^{\alpha,\gamma} \) can be approximated by eigenvalues of the regular Sturm-Liouville operator \( A_{b}^{\alpha,\gamma} \) as \( b \to c \).

### 4 Boundary Functionals

In their encyclopedic volumes on Linear Operators N. Dunford and J.T. Schwartz take a Hilbert-Space approach to the spectral theory associated with nth-order differential equations, an approach which was first taken by M.H. Stone for

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20 There is a one-to-one correspondence between the \( \gamma \) and \( \beta \) parameters such that \( A_{b}^{\alpha,\gamma} = L_{b}^{\alpha,\beta} \). Compare Corollary 4.1, Lemma 6.1, and Remark 6.3 below.
second-order differential equations in [29, Chapter 10]. The notion of 'boundary values' used by Dunford and Schwartz in [5, Chapter XIII], will be particularly helpful to elucidate the connection between the boundary conditions in (7) and the boundary conditions given by H. Weyl, M.H. Stone, E.C. Titchmarsh, and K. Kodaira. For this reason we give in this section the definitions and results which will be needed. For results not proved here reference will be made to the book of Dunford and Schwartz. Also we will use the terminology 'boundary functional' instead of 'boundary value'.

**Definition 3.**

(i) A 'boundary functional for $\tau$' is a continuous linear functional on the (complete) Hilbert Space $(D(L_1), ||\cdot||^*)$ which vanishes on $D(L_0)$. Here $||\cdot||^*$ denotes the graph norm, i.e.

$$||f||^* = \left( \int_a^c |\tau f|^2 dx + \int_a^c |f|^2 dx \right)^{1/2}$$

A boundary functional for $\tau$, $b(\cdot)$, is called 'real' if

$$b(f) = b(f)$$

for all $f \in D(L_1)$.

(ii) A boundary functional 'for $\tau$ at $c$' is a boundary functional for $\tau$, $b(\cdot)$, such that,

$$f \in D(L_1)$$

and $f = 0$ in a Nbd. of $c$ implies $b(f) = 0$.

Boundary functionals 'for $\tau$ at $a$' are similarly defined.

(iii) If $b(\cdot)$ is a boundary functional for $\tau$ at $c$ then

$$b(f) = 0, f \in D(L_1)$$

is called a boundary condition 'for $\tau$ at $c$'. This boundary condition is called 'real' if $b(\cdot)$ is a real boundary functional.

(iv) Two boundary functionals for $\tau$, $b_1(\cdot)$ and $b_2(\cdot)$, are called 'equivalent' if they are linearly dependent.

**Remark 3.1** A boundary functional for $\tau$ necessarily vanishes on $D(L_0)$ since it vanishes on $D(L_0)$ and is continuous in the graph norm. From the decomposition,

$$D(L_1) = D(L_0) \oplus D^+ \oplus D^-$$

(where $D^+$ and $D^-$ are the positive and negative deficiency spaces, cf. [5, p.1226-27, Definition 9 and Lemma 10]) it follows that the space $M$ of boundary values is due to J.W. Calkin, [3].
functionals for \( \tau \) can be identified as the 4-dimensional space of linear functionals on the 4-dimensional Hilbert Space \((D^+ \oplus D^-, \|\cdot\|)\).

**Lemma 3.1**

(i)\(^{23}\) If \( a \) is a regular endpoint, then \( b_a (\cdot) \) is a boundary functional for \( \tau \) at \( a \) if and only if
\[
 b_a (f) = z_1 f (a) + z_2 f' (a) \quad \text{for some } z_1, z_2 \in \mathbb{C}. \tag{1} \]

(ii)\(^{24}\) If \( c \) is a singular end point, then \( b_c (\cdot) \) is a boundary functional for \( \tau \) at \( c \) if and only if there exists a function \( w(x) \in D(L_1) \) such that
\[
 b_c (f) = W_c (w, f) \quad \text{for all } f \in D(L_1). \tag{2} \]

(iii) If \( M \) is the space of boundary functionals for \( \tau \) and \( M_a, M_c \) the spaces of boundary functionals for \( \tau \) at \( a \) and \( c \) respectively, then \( M_a \) and \( M_c \) are two-dimensional spaces\(^{25}\) and
\[
 M = M_a \oplus M_c. \tag{3} \]

**Remark 3.2** For each \( f \in D(L_1) \) let
\[
 b_{1,c} (f) := (Sf)_1 (c) \quad \text{and } b_{2,c} (f) := (Sf)_2 (c). 
\]

From (2) and (3,4) it is clear that \( b_{1,c} (\cdot) \) and \( b_{2,c} (\cdot) \) are boundary functionals for \( \tau \) at \( c \). Also because of (1),
\[
 z_1 b_{1,c} (\cdot) + z_2 b_{2,c} (\cdot) = 0 \quad \text{implies } z_1 = z_2 = 0. 
\]

Hence \( b_{1,c} (\cdot) \) and \( b_{2,c} (\cdot) \) are linearly independent boundary functionals for \( \tau \) at \( c \) and therefore span the space \( M_c \). We note also that they are real boundary functionals since the functions \( u(x) \) and \( v(x) \) in (1) were taken to be real-valued.

From Lemma 3.1 (i) and (iii) and from Remark 3.2 it is clear that any boundary functional for \( \tau, b(\cdot) \), can be represented in the form,
\[
 b(f) = z_1 f (a) + z_2 f' (a) + z_3 (Sf)_1 (c) + z_4 (Sf)_2 (c), \tag{4} 
\]
where \( z_1, z_2, z_3, \) and \( z_4 \) are complex numbers.

**Definition 4.**\(^{27}\)

(i) If in the representation (4), \( |z_1|^2 + |z_2|^2 > 0 \) and \( |z_3|^2 + |z_4|^2 > 0 \), the boundary functional \( b(\cdot) \) is called a ‘mixed’ boundary functional.

\(^{23}\) Dunford and Schwarz [5, p.1301, Corollary 23.]
\(^{24}\) ibid. p. 1302, Theorem 27, and p. 1303, Corollary 28.
\(^{25}\) ibid. p. 1305-06.
\(^{26}\) ibid. p. 1298, Theorem 19.
Two boundary functionals $b_1(\cdot)$ and $b_2(\cdot)$ are called ‘separated’ if neither
is a mixed boundary functional. The corresponding boundary conditions
are then also called ‘separated boundary conditions’.

The following theorem gives a necessary and sufficient condition for two bound-
dary functionals for $\tau$ at $c$ to be equivalent.

**Theorem 3.**
Let $w_1(x)$ and $w_2(x)$ be functions in $D(L_1)$ such that

$$b_1(f) := W_c(w_1, f) \quad \text{and} \quad b_2(f) := W_c(w_2, f)$$

are nontrivial (that is, nonzero) boundary functionals for $\tau$ at $c$ (i.e. $z_1 = z_2 = 0$
and $|z_3|^2 + |z_4|^2 > 0$ in (4)). Then: The boundary functionals $b_1(\cdot)$ and $b_2(\cdot)$
are equivalent if and only if

$$W_c(w_1, w_2) = 0.$$

In this event we have,

$$W_c(w_1, f) = KW_c(w_2, f) \quad \text{for all} \quad f \in D(L_1) \quad \text{with} \quad K \neq 0 \quad \text{where} \quad (5)$$

$$K = \begin{cases} 
\frac{(Sw_1)_1(c)}{(Sw_2)_1(c)} = \frac{(Sw_1)_2(c)}{(Sw_2)_2(c)}, \quad \text{if} \quad (Sw_2)_1(c) \neq 0 \quad \text{and} \quad (Sw_2)_2(c) \neq 0 \\
\frac{(Sw_1)_1(c)}{(Sw_2)_1(c)}, \quad \text{if} \quad (Sw_2)_2(c) = 0 \\
\frac{(Sw_1)_2(c)}{(Sw_2)_2(c)}, \quad \text{if} \quad (Sw_2)_1(c) = 0
\end{cases}$$

**Proof.** Applying (5) we have for all $f \in D(L_1)$:

$$b_1(f) = (Sw_1)_1(c)(Sf)_2(c) - (Sw_1)_2(c)(Sf)_1(c)$$

$$b_2(f) = (Sw_2)_1(c)(Sf)_2(c) - (Sw_2)_2(c)(Sf)_1(c)$$

$$W_c(w_1, w_2) = (Sw_1)_1(c)(Sw_2)_2(c) - (Sw_1)_2(c)(Sw_2)_1(c)$$

**Observations:**

1. If $W_c(w_1, w_2) = 0$, the null spaces of $b_1(\cdot)$ and $b_2(\cdot)$ are identical and they
   are therefore linearly dependent.
2. Now suppose \( b_1 (\cdot) \) and \( b_2 (\cdot) \) are equivalent boundary functionals and that 
\[ W_c (w_1, w_2) \neq 0. \]
The \( w_2 (x) \) is in the null space of \( b_2 (\cdot) \), but not in the null space of \( b_1 (\cdot) \), which contradicts the assumption that \( b_1 (\cdot) \) and \( b_2 (\cdot) \) are linearly dependent.

To obtain the value of \( K \) in (5) the above expressions for \( b_1 (f) \) and \( b_2 (f) \) can be substituted on both sides of (5) and the value of \( K \) determined in terms of \( (Sw_j)_1, i = 1, 2 \) and \( j = 1, 2 \), by making use of the fact that \( (Sw_j)_1 \) and \( (Sw_j)_2 \), \( j = 1, 2 \) are not both zero for \( j = 1, 2 \).

5 The Characteristic Functions of K. Kodaira and the Rellich Initial Numbers

In his famous paper on the spectral theory on the spectral theory of second-order differential equations [16] K.Kodaira defines \( m (\lambda) \) – functions in the limit circle case which are of the same type introduced in section 2. He does not, however, give a parametrization of his ‘characteristic functions’ of the type (1.4).

In this section we show that each of Kodaira’s characteristic functions is one of the functions we have defined in (31).

Kodaira’s boundary conditions at the singular point \( c \) (cf. [16, p.924, Equa. 1.7]) are,
\[ W_c (w_c, f) = 0, \]
where \( w_c (x) \) is an arbitrary real-valued function in \( D (L_1) \) satisfying a condition of ‘nontriviality’ [16, p.924, Equa. 1.6]) which can be expressed by,
\[ (Sw_c)_1^2 (c) + (Sw_c)_2^2 (c) \neq 0. \]

For the interval \([a, c)\) the corresponding Weyl Spaces are
\[ U_{3^\alpha,w_c} := \{ f \in D (L_1) \mid R_\alpha (f) = 0, W_c (w_c, f) = 0 \}, \alpha \in [0, \pi). \]
Kodaira associates with this Weyl Space a ‘characteristic function’, which is defined [16, p.925, Equa. (1.9) and p.934, Section 4] by,
\[ f_c (\lambda) := \frac{W_c (w_c, \theta^\alpha_{a,\lambda})}{W_c (w_c, \phi^\alpha_{a,\lambda})}. \]

It will be noted that the boundary functional \( R_\alpha (\cdot) \) of (3.7) is of the type (1) with
\[ w_c (x) = -\cos \gamma v (x) + \sin \gamma u (x) \]
and that with this choice of \( w_c (x) \) Kodaira’s characteristic function \( f_c (\lambda) \) is precisely the function \( m^{\alpha, \gamma} (\lambda) \) in (3.39). Because of the arbitrariness which Kodaira allows in the choice of the function \( w_c (x) \) it is not exactly clear that
every Weyl Space of the type (3) is of the type (3.6) for some value of $\gamma$. To see that this is the case we prove the following theorem:

**Theorem 4.**

(i) The boundary functionals, $R_\gamma^w := (S(\cdot))_1(c)\cos\gamma + (S(\cdot))_2(c)\sin\gamma$ and $R_\gamma^{w,c}(\cdot) := W_c(w_c,\cdot)$ with $w_c \in D(L_1)$ and real-valued are equivalent if and only if

$$\gamma = \text{Arc}
\cot \left( \frac{(Sw_c)_2(c)}{(Sw_c)_1(c)} \right) \quad \text{(Here Arc\cot(\pm\infty) = 0.)} \quad (5)$$

(ii) If $\gamma$ is as in (5) then

$$W_c(w_c,f) = KR_\gamma^c(f) \text{ for all } f \in D(L_1) \text{ with } K \neq 0 \quad (6)$$

where

$$K = \begin{cases} 
\frac{(Sw_c)_1(c)}{\sin\gamma} = \frac{(Sw_c)_2(c)}{-\cos\gamma}, & \text{if } \gamma \neq 0, \pi/2 \\
-(Sw_c)_2(c), & \text{if } \gamma = 0 \\
(Sw_c)_1(c), & \text{if } \gamma = \pi/2 
\end{cases}$$

Moreover,

$$f_c(\lambda) = m^{\alpha,\gamma}(\lambda) \text{ for all } \lambda \text{ not zeros of } w^{\alpha,\gamma}(\lambda). \quad (7)$$

**Proof.** Applying Theorem 3 and (5) we observe that $W_c(w_c, u \sin\gamma - v \cos\gamma) = 0$ if and only if

$$\cot\gamma = -\frac{(Sw_c)_2(c)}{(Sw_c)_1(c)}.$$

Since $w_c(x)$ is real-valued $(Sw_c)_1(c)$ and $(Sw_c)_2(c)$ are real numbers and they cannot both be zero because of the nontriviality condition (2). Therefore (5) defines a unique value of $\gamma \in [0, \pi)$. The formula for $K$ in (6) follows at once from (5) with $w_1(x) = w_c(x)$ and $w_2(x) = -\cos\gamma v(x) + \sin\gamma u(x)$. To see (7) we observe from (4) and (6) that,

$$f_c(\lambda) = \frac{-KR_c^\gamma(\theta_{\alpha,\lambda})}{KR_c^\gamma(\phi_{\alpha,\lambda})} = \frac{-R_c^\gamma(\theta_{\alpha,\lambda})}{R_c^\gamma(\phi_{\alpha,\lambda})},$$

which, comparing with (3.39), is readily seen to be $m^{\alpha,\gamma}(\lambda).$ ■

We now make use of Theorem 4 to establish the connection between the boundary functionals of (3.7) and the Rellich Initial Numbers. In [22] F. Rellich
showed [22, p.354, Equa. 9] that the boundary conditions of H.Weyl could be written in the form,

\[ f_1 \cos \nu + f_2 \sin \nu = 0, \nu \in [0, \pi), \]

(8)

where \( f_1 \) and \( f_2 \) are 'initial numbers' associated with each \( f \in D(L_1) \) which are defined as follows:

Let \( \mu \) be a real number and \( \{ \psi_\mu (x), \omega_\mu (x) \} \) a real-valued fundamental system of (8) for \( \lambda = \mu \) with

\[ W (\psi_\mu, \omega_\mu) = \Delta \neq 0. \]

(9)

For arbitrary \( f \in D(L_1) \), \( f \) is a solution of the inhomogeneous equation,

\[ \tau u - \mu u = g(x) := \tau f(x) - \mu f(x), \]

and is therefore uniquely representable in the form,

\[
\begin{align*}
f(x) &= f_1 \psi_\mu (x) + f_2 \omega_\mu (x) \\
&\quad + \omega_\mu (x) \int_x^c \psi_\mu (y) [\tau f(y) - \mu f(y)] dy \\
&\quad - \psi_\mu (x) \int_x^c \omega_\mu (y) [\tau f(y) - \mu f(y)] dy
\end{align*}
\]

(10)

with \( f_1 \) and \( f_2 \) in \( \mathbb{C} \). The numbers \( f_1 \) and \( f_2 \) which are uniquely defined by (10) for each \( f \in D(L_1) \) are called the 'initial numbers' for \( f \) with respect to \( \mu, \psi_\mu \) and \( \omega_\mu \). The Weyl Spaces associated with the boundary conditions in (8) are

\[ U_4^{\alpha, \nu} := \{ f \in D(L_1) | R_4^{\alpha} (f) = 0, f_1 \cos \nu + f_2 \sin \nu = 0 \}, \]

(11)

with \( \alpha \in [0, \pi) \) and \( \nu \in [0, \pi) \). Calculating the wronskians, \( W_\mu (f, w_\mu) \) and \( W_\mu (f, \psi_\mu) \) from (10) and making use of the fact that \( W_\mu (x) \) and \( \psi_\mu (x) \) and \( \tau f - \mu f \) are in \( L_2(a,c) \), it is readily shown that

\[ f_1 = \Delta \cdot W_\mu (f, w_\mu) \quad \text{and} \quad f_2 = \Delta \cdot W_\mu (f, -\psi_\mu). \]

(12)

Comparing (12) with (3.3) it will be noticed that if \( \mu = 0 \) and if the fundamental system \( \{ \psi_0 (x), \omega_0 (x) \} \) is taken to be the fundamental system \( \{ u(x), v(x) \} \) chosen in (3.1), then

\[ f_1 = (Sf)_1 (c) \quad \text{and} \quad f_2 = (Sf)_2 (c), \]

\[ 28 \text{In [22] Rellich assumes that (1.8) is nonoscillatory for real } \mu \text{ at the singular point (which means – p.349, Satz 2b – that the self-adjoint operators associated with } \tau \text{ are bounded below), and takes } w_\mu (x) \text{ and } \psi_\mu (x) \text{ to be principal and nonprincipal solutions respectively. These additional restrictions are not, however, necessary for (4.8).} \]

\[ 29 \text{Compare (4.8) with K.Jörgens [15, p. 9.10, Corollary to Theorem 4].} \]
so that the boundary conditions in (8) are identical to those in (3.7) with \( \nu = \gamma \). In this case the Weyl Spaces in (11) are identical with those in (3.6), and the associated ‘\( m(\lambda) \)' - functions of (3.39) can be written as,

\[
m^{\alpha, \gamma}(\lambda) = -\left\{ \theta^\alpha_{a, \lambda} \right\}_1 \cos \gamma + \left\{ \phi^\alpha_{a, \lambda} \right\}_2 \sin \gamma, \quad \gamma \in [0, \pi),
\]

(13)

where \( \left\{ \theta^\alpha_{a, \lambda} \right\}_1, \left\{ \theta^\alpha_{a, \lambda} \right\}_2 \) and \( \left\{ \phi^\alpha_{a, \lambda} \right\}_1, \left\{ \phi^\alpha_{a, \lambda} \right\}_2 \) are the Rellich initial numbers for \( \theta^\alpha_{a, \lambda} \) and \( \phi^\alpha_{a, \lambda} \) at \( c \) with respect to \( u(x), v(x) \).

If \( \mu \neq 0 \), or if \( \mu = 0 \) and

\[
\begin{pmatrix}
\psi_0(x) \\
\omega_0(x)
\end{pmatrix} \neq \begin{pmatrix}
u(x) \\
v(x)
\end{pmatrix},
\]

we have in any case the following corollary of Theorem 4:

**Corollary 4.1**

(i) The boundary functionals

\[
R^\alpha_c(\cdot) : = (S(\cdot))_1(c) \cos \gamma + (S(\cdot))_2(c) \sin \gamma \text{ and}
\]

\[
P^\nu_e(f) : = f_1 \cos \nu + f_2 \sin \nu
\]

are equivalent if and only if

\[
\gamma = \text{Arc cot} \left( \frac{(S\omega_\mu)_2(c) \cos \nu - (S\psi_\mu)_2(c) \sin \nu}{-(S\omega_\mu)_1(c) \cos \nu + (S\psi_\mu)_1(c) \sin \nu} \right)
\]

(14)

(ii) Moreover, (14) defines a one-to-one correspondence between the parameters \( \gamma \) and \( \nu \) under which \( U_2^{\alpha, \gamma} = U_4^{\alpha, \nu} \) and

\[
m^{\alpha, \gamma}(\lambda) = -\left\{ \theta^\alpha_{a, \lambda} \right\}_1 \cos \nu + \left\{ \phi^\alpha_{a, \lambda} \right\}_2 \sin \nu
\]

\[
\left\{ \phi^\alpha_{a, \lambda} \right\}_1 \cos \nu + \left\{ \phi^\alpha_{a, \lambda} \right\}_2 \sin \nu.
\]

(15)

**Proof.** (i) Because of (12) the boundary condition (8) may be written as,

\[
W_e(-\cos \nu \omega_\mu + \sin \nu \psi_\mu, f) = 0,
\]

which is of the type (1) since \( \psi_\mu(x) \) and \( \omega_\mu(x) \) are real-valued. Thus (i) follows immediately from Theorem 4 with

\[
w_e(x) = \Delta \cdot \left[ -\cos \nu \omega_\mu(x) + \sin \nu \psi_\mu(x) \right].
\]

To see that (14) defines a one-to-one correspondence between \( \gamma \in [0, \pi) \) and \( \nu \in [0, \pi) \) it suffices to show (since \( (S\omega_\mu)(c) \) and \( (S\psi_\mu)(c) \) are real-valued) that

\[
A(\gamma) := \frac{(S\omega_\mu)_2(c) \gamma - (S\psi_\mu)_2(c)}{-((S\omega_\mu)_1(c) \gamma + (S\psi_\mu)_1(c))}
\]

27
defines an invertible linear fractional transformation. The invertibility follows from (9) because of (3.5).

(ii) Because of the equivalence of the boundary functionals in (i) we have,

\[ P^\nu_c (\cdot) = KR^\gamma_c (\cdot) \text{ for some constant } K \neq 0 \]

when \( \nu \) and \( \gamma \) are related as in (14). Hence

\[ -\frac{P^\nu_c (\theta_{a,\lambda})}{P^\nu_c (\phi_{a,\lambda})} = -\frac{KR^\gamma_c (\theta_{a,\lambda})}{KR^\gamma_c (\phi_{a,\lambda})} = -\frac{R^\gamma_c (\theta_{a,\lambda})}{R^\gamma_c (\phi_{a,\lambda})}, \]

which proves (15). ■

6 The Weyl Spaces of Hermann Weyl

Boundary conditions of the type originally given by H.Weyl in his famous paper [37] are as follows:30

Let \( u_\lambda (x) \) be a solution of (2.8) for \( \mathrm{Im} \lambda \neq 0 \) satisfying

\[ W_c (u_\lambda, \bar{u}_\lambda) = 0. \quad (1) \]

Then

\[ b (f) := W_c (u_\lambda, f) \text{ for } f \in D (L_1) \quad (2) \]

is a boundary functional for \( \tau \) at \( c \) and the corresponding Weyl Spaces are,

\[ U_5^{\tau, u_\lambda} := \{ f \in D (L_1) | R^\alpha_a (f) = 0, W_c (u_\lambda, f) = 0 \}. \quad (3) \]

Remark 5.1 For \( \lambda = i \) a direct proof of the self-adjointness of operators having domains of the type (3) was given by F.Rellich in his 1950-51 lecture notes [23, p.39]. This proof was later improved by A.Schneider in [28] and was included in the book of G.Hellwig [11, p.238, Theorem 1]. The proof can be modified to obtain the self-adjointness of spaces of the type (3) for any \( \lambda, \mathrm{Im} \lambda \neq 0 \).

In this section we show that each boundary functional of the type (2) is equivalent to one of the boundary functionals in (3.7) for some value of \( \gamma \in [0, \pi) \). It will be noted that this does not follow directly from Theorem 4 because the function \( u_\lambda (x) \) in (1) is complex-valued. We require the following two lemmas:

Lemma 5.1

---

30In [37], H.Weyl considers only the case \( \lambda = i \).
(i) For each \( \lambda, \Im \lambda \neq 0 \),

\[
T_{c,\lambda}^{\alpha} (z) := -\frac{(S\theta_{a,\lambda})_1 (c) z + (S\theta_{a,\lambda})_2 (c)}{(S\phi_{a,\lambda})_1 (c) z + (S\phi_{a,\lambda})_2 (c)}
\]

is an invertible linear fractional transformation which maps the real \( z \)-axis onto the \( \lambda \)-limit circle, \( C_{c,\lambda}^{\alpha} \).

(ii) If \( z = (T_{c,\lambda}^{\alpha})^{-1} (w) \), then for all \( \lambda \) with \( \Im \lambda \neq 0 \) and all \( w \in \mathbb{C} \) we have:

\[
\left| (S\theta_{a,\lambda})_1 (c) + w (S\phi_{a,\lambda})_1 (c) \right|^2 \frac{\Im z}{\nu} = \frac{\bar{W}_c (S (\theta_{a,\lambda} + w\phi_{a,\lambda}), S (\theta_{a,\lambda} + w\phi_{a,\lambda}))}{2iv} = \int_a^c \left| \theta_{a,\lambda} (x) + w\phi_{a,\lambda} (x) \right|^2 \, dx + \frac{\Im w}{\nu}
\]

where \( \theta_{a,\lambda}^\alpha \) and \( r_{c,\lambda}^\alpha \) are the center of radius of the limit circle, \( C_{c,\lambda}^{\alpha} \), cf. equation (24).

**Proof.** (i) The fact that \( T_{c,\lambda}^{\alpha} \) is invertible follows from (3.16) because of (3.5). The image of the real \( z \)-axis under \( T_{c,\lambda}^{\alpha} \) can be computed (Ahlfors [2, p.79], for example) and is found to be the circle with

\[
\text{center} = -\frac{\bar{W}_c (S\theta_{a,\lambda}, S\theta_{a,\lambda})}{W_c (S\phi_{a,\lambda}, S\phi_{a,\lambda})}, \quad \text{and}
\]

\[
\text{radius} = \frac{1}{\left| W_c (S\phi_{a,\lambda}, S\phi_{a,\lambda}) \right|}.
\]

This circle is the \( \lambda \)-limit circle, \( C_{c,\lambda}^{\alpha} \), because of (3.5) and (2.24).

(ii) The first part of (5) follows by computation of \( \Im z \). The rest follows by letting \( b \to c \) in (2.23) and using (3.5).

**Lemma 5.2** For fixed \( \alpha \in [0, \pi) \) and fixed \( \lambda, \Im \lambda \neq 0 \),

\[
f_{a,\lambda} : \gamma \to m^{\alpha,\gamma} (\lambda)
\]

defines a one-to-one mapping from \([0, \pi)\) onto the \( \lambda \)-limit circle, \( C_{c,\lambda}^{\alpha} \).
Proof. From the representation (3.39) we have,

$$m^{\alpha,\gamma}(\lambda) = T_{c,\lambda}^\alpha (\cot \gamma) \text{ for all } \gamma \in [0, \pi),$$

so the result is immediate from Lemma 5.1 (i). \qed

Remark 5.2 There are obvious analogous of Lemma 5.1 and 5.2 associated with the functions $m^{\alpha,\gamma}_b(\lambda)$ defined in Remark 2.12 for $b < c$. In this case the linear fractional transformation, $T_{b,\lambda}^\alpha$, maps the real $z$-axis onto the circle $C_b^\alpha$ (defined by (2.22)) for each $\lambda$, $\text{Im } \lambda \neq 0$.

Remark 5.3 There are also analogous of Lemmas 5.1 and 5.2 associated with the left-handed $m(\lambda)$- functions of Remark 2.8.

From (5) and (3.5) it is clear that $w \in C_{c,\lambda}^\alpha$ if and only if

$$W_c (\theta_{a,\lambda} + w\phi_{a,\lambda}, \theta_{a,\lambda} + w\phi_{a,\lambda}) = 0. \quad (8)$$

Now fix $\alpha \in [0, \pi)$, and consider the family of Weyl Spaces, $U_{S}^{\alpha,a,\lambda}$, for different choices of $u_\lambda$. Because of (8) a solution of (2.8) for $\text{Im } \lambda \neq 0$ satisfying (1) must expressible in the form,

$$u_\lambda (x) = K \cdot [\theta_{a,\lambda}^\alpha (x) + w\phi_{a,\lambda}^\alpha (x)] \text{ with } w \in C_{c,\lambda}^\alpha \quad (9)$$

and $K \neq 0$. (The possibility that $u_\lambda (x)$ could be linearly dependent with $\phi_{a,\lambda}^\alpha (x)$ is ruled out by (2.24).) Hence for each $\lambda$, $\text{Im } \lambda \neq 0$, the class $F_{c,\lambda}^\alpha$ of nonequivalent boundary functionals type (2) associated with $\lambda$ can be described as follows:

$$F_{c,\lambda}^\alpha := \{ b(\cdot) \mid b(\cdot) = W_c (\theta_{a,\lambda}^\alpha + Q\phi_{a,\lambda}^\alpha, \cdot) \text{ and } Q \in C_{c,\lambda}^\alpha \} \quad (10)$$

We now show that the one-to-one correspondence in (6) identifies, for each $\gamma \in [0, \pi)$, the Weyl-boundary-functional in $F_{c,\lambda}$ which is equivalent to the boundary functional $R_c^\gamma (\cdot)$ of (3.7). More precisely we have:

Theorem 5. Fix $\lambda_0$, $\text{Im } \lambda_0 \neq 0$ and fix $\alpha \in [0, \pi)$. Then the boundary functional,

$$R_{c,\lambda_0}^Q (\cdot) := W_c (\theta_{a,\lambda_0}^\alpha + Q\phi_{a,\lambda_0}^\alpha, \cdot) \text{ with } Q \in C_{c,\lambda_0}^\alpha \quad (11)$$

is equivalent to the boundary functional,

$$R_c^\gamma (\cdot) := (S(\cdot))_1 (c) \cos \gamma + (S(\cdot))_2 (c) \sin \gamma \quad (12)$$

if and only if

$$Q = m^{\alpha,\gamma}(\lambda_0). \quad (13)$$

30
Proof. If \( Q = m^{\alpha, \gamma} (\lambda_0) \) then from (3.33) and (3.19) we have,
\[
R^Q_{c, \lambda_0} (\cdot) = \frac{1}{w^{\alpha, \gamma} (\lambda_0)} W_c \left( \phi_{c, \lambda_0}^{\gamma}, \cdot \right) = \frac{1}{w^{\alpha, \gamma} (\lambda_0)} R^\gamma_c (\cdot),
\]
and therefore \( R^Q_{c, \lambda_0} (\cdot) \) and \( R^\gamma_c (\cdot) \) are linearly dependent since \( w^{\alpha, \gamma} (\lambda_0) \neq 0 \).
Now suppose \( Q \neq m^{\alpha, \gamma} (\lambda_0) \). By assumption \( Q \in C^{\alpha}_{c, \lambda_0} \) and therefore by Lemma 5.2 there exists \( \gamma' \in [0, \pi) \) such that
\[
Q = m^{\alpha, \gamma'} (\lambda_0).
\]
From (3.33) and (3.19) we again have,
\[
R^Q_{c, \lambda_0} (\cdot) = \frac{1}{w^{\alpha, \gamma'} (\lambda_0)} W_c \left( \phi_{c, \lambda_0}^{\gamma'}, \cdot \right) = \frac{1}{w^{\alpha, \gamma'} (\lambda_0)} R^{\gamma'}_c (\cdot),
\]
but \( \gamma' \neq \gamma \) then yields a contradiction because of (3.21). □

Remark 5.4 The choice of a single point \( Q \) on the circle \( C^{\alpha}_{c, \lambda_0} \) for a single \( \lambda_0 \), \( \text{Im} \lambda_0 \neq 0 \), uniquely fixes points \( P_{\lambda} (Q) \) on \( C^{\alpha}_{c, \lambda} \) for every other \( \lambda, \text{Im} \lambda \neq 0 \), such that the boundary functionals
\[
R^Q_{c, \lambda_0} (\cdot) \text{ and } R^{P_{\lambda} (Q)}_{c, \lambda} (\cdot)
\]
are equivalent. The collection of all these points \( P_{\lambda} (Q) \) is the function \( m^{\alpha, \gamma} (\lambda) \) where the value of \( \gamma \) is determined by \( Q \) according to the equation (13).

Remark 5.5 N. Levinson defines Weyl Spaces in the limit circle case with boundary functionals of the type (11), cf. [4, p.242, Theorem 4.1, and p.244, Equa. (4.10)]. For each boundary functional of the type (11) Levinson defines an \( \ell (\lambda)' \)-function (depending on \( \lambda_0 \) and \( Q \)) by,
\[
\ell (\lambda) := \frac{Q - (\lambda - \lambda_0) \int_a^c \theta_{a, \lambda} (x) \left[ \theta_{a, \lambda} (x) + Q \phi_{a, \lambda} (x) \right] dx}{1 + (\lambda - \lambda_0) \int_a^c \phi_{a, \lambda} (x) \left[ \theta_{a, \lambda} (x) + Q \phi_{a, \lambda} (x) \right] dx} \tag{14}
\]
The meromorphic character of this \( \ell (\lambda) \) function arises easily from the fact that both numerator and denominator are entire in \( \lambda \). The representation (14) is however, nothing special. For if one solves equation (3.35) for \( m^{\alpha, \gamma} (\lambda) \) in terms of \( m^{\alpha, \gamma} (\lambda') \) one finds,
\[
m^{\alpha, \gamma} (\lambda) = \frac{m^{\alpha, \gamma} (\lambda') - (\lambda - \lambda') \int_a^c \theta_{a, \lambda'} (x) \left[ \theta_{a, \lambda'} (x) + m^{\alpha, \gamma} (\lambda') \phi_{a, \lambda'} (x) \right] dx}{1 + (\lambda - \lambda') \int_a^c \phi_{a, \lambda'} (x) \left[ \theta_{a, \lambda'} (x) + m^{\alpha, \gamma} (\lambda') \phi_{a, \lambda'} (x) \right] dx} \tag{15}
\]
\[31\text{Sign differences arise from Levinson's choice of the parameter in the boundary condition at } a, \text{ cf. [4, p.231].} \]
\[32\text{The representation (5.15) and the argument that the right hand side is meromorphic in } \lambda \text{ was noticed by D.B.Sears in [25, p.209, section 4].} \]
which holds arbitrarily in $\lambda, \lambda'$ so long as $\lambda, \lambda'$ are not zeros of $w^{\alpha,\gamma}(\lambda)$. Putting $\lambda' = \lambda_0$ and letting $\gamma$ be defined by (13) it is readily seen that (15) reduces to (14). Hence the $'m(\lambda)'$-function defined by the formula (14) is precisely the function $m^{\alpha,\gamma}(\lambda)$ where $Q$ determines $\gamma$ as in (13). It will be noted that Levinson allows his Weyl Space to depend on the choice of $\lambda_0$ as well as on the choice of $Q \in C^a_{c,\lambda_0}$. As we have already noted in Remark 5.4 the Weyl Space and corresponding $m(\lambda)$-function is fixed merely by the choice of $Q$ on $C^a_{c,\lambda_0}$. Thus the Weyl Space does not depend in any special way on $\lambda_0$ itself.

Remark 5.6 The choice of $\alpha$ in the representation (10) has no affect on the class $F_{c,\lambda}$ of (nonequivalent) boundary functionals Weyl type. This follows from Theorem 5 since the functionals $R^a_a(\cdot)$ depend on $\gamma$ only.

Remark 5.7 There are obvious analogous of Theorem 5 and Remarks 5.4, 5.5, and 5.6 relating the boundary functionals $R^a_a(\cdot)$ of (2.6) to boundary functionals of Weyl type at $a$. Similarly there are analogous of Theorem 5 and Remarks 5.4, 5.5, and 5.6 relating the boundary functionals of (2.7)$'$ for $a < b < c$ (cf. Remark 2.12) to boundary functionals of Weyl type at $b$.

From Theorem 5, Remark 5.4, and Remark 5.7 the following corollary is self-evident:

**Corollary 5.3** Let $m^\alpha_1(\lambda)$ and $m^\alpha_2(\lambda)$ denote the right-handed and left-handed $m(\lambda)$-functions for $L^{\alpha,\gamma}$ defined in (3.31) and (3.31)$'$ respectively. For fixed $\alpha, \gamma \in [0, \pi)$ we have:

$$\{ f \in D(L_1) | R^a_a(\cdot) = 0, \; R^b_b(\cdot) = 0 \}$$

$$= \{ f \in D(L_1) | W_a \left( \theta^\gamma_{c,\lambda} + m^\alpha_2(\lambda) \phi^\gamma_{c,\lambda}; f \right) = 0$$

and $W_c \left( \theta^\alpha_{a,\lambda'} + m^\alpha_1(\lambda') \phi^\gamma_{a,\lambda'}; f \right) = 0$

where $\lambda$ and $\lambda'$ are fixed complex numbers which are not zeros of $w^{\alpha,\gamma}(\lambda)$

$$= \{ f \in D(L_1) | W_a \left( \theta^\gamma_{c,\lambda} + m^\alpha_2(\lambda) \phi^\gamma_{c,\lambda}; f \right) = 0$$

and $W_c \left( \theta^\alpha_{a,\lambda'} + m^\alpha_1(\lambda') \phi^\gamma_{a,\lambda'}; f \right) = 0$ \hspace{1cm} (16)

for all $\lambda, \; \text{Im} \lambda \neq 0$.

---

33 For the interval $[a, c), R^a_a(\cdot)$ is defined as in (1.6), and for the interval $(a, c)$ it is defined as in (1.6)$'$, Remark 2.9.
It should be emphasized that arbitrary parametrization of the limit circles, $C_{c,\lambda}$, do not necessarily yield, for fixed value of the parameter, one of the functions defined in (3.31). As a matter of fact any parametrization of the limit circles which does produce the functions in (3.31) for fixed values of the parameter can be obtained from (4) by choosing $z$ to be a suitable real valued function of the parameter. More precisely we have the following corollary:

**Corollary 5.4** Let $J$ be an interval of the real axis, and let a function, $$\eta (\cdot , \cdot ) : \{ \lambda \in \mathbb{C} | \text{Im} \lambda \neq 0 \} \times J \to C_{c,\lambda}^\alpha$$ be defined which has the following properties:

(i) For each $\lambda$, $\text{Im} \lambda \neq 0$, $n (\lambda, \cdot)$ maps $J$ one-to-one onto $C_{c,\lambda}^\alpha$.

(ii) For each $\delta \in J$, The boundary functionals,

$$R_{c,\lambda}^\delta (\cdot ) : = W_c (\theta_{a,\lambda} + \eta (\lambda, \delta) \phi_{a,\lambda'}, \cdot ) , \text{Im} \lambda \neq 0,$$

$$R_{c,\lambda'}^\delta (\cdot ) : = W_c (\theta_{a,\lambda'} + \eta (\lambda', \delta) \phi_{a,\lambda}, \cdot ) , \text{Im} \lambda' \neq 0,$$

are equivalent.

Then there exists a one-to-one correspondence between $J$ and $[0, \pi)$, say,

$$f : J \to [0, \pi),$$

such that for all $\lambda$, $\text{Im} \lambda \neq 0$, and all $\delta \in J$ we have,

$$\eta (\lambda, \delta) = T_{c,\lambda}^\alpha (\cot f (\delta)) \quad (17)$$

**Proof.** Suppose the assertion is false. Then there exists $\delta_0 \in J$ and $\lambda_1$, $\lambda_2$ with $\text{Im} \lambda_1 \neq 0$ and $\text{Im} \lambda_2 \neq 0$, such that the points $\eta (\lambda_1, \delta_0)$ and $\eta (\lambda_2, \delta_0)$ on $C_{c,\lambda_1}^\alpha$ and $C_{c,\lambda_2}^\alpha$ do not correspond to the same value of $z$ in (4). That is, we have

$$\eta (\lambda_1, \delta_0) = T_{c,\lambda_1}^\alpha (x_1) \quad \text{and} \quad \eta (\lambda_2, \delta_0) = T_{c,\lambda_2}^\alpha (x_2)$$

where $x_1$ and $x_2$ are in $(-\infty, \infty]$ and $x_1 \neq x_2$. With $\gamma_1 := \text{Arc} \cot x_1$ and $\gamma_2 := \text{Arc} \cot x_2$ it is clear from Theorem 5 that the boundary functionals

$$R_{c,\lambda_1}^{\delta_0} (\cdot ) \quad \text{and} \quad R_{c,\lambda_2}^{\delta_0} (\cdot )$$

and equivalent to the boundary functionals

$$R_{c,\lambda_1}^{\gamma_1} (\cdot ) \quad \text{and} \quad R_{c,\lambda_2}^{\gamma_2} (\cdot )$$

respectively. Thus the assumption (ii) implies that $R_{c,\lambda_1}^{\gamma_1} (\cdot )$ and $R_{c,\lambda_2}^{\gamma_2} (\cdot )$ are equivalent boundary functionals, a contradiction since $\gamma_1 \neq \gamma_2$. ■

**Remark 5.8** It will be observed that the parametrization of the circles, $C_{c,\lambda}^\alpha$, given by the linear fractional transformation $T_{c,\lambda}^\alpha$, depends on the choice of the
fundamental system \{u(x), v(x)\} in (3.1). If \( V_{c,\lambda}^\alpha \) is the linear fractional transformation in (4) associated with a different choice of the fundamental system, say \( \{\psi_0(x), \omega_0(x)\} \), then as in Corollary 4.1 (ii) we have,

\[
m^{\alpha,\gamma}(\lambda) = T_{c,\lambda}^\alpha (\cot \gamma) = V_{c,\lambda}^\alpha (\cot \nu)
\]

if and only if \( \gamma \) is related to \( \nu \) as in (5.14). Thus (5.14) gives the one-to-one correspondence between \( \gamma \) and \( \nu \), the existence of which is asserted by Corollary 5.4.

7 Titchmarsh’s \( m(\lambda) \)- Functions in the Limit Circle Case

Titchmarsh defines ([31, p.31, Equa. 2.7.2]) a boundary condition at the singular point by requiring,

\[
W_c(\psi_\lambda, f) = 0 \text{ for all } \lambda, \text{Im } \lambda \neq 0,
\]

where 

\[
\psi_\lambda(x) := \theta_{a,\lambda}(x) + m(\lambda) \phi_{a,\lambda}(x),
\]

and \( m(\lambda) \) is a function defined for Im \( \lambda \neq 0 \) as follows: One chooses a real-valued function 

\[
\beta(\cdot) : [a, c) \to [0, \pi) \text{ such that } \\
\lim_{b \to c^-} \ell_b^{(\alpha,\beta(b))}(\lambda) \text{ exists for all } \lambda, \text{ Im } \lambda \neq 0,
\]

and then one puts,

\[
m(\lambda) := \lim_{b \to c^-} \ell_b^{(\alpha,\beta(b))}(\lambda) \text{ for Im } \lambda \neq 0.
\]

The associated Weyl Space is then,

\[
U_b^{(\alpha,\beta)} := \{f \in D(L_1) | R^\alpha_a (f) = 0, W_c(\psi_\lambda, f) = 0 \text{ for all } \lambda, \text{ Im } \lambda \neq 0\}.
\]

The existence of functions \( \beta(b) \) (independent of \( \lambda \)) such that the limit in (3) exists is by no means obvious. The following observation can, however, be made: Putting \( w = \ell_b^{(\alpha,\beta)}(\lambda) \) in equation (2.23) and using simple estimates one obtains ([31, p.25-26]) for \( \nu = \text{Im } \lambda \neq 0, \)

\[
|\ell_b^{(\alpha,\beta)}(\lambda)| \leq \frac{1}{|\nu| \int_a^b |\phi_{a,\lambda}(x)|^2 dx} + \left( \frac{2 \int_a^b |\theta_{a,\lambda}(x)|^2 dx}{|\nu| \int_a^b |\phi_{a,\lambda}(x)|^2 dx} + \frac{1}{|\nu| \int_a^b |\phi_{a,\lambda}(x)|^2 dx} \right)^{1/2}
\]

34
From this estimate it follows that (for fixed $\alpha$) the functions $\ell_b^{\alpha, \beta}(\lambda)$ are uniformly bounded on compact subsets of $\{\lambda | \text{Im} \lambda > 0\}$ and $\{\lambda | \text{Im} \lambda < 0\}$, independently of $b$ and $\beta$. Therefore (cf. Ahlfors [2, p.216, Theorem 12])

$$F^\alpha := \left\{ \ell_b^{\alpha, \beta}(\cdot) | b \in [a, c) \text{ and } 0 \leq \beta < \pi \right\}$$

is a normal family of analytic functions in the upper and lower half $\lambda$-planes. This means that for every sequence $(b_n, \beta_n)$ there exists a subsequence $(b_{nk}, \beta_{nk})$ such that

$$\lim_{k \to \infty} \ell_{b_{nk}}^{\alpha, \beta_{nk}}(\lambda) \text{ exists for } \text{Im} \lambda \neq 0,$$

and converges uniformly on compact sets in the upper and lower half $\lambda$-plane. This observations was made by Titchmarsh in his original papers\(^{34}\) (cf. [32, p.40, section 5]) where he writes (our modification in notation):

"For a given $\beta$ the function $\ell_b^{\alpha, \beta}(\lambda)$ is an analytic function of $\lambda$ regular and bounded for $\text{Im} \lambda \geq \delta > 0$ (or $\leq -\delta$) as $b \to \infty$. Hence by Vitali’s Theorem there is a sequence of values of $b$, say $b_n$, such that $\ell_b^{\alpha, \beta}(\lambda)$ converges to a limit $\ell_1(\lambda)$, which is an analytic function of $\lambda$, regular in the upper (or lower) half-plane. In the limit point case $\ell_1(\lambda)$ is the limit point. In the limit circle case $\ell_1(\lambda)$ is a point on the limit circle, which may or may not depend on $\beta.""}

In the first edition of his book Titchmarsh omitted the above definition of the limit circle ‘$m(\lambda)$’ function, cf. [30, p.22, line 1-5]. In the Corrections and Additions which were added to the first edition ([30, 1950 printing]) the following amendment was added to p.22:

"In the limit circle case, $m(\lambda)$ is the limit of $\ell_b^{\alpha, \beta}(\lambda)$ where $b \to \infty$ and $\beta$ may be either a constant or a function of $b$."

This statement was modified in the second edition ([31, p.26, line 6-8]) to read:

"In the limit circle case, points $m(\lambda)$ on the limit circle can be obtained as the limit as $b \to \infty$ of $\ell_b^{\alpha, \beta}(\lambda)$ where $\beta$ is a suitable function of $b$."

As we shall show in example 1 in section 7 below the assertion that

$$\lim_{b \to \infty} \ell_b^{\alpha, \beta}(\lambda) \text{ exists for fixed } \beta$$

is false. Titchmarsh’s first statement in which a limit-circle $m(\lambda)$-function is obtained via Vitali’s Theorem is, however, correct\(^{35}\). The statement occurring

\(^{34}\)The same observation is also made by K. Yosida in [38, p.171-72].\(^{35}\)The meromorphic character of the limit function in the limit circle case is not an immediate consequence of Vitali’s Theorem, since there is no guarantee that the limit function can be continued analytically across the real axis. This may be the reason why Titchmarsh did not
in the second edition is to vague since the ‘suitable functions’, \( \beta(b) \), are not prescribed.\(^{36}\)

In this section we derive necessary and sufficient conditions on the function \( \beta(b) \) such that the limit in (3) exists for all \( \lambda \neq 0 \), and we then show that the limit function is necessarily one of the functions \( m^{\alpha, \gamma} (\lambda) \) of (3.31) for some \( \gamma \in [0, \pi) \). Let the functions defined in (3.22) and (3.31) be denoted by \( w_c^{\alpha, \gamma} (\lambda) \), \( p_1^{\alpha, \gamma} (\lambda) \), \( p_2^{\alpha, \gamma} (\lambda) \), \( v_c^{\alpha, \gamma} (\lambda) \) and \( m^{\alpha, \gamma} (\lambda) \), and let the corresponding functions for \( a < b < c \) be denoted by \( w_b^{\alpha, \gamma} (\lambda) \), \( p_1^{\alpha, \gamma} (\lambda) \), \( p_2^{\alpha, \gamma} (\lambda) \), \( v_b^{\alpha, \gamma} (\lambda) \) and \( m_b^{\alpha, \gamma} (\lambda) \) as in Remark 2.12. We require the following theorem:

**Theorem 6**

(i) Let \( \gamma(\cdot): [a, c) \to [0, \pi) \) be any function which has a limit at \( c \), say

\[
\gamma^* := \lim_{b \to c} \gamma(b). \tag{5}
\]

Then

\[
\lim_{b \to c} m_b^{\alpha, \gamma(b)} (\lambda) = m_c^{\alpha, \gamma^*} (\lambda) \text{ for all } \lambda \text{ not zeros of } w_c^{\alpha, \gamma^*} (\lambda),
\]

and the convergence is uniform on compact subsets of

\[
\mathbb{C} \setminus \{ \lambda \mid w_c^{\alpha, \gamma^*} (\lambda) = 0 \}. \tag{6}
\]

(ii) Moreover, if \( \lambda_0 \) is a complex number with \( \text{Im} \lambda_0 \neq 0 \), and if \( \gamma(\cdot): [a, c) \to [0, \pi) \) is any function such that

\[
\lim_{b \to c} m_b^{\alpha, \gamma(b)} (\lambda_0) \text{ exists}
\]

then \( \gamma^* := \lim_{b \to c} \gamma(b) \) exists and therefore (6) holds.

**Remark 6.1** Since the function \( \gamma(b) \) is required only to have the limit at \( c \), but is otherwise arbitrary, the theorem can also be stated in terms of sequences \( (b_j, \gamma_j) \) with \( b_j \to c \).

**Remark 6.2** It will be noted that the statement,

\[
\lim_{b \to c} m_b^{\alpha, \gamma(b)} (\lambda) = m_c^{\alpha, \gamma^*} (\lambda) \text{ uniformly on compact subsets of } \mathbb{C} \setminus \{ \lambda \mid w_c^{\alpha, \gamma^*} (\lambda) = 0 \}, \tag{7}
\]

recognize the meromorphic character of the limit-circle \( m(\lambda) \)-functions in the first edition of his book. In the second edition he gives a proof that the limit-circle \( m(\lambda) \)-functions are meromorphic (with poles only on the real axis), cf. [31, p.125-26, section 5.12]. It will be noted that this proof (as well as other statements involving the limit-circle case – notably Lemma 2.3) depends on knowing the existence of the ‘suitable functions’ \( \beta(b) \) used in the definition (6.3).

\(^{36}\)The existence of such functions can, however, be deduced by slight modification of an argument of N. Levinson, cf. [4, p242, Theorem 4.1]. Compare also Remark 5.5.
is a special case of (6).

**Proof.** (i) To prove (6) we estimate the difference,
\[
\begin{align*}
&\left| m^\alpha_{c^*} (\lambda) - m^\alpha_{b^*} (\lambda) \right| \\
&= \left| \frac{w^\alpha_{c^*} (\lambda) p^\alpha_{1b^*} (\lambda) - p^\alpha_{1c^*} (\lambda) w^\alpha_{b^*} (\lambda)}{w^\alpha_{b^*} (\lambda) w^\alpha_{c^*} (\lambda)} \right|.
\end{align*}
\]
(8)

Observe:
\[
\text{Numerator } = \quad w^\alpha_{c^*} (\lambda) \left[ p^\alpha_{1b^*} (\lambda) - p^\alpha_{1c^*} (\lambda) \right] \\
+ w^\alpha_{c^*} (\lambda) \left[ p^\alpha_{1c^*} (\lambda) - p^\alpha_{1c^*} (\lambda) \right] \\
+ p^\alpha_{1c^*} (\lambda) \left[ w^\alpha_{c^*} (\lambda) - w^\alpha_{c^*} (\lambda) \right] \\
+ p^\alpha_{1c^*} (\lambda) \left[ w^\alpha_{c^*} (\lambda) - w^\alpha_{c^*} (\lambda) \right].
\]

From (3.23) we have
\[
\left| w^\alpha_{c^*} (\lambda) - w^\alpha_{b^*} (\lambda) \right| \leq \left| (S^b_{a,\lambda})_1 (c) - (S^b_{a,\lambda})_1 (b) \right| \\
+ \left| (S^b_{a,\lambda})_2 (c) - (S^b_{a,\lambda})_2 (b) \right|
\]
and
\[
\left| p^\alpha_{1c^*} (\lambda) - p^\alpha_{1b^*} (\lambda) \right| \leq \left| (S^c_{a,\lambda})_1 (c) - (S^c_{a,\lambda})_1 (b) \right| \\
+ \left| (S^c_{a,\lambda})_2 (c) - (S^c_{a,\lambda})_2 (b) \right|
\]

independently of the choice of the function $\gamma (b)$. Hence it follows from (13) that these terms can be made arbitrarily small on a given compact $\lambda$-set by choosing $b$ sufficiently close to $c$. Similarly from (3.23) we also have
\[
\left| w^\alpha_{c^*} (\lambda) - w^\alpha_{c^*} (\lambda) \right| \leq \left| (S^c_{a,\lambda})_1 (c) \right| \left| \cos (b) - \cos \gamma^* \right| \\
+ \left| (S^c_{a,\lambda})_2 (c) \right| \left| \sin (b) - \sin \gamma^* \right|
\]
and
\[
\left| p^\alpha_{1c^*} (\lambda) - p^\alpha_{1c^*} (\lambda) \right| \leq \left| (S^c_{a,\lambda})_1 (c) \right| \left| \cos (b) - \cos \gamma^* \right| \\
+ \left| (S^c_{a,\lambda})_2 (c) \right| \left| \sin (b) - \sin \gamma^* \right|.
\]
Since the quantities depending on $\lambda$ are bounded on compact $\lambda$-sets these terms can also be made small on a given compact $\lambda$-set because of the existence of the limit in (5). Since the functions $w_{c,\gamma}^\alpha (\lambda)$ and $p_{c,\gamma}^\alpha (\lambda)$ are bounded on compact sets it follows from the above remarks that the numerator in (8) can be made arbitrarily small on a given compact $\lambda$-set by choosing $b$ sufficiently close to $c$. Now let $D$ be a compact $\lambda$-set not containing any zeros of $w_{c,\gamma}^\alpha (\lambda)$. To complete the proof it suffices to demonstrate that the denominator in (8) is bounded away from zero on $D$ for $b$ sufficiently close to $c$. Let

$$d := \min_{\lambda \in D} \left| w_{c,\gamma}^\alpha (\lambda) \right| > 0$$

and choose $b$ sufficiently close to $c$ so that

$$\left| w_{b,\gamma}^\alpha (\lambda) - w_{c,\gamma}^\alpha (\lambda) \right| < d/2$$

for all $\lambda \in D$. (This is possible by the above remarks). Then,

$$\left| w_{b,\gamma}^\alpha (\lambda) \right| \geq \left| w_{c,\gamma}^\alpha (\lambda) \right| - \left| w_{b,\gamma}^\alpha (\lambda) - w_{c,\gamma}^\alpha (\lambda) \right| \geq d - \frac{1}{2} d = \frac{1}{2} d$$

for all $\lambda \in D$, and therefore

$$\left| w_{b,\gamma}^\alpha (\lambda) w_{c,\gamma}^\alpha (\lambda) \right| \geq \frac{1}{2} d^2$$

for all $\lambda \in D$.

(ii) The points $m_{b,\gamma}^\alpha (\lambda_0)$ lie on the circle $C_{b,\lambda_0}^\alpha$. (This follows from Lemma 5.1 and 5.2 with $b$ in place of $c$ as noted in Remark 5.2) Since the circles $C_{b,\lambda_0}^\alpha$ converge to the circles $C_{c,\lambda_0}^\alpha$ as $b \rightarrow c$ it follows that

$$\lim_{b \rightarrow c} m_{b,\gamma}^\alpha (\lambda_0)$$

is a point on $C_{c,\lambda_0}^\alpha$.

Hence by Lemma 5.2 there exists $\gamma^* \in [0, \pi)$ such that

$$\lim_{b \rightarrow c} m_{b,\gamma}^\alpha (\lambda_0) = m_{c,\gamma^*}^\alpha (\lambda_0),$$

i.e.

$$\lim_{b \rightarrow c} T_{b,\lambda_0}^\alpha (\cot \gamma (b)) = T_{c,\lambda_0}^\alpha (\cot \gamma^*).$$

This implies

$$\cot \gamma^* = (T_{c,\lambda_0}^\alpha)^{-1} \left( \lim_{b \rightarrow c} T_{b,\lambda_0}^\alpha (\cot \gamma (b)) \right)$$

$$= \lim_{b \rightarrow c} (T_{c,\lambda_0}^\alpha)^{-1} \left( T_{b,\lambda_0}^\alpha (\cot \gamma (b)) \right).$$

(9)

The composite linear fractional transformation $(T_{c,\lambda_0}^\alpha)^{-1} T_{b,\lambda_0}^\alpha$ is

$$(T_{c,\lambda_0}^\alpha)^{-1} \circ T_{b,\lambda_0}^\alpha (z) = \frac{A (b) z + B (b)}{C (b) z + D (b)}.$$
where $A(b), B(b), C(b),$ and $D(b)$ satisfy (by (3.5) and (2.10),
\[
\lim_{b \to c} \begin{pmatrix} A(b) & B(b) \\ C(b) & D(b) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Now in order to establish the existence of $\lim_{b \to c} \gamma(b)$, let
\[
x(b) := A(b) \cot \gamma(b) + B(b) \\
C(b) \cot \gamma(b) + D(b)
\]
Then
\[
\cot \gamma(b) = -D(b) x(b) + B(b) \\
C(b) x(b) - A(b).
\]

By (9),
\[
\lim_{b \to c} x(b) = \cot \gamma^*.
\]
Therefore, letting $b \to c$ in (10) we have
\[
\lim_{b \to c} \gamma(b) = \cot \gamma^*
\]
and hence
\[
\lim_{b \to c} \cot \gamma(b) = \cot \gamma^*.
\]

In order to characterize the functions $\beta(b)$ which are admissible in (3) we require the following lemma which gives the connection between the functions $\ell_b^\alpha, \beta(\lambda)$ defined in (2.17) and the functions $m_b^{\alpha, \gamma}(\lambda)$ of Remark 2.12.

**Lemma 6.1**  For fixed $b \in (a, c),$
\[
\beta = \text{Arc cot} \left( -v'(b) \cos \gamma + u'(b) \sin \gamma \over v(b) \cos \gamma - u(b) \sin \gamma \right), \quad \gamma \in [0, \pi), \tag{11}
\]
defines a one-to-one correspondence\footnote{14}{The fact that there must be a one-to-one correspondence between $\beta$ and $\gamma$ follows from Remark 1.1 and Remark 5.2.} between $\beta$ and $\gamma$ under which
\[
\ell_b^\alpha, \beta(\lambda) = m_b^{\alpha, \gamma}(\lambda) \quad \text{for all } \lambda, \ \text{Im } \lambda \neq 0. \tag{12}
\]

**Proof.** From (2.25) and (6.7) with $b$ in place of $c$ we have
\[
\ell_b^\alpha, \beta(\lambda) = m_b^{\alpha, \gamma}(\lambda) \quad \text{if and only if}
\]
\[
\cot \beta = (A_{b,\lambda})^{-1} T_{b,\lambda}^{\alpha} (\cot \gamma),
\]
and a simple computation shows that the composite transformation is
\[
(A_{b,\lambda})^{-1} \circ T_{b,\lambda}^{\alpha} (z) = {A z + B \over C z + D}
\]
with
\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = 
\begin{pmatrix}
-v'(b) & u'(b) \\
v(b) & -u(b)
\end{pmatrix}
\]

Remark 6.3. Note that if the fundamental system \{u(x), v(x)\} is defined by the initial conditions,
\[
\begin{pmatrix}
u(b)
\end{pmatrix}
\begin{pmatrix}
u(x)
\end{pmatrix} = 
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]
then \(\beta = \gamma\), so that the function \(m_{\beta}^{\alpha, \gamma}(\lambda)\) is in fact \(e_{\alpha, \beta}^{\lambda}(\lambda)\).

Remark 6.4. From Remark 6.2 and Lemma 6.1 it follows (for fixed \(\gamma \in [0, \pi]\)) that
\[
\lim_{b \to c} e_{\alpha, \beta}^{\gamma}(b) = m_{\gamma}^{\alpha, \gamma}(\lambda)
\]
uniformly on compact subsets of \(\mathbb{C} \setminus \{\lambda | w_{\gamma}^{\alpha, \gamma}(\lambda) = 0\}\) where \(\beta(b)\) is defined for \(b \in (a, c)\) by,
\[
\beta(b) := \text{Arc cot} \left( -\frac{v(b) \cos \gamma - u'(b) \sin \gamma}{v(b) \cos \gamma - u(b) \sin \gamma} \right).
\]
(Here as usual we adopt the convention \(\text{Arc cot}(\pm \infty) = 0\).)

We now have the following corollary of Theorem 6 and Lemma 6.1:

Corollary 6.2

(i) Let \(\beta(\cdot) : [a, c) \to [0, \pi)\) be a real valued function such that
\[
\lim_{b \to c} \begin{pmatrix}
u(b)
\end{pmatrix} \begin{pmatrix}
u(x)
\end{pmatrix} = 
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]
exists (with finite or infinite limit). Put
\[
\gamma^{*} := \text{Arc cot} \left( \lim_{b \to c} \frac{u(b) \cos \beta(b) + u'(b) \sin \beta(b)}{v(b) \cos \beta(b) + v'(b) \sin \beta(b)} \right)
\]
(We assume the convention \(\text{Arc cot}(\pm \infty) = 0\)).

Then
\[
\lim_{b \to c} e_{\alpha, \beta}^{\gamma^*}(b) = m_{\gamma^*}^{\alpha, \gamma^*}(\lambda) \quad \text{for all } \lambda \not= 0 \text{ not zeros of } w_{\gamma^*}^{\alpha, \gamma^*}(\lambda) \text{ and the convergence is uniform on compact subsets of } \mathbb{C} \setminus \{\lambda | w_{\gamma^*}^{\alpha, \gamma^*}(\lambda) = 0\}.
\]

(ii) Moreover, if for fixed \(\lambda_0, \text{Im } \lambda_0 \neq 0, \beta(\cdot) : [a, c) \to [0, \pi)\) is any function such that \(\lim_{b \to c} e_{\alpha, \beta}^{\gamma^*}(b) = m_{\gamma^*}^{\alpha, \gamma^*}(\lambda_0)\) exists, then
\[
\gamma^{*} := \text{Arc cot} \left( \lim_{b \to c} \frac{u(b) \cos \beta(b) + u'(b) \sin \beta(b)}{v(b) \cos \beta(b) + v'(b) \sin \beta(b)} \right)
\]
exists and therefore (16) holds.

Proof. (i) By Lemma 6.1, \( \ell_b^{\alpha,\beta} (\lambda) = m_b^{\alpha,\gamma} (\lambda) \) for all \( \lambda \), \( \text{Im} \lambda \neq 0 \) if and only if
\[
\gamma = \text{Arc} \cot \left( \frac{u'(b) \cos \beta + u'(b) \sin \beta}{v(b) \cos \beta + v'(b) \sin \beta} \right).
\]

So if \( \beta = \beta(b) \) is a function satisfying (15), then
\[
\gamma(b) := \text{Arc} \cot \left( \frac{u'(b) \cos \beta(b) + u'(b) \sin \beta(b)}{v(b) \cos \beta(b) + v'(b) \sin \beta(b)} \right) \tag{18}
\]
is a function satisfying (5). The statement (16) therefore follows from (6).

(ii) The existence of
\[
\lim_{b \to c} \ell_b^{\alpha,\beta(b)} (\lambda_0)
\]
implies the existence of
\[
\lim_{b \to c} m_b^{\alpha,\gamma(b)} (\lambda_0)
\]
where \( \gamma(b) \) is defined as in (18). Hence the statement (17) follows from Theorem 6 (ii). \( \blacksquare \)

Remark 6.5 It follows from Corollary 6.2 that the class of functions \( \beta(b) \) which are admissible in (3) is characterized by the condition (15). That is, if \( \beta(b) \) is any function for which the limit in (3) exists, then the limit in (15) exists. It will be noted that the existence or nonexistence of the limit in (15) depends only on the function \( \beta(b) \) and not on the choice of the fundamental system \( \{u(x), v(x)\} \). This follows from Corollary 6.2 (ii), the assumptions of which do not depend on the choice of \( \{u(x), v(x)\} \). Thus the existence of the limit in (15) for one choice of the fundamental system implies the existence for all choices.

Remark 6.6 In the paper of D.B. Sears and E.C. Titchmarsh, "Some Eigenfunction Formulae," in which a representation of the limit-circle \( m(\lambda) \)-functions of the type (4) is obtained (under additional assumptions on \( q(x) \)) it will be noted ([27, p.168]) that a one-parameter family of functions \( \beta^k(b) \) is defined in such a way that
\[
\lim_{b \to \infty} \ell_b^{\alpha,\beta^k(b)} (\lambda) \text{ exists for } \text{Im} \lambda \neq 0.
\]
It follows from Corollary 6.2 (ii) that for each \( k \) the limit function is necessarily one of the functions \( m_{\infty}^{\alpha,\gamma}(\lambda) \) of (3.31).

Because of Corollary 6.2 we may conclude that each \( m(\lambda) \)-function of the type (3) is necessarily one of the functions \( m_{\infty}^{\alpha,\gamma}(\lambda) \) of (3.31) for some \( \gamma \in [0, \pi) \). The value of \( \gamma \) is given by the formula (15). Comparing (4) with Corollary 5.3 it is also readily seen that each Weyl Space of the type (4) is one of the spaces
in (6). For the sake of later reference we now formulate Corollary 6.2 in terms of sequences:

**Corollary (6.2)**

(i) Let \((b_n, \beta_n)\) be a sequence with \(\lim_{n \to \infty} b_n = c\) such that

\[
\lim_{n \to \infty} \left( \frac{u(b_n) \cos \beta_n + u'(b_n) \sin \beta_n}{v(b_n) \cos \beta_n + v'(b_n) \sin \beta_n} \right) \quad (6.15)
\]

exists (with a finite or infinite limit). Put

\[
\gamma^* := \text{Arc}\cot \left( \lim_{n \to \infty} \frac{u(b_n) \cos \beta_n + u'(b_n) \sin \beta_n}{v(b_n) \cos \beta_n + v'(b_n) \sin \beta_n} \right)
\]

(where by convention \(\text{Arc}\cot (\pm \infty) = 0\)). Then

\[
\lim_{n \to \infty} L^{\alpha, \beta, \gamma^*}_{b_n} = m_c^{\alpha, \gamma^*} (\lambda) \quad \text{for all } \lambda \text{ not zeros of } w_c^{\alpha, \gamma^*} (\lambda) \text{ and the convergence is uniform on compact subsets of } C \setminus \{ \lambda | w_c^{\alpha, \gamma^*} (\lambda) = 0 \}. \quad (6.16)
\]

(ii) Moreover, if for fixed \(\lambda_0, \text{ Im } \lambda_0 \neq 0, (b_n, \beta_n)\) is a sequence with

\[
\lim_{n \to \infty} b_n = c
\]

such that

\[
\lim_{n \to \infty} L^{\alpha, \beta, \gamma^*}_{b_n} (\lambda_0) \text{ exists},
\]

then the limit in (6.15) exists and therefore (6.16) holds.

**Remark 6.7** There are obvious analogues of Theorem 6, Lemma 6.1 and Corollaries (6.2) and (6.2)' involving the left-handed \(m(\lambda)\)-functions of Remarks 2.8 and 2.9.

### 8 Examples

We give three examples to illustrate the above theory. The first example is a counterexample to the assertion,

\[
\lim_{b \to \infty} L^{\alpha, \beta}_{b} (\lambda) \text{ exists for } \text{Im } \lambda \neq 0 \text{ and for fixed } \beta, \quad (1)
\]
and was suggested to me by Professor Dr. J. Walter. It will be noted that if this assertion were true (and if for each \( \beta \in [0, \pi] \) the limit function were different) then \( m(\lambda) \) functions in (7.3) could be parametrized by the parameter \( \beta \) and there would be no need to introduce the parameter \( \gamma \) as we have done in Section 2.

**Example 1.** Consider

\[ q(x) := \frac{2}{x^2} - x^4 \text{ in the interval } [a, \infty) \]  

with \( a > 0 \). We choose the fundamental system \( \{u(x), v(x)\} \) satisfying (3.1) to be,

\[ u(x) = \frac{1}{x} \cos \left( \frac{x^3}{3} \right), \quad v(x) = \frac{1}{x} \sin \left( \frac{x^3}{3} \right). \]  

(3)

With this choice of fundamental system let \( L^{\alpha, \gamma} \) denote the operator having the domain \( U_2^{\alpha, \gamma} \) of (3.6). Now letting

\[ F(b, \beta) := \frac{u(b) \cos \beta + u'(b) \sin \beta}{v(b) \cos \beta + v'(b) \sin \beta} \]  

(4)

it is readily seen that for fixed \( \beta \in [0, \pi] \), \( F(b, \beta) \) does not have a limit \( b \to \infty \).

It follows from Corollary 6.2 (ii) that

\[ \lim_{b \to \infty} \ell^{0, \beta}_b(\lambda) \]  

does not exist for any \( \lambda, \) \( \text{Im} \lambda \neq 0 \).

Thus a constant value of \( \beta \) in (15) is not admissible. We note, however, that there are sequences of the form \( (b_n, \beta) \) with \( \beta \) fixed for which

\[ \lim_{n \to \infty} \ell^{0, \beta}_{b_n}(\lambda) \]  

exists for all \( \lambda, \text{Im} \lambda \neq 0 \).

For if \( b_n^{(\theta)} \) is defined by,

\[ b_n^{(\theta)} := \sqrt[3]{3(\theta + 2n\pi)}, \text{ with } \theta \in [0, \pi), \]  

(5)

it follows from (4) that

\[ \text{For } \beta \neq 0, \frac{\pi}{2}, \quad \lim_{n \to \infty} F(b_n^{(\theta)}, \beta) = \begin{cases} -\tan \theta, & \theta \neq \frac{\pi}{2} \\ -\infty, & \theta = \frac{\pi}{2} \end{cases} \]  

(6)

\[ \text{for } \beta = \frac{\pi}{2}, \quad \lim_{n \to \infty} F(b_n^{(\theta)}, \frac{\pi}{2}) = \begin{cases} -\tan \theta, & \theta \neq \frac{\pi}{2} \\ +\infty, & \theta = \frac{\pi}{2} \end{cases} \]

---

\footnote{This example is obtained by choosing \( g(x) = \frac{1}{x} \) (and \( p = 1 \)) in J. Walter [34, Equa. (3)], and then computing \( q(x) \). The fact that \( q(x) \) belongs to the limit circle case at \( \infty \) then follows from [34, Satz 1, p. 346] since \( g(x) \in L_2[a, \infty) \).}

\footnote{Ibid. [34, p. 346, Equa. (9)].}
and for $\beta = 0$, 
$$
\lim_{n \to \infty} F\left( b_n^{(\theta)}, 0 \right) = \begin{cases} 
\cot \theta, & \theta \neq 0 \\
+\infty, & \theta = 0 
\end{cases}.
$$

Because of the existence of the limits in (6) it follows from Corollary (6.2)' that the sequences $(b_n^{(\theta)}, \beta)$ are admissible in (6.15)' for each $\beta \in [0, \pi)$ and we therefore have,
$$
\lim_{n \to \infty} \ell_{b_n^{(\theta)}}^{\alpha, \beta} (\lambda) = m_{\infty}^{\alpha, \gamma} (\lambda)
$$
where $\gamma^*$ is determined by $\theta$ via (6.15)', that is,
$$
\gamma^* = \text{Arc}\cot \left( \lim_{n \to \infty} F\left( b_n^{(\theta)}, \beta \right) \right). \quad (7)
$$

**Remark 7.1** The assertion of Theorem 6, and particularly the special case (7.7), is an interesting contrast to the false assertion in (1). The parametrization of the $C_{b, \lambda}$ circle afforded by the parameter $\gamma$ (cf. Lemma 5.2 and Remark 5.2) guarantees that points $m_{\infty}^{\alpha, \gamma} (\lambda)$ corresponding to a fixed value of $\gamma$ converge as $b \to c$ to a point on the limit circle, $C_{c, \lambda}$, while the parametrization afforded by the the parameter $\beta$ (cf. Remark 1.1) evidently allows the possibility that points $\ell_{b, \beta}^{\alpha, \beta} (\lambda)$ corresponding to a fixed value of $\beta$ do not converge as $b \to c$.

**Example 2.** Let $0 < \nu < 1$, $\nu \neq \frac{1}{2}$, and consider
$$
q(x) = \frac{\nu^2 - 1/4}{x^2} \quad \text{in the interval } (0, c] \quad (8)
$$
with $0 < c < \infty$. We choose a fundamental system $\{u(x), v(x)\}$ satisfying (3.1) by,
$$
u(x) = \frac{1}{\sqrt{2\nu}} x^{\frac{1}{2} - \nu}, \quad v(x) = \frac{1}{\sqrt{2\nu}} x^{\frac{1}{2} + \nu}, \quad (9)
$$
and consider the Weyl Spaces,
$$
U_{x}^{\nu} := \{ f \in L_{2}(0, c] \mid f \text{ and } f' \text{ absolutely continuous on compact subsets of } (0, c] \text{ and } \tau f \in L_{2}(0, c], \quad 
f (c) = 0 \text{ and } \\
R_{0}^{\nu} (f) := (Sf)_{1} (0) \cos \alpha + (Sf)_{2} (0) \sin \alpha = 0 \}, \quad (10)
$$
where $Sf$ is as in (3.3) with the above choice of the fundamental system for $\lambda = 0$. We wish to calculate the left-handed $m(\lambda)$-functions associated with the spaces $U_{x}^{\nu}$, $\alpha \in [0, \pi)$, and thus demonstrate that our theory yields the same result obtained by Titchmarsh for this problem (cf. [31, p. 83, Equa. 4.8.5]). Let $\{\phi_{c, \lambda}, \theta_{c, \lambda}(x)\}$ be the fundamental system of (2.8) defined by the initial conditions,
$$
\begin{pmatrix} 
\phi_{c, \lambda}(c) & \theta_{c, \lambda}(c) \\
\phi'_{c, \lambda}(c) & \theta'_{c, \lambda}(c)
\end{pmatrix} = 
\begin{pmatrix} 
0 & 1 \\
-1 & 0
\end{pmatrix} \quad (11)
$$
for all $\lambda \in \mathbb{C}$. It is easily verified (compare also [31, p.82]) that the functions $\phi_{c,\lambda}(x), \theta_{c,\lambda}(x)$ defined by (11) are

$$\phi_{c,\lambda}(x) = -\frac{\pi (cx)^{1/2}}{2 \sin (\nu \pi)} [J_{\nu}(xs) J_{-\nu}(cs) - J_{-\nu}(xs) J_{\nu}(cs)]$$

$$\theta_{c,\lambda}(x) = -\frac{s \pi (cx)^{1/2}}{2 \sin (\nu \pi)} [J_{\nu}(xs) J'_{-\nu}(cs) - J'_{-\nu}(xs) J_{\nu}(cs)] + \frac{\phi_{c,\lambda}(x)}{2c}. \quad (12)$$

where $J_{\nu}(z)$ is Bessel function of order $\nu$ and $s := \sqrt{\lambda}$. Making use of the formulas,

$$J_{\nu}(xs) = \left(\frac{xs}{2}\right)^{\nu} \left[\frac{1}{\Gamma(1+\nu)} + O(x^2)\right]$$

$$J'_{\nu}(xs) = \frac{\nu}{2} \left(\frac{xs}{2}\right)^{\nu-1} \left[\frac{1}{\Gamma(1+\nu)} + O(x^2)\right],$$

the quantities occurring in the formula for the left-handed $m(\lambda)$-functions (cf. equation (2.39)00, Remark 2.9) can be computed and are found to be:

$$(S\phi_{c,\lambda})_1(0) = \frac{\sqrt{2\nu \pi \sqrt{c}}}{2 \sin (\nu \pi)} \left(\frac{s}{2}\right)^{-\nu} \frac{1}{\Gamma(1+\nu)} J_{\nu}(cs)$$

$$(S\phi_{c,\lambda})_2(0) = -\frac{\sqrt{2\nu \pi \sqrt{c}}}{2 \sin (\nu \pi)} \left(\frac{s}{2}\right)^{\nu} \frac{1}{\Gamma(1+\nu)} J'_{-\nu}(cs)$$

$$\begin{align*}
(S\theta_{c,\lambda})_1(0) &= \frac{\sqrt{2\nu \pi \sqrt{c}}}{2 \sin (\nu \pi)} \left(\frac{s}{2}\right)^{-\nu} \frac{1}{\Gamma(1+\nu)} \left[s J'_{\nu}(cs) + \frac{1}{2c} J_{\nu}(cs)\right] \\
(S\theta_{c,\lambda})_2(0) &= -\frac{\sqrt{2\nu \pi \sqrt{c}}}{2 \sin (\nu \pi)} \left(\frac{s}{2}\right)^{\nu} \frac{1}{\Gamma(1+\nu)} \left[s J'_{-\nu}(cs) + \frac{1}{2c} J_{-\nu}(cs)\right]
\end{align*}\quad (13)$$

Substituting these expressions in (2.39)00 and simplifying we obtain the left-handed $m(\lambda)$-functions,

$$m_2^\alpha(\lambda) = \frac{s^{-\nu} [2cs J'_{\nu}(cs) + J_{\nu}(cs)]}{s^{-\nu} [2cs J'_{-\nu}(cs) + J_{-\nu}(cs)]} \frac{K - s^{\nu} [2cs J'_{-\nu}(cs) + J_{-\nu}(cs)]}{K - s^{\nu} [2cs J_{\nu}(cs)]} \quad (14)$$

where

$$K := \frac{2\nu \Gamma (1+\nu)}{2^{-\nu} \Gamma (1-\nu)} \cot \alpha. \quad (15)$$

The formula (14) agrees with Titchmarsh [31, Equation 4.8.5], where Titchmarsh’s parameter $c$ is the parameter $K$ defined in (15).
In contrast to example 1 it will be noted that

\[
\text{Arc cot} \left( \lim_{b \to 0} \frac{u(b) \cos \beta + u'(b) \sin \beta}{v(b) \cos \beta + v'(b) \sin \beta} \right) = 0
\]  

(16)

for all \( \beta \in [0, \pi) \). Hence by (the left-handed analogue of) Corollary 6.2 and by (14) we have\(^{40}\)

\[
\lim_{b \to 0} e_\beta^b(\lambda) = \lim_{b \to 0} \left( \frac{\theta_{c,\lambda}(b) \cos \beta + \theta'_{c,\lambda}(b) \sin \beta}{\phi_{c,\lambda}(b) \cos \beta + \phi'_{c,\lambda}(b) \sin \beta} \right) = -\frac{2cJ'_\nu(cs) + J_\nu(cs)}{2cJ_\nu(cs)}, \quad s = \sqrt{\lambda},
\]  

(17)

for each fixed \( \beta \in [0, \pi) \).

The following example is an extension of example 2 and illustrates the above theory for a case when (2.8) is oscillatory at 0 and thus corresponding operators unbounded below (cf. Rellich [22, p.349, Satz 2b]).

**Example 3**\(^{41}\). Let \( \nu = i\mu, \mu > 0 \), and consider

\[
q(x) = \frac{(i\mu)^2 - 1/4}{x^2} = -\frac{\mu^2 - 1/4}{x^2} \quad \text{in} \ (0, 1].
\]

A fundamental system which is real-valued and satisfies (3.1) is

\[
u(x) = \sqrt{x} \cos(\mu \log x), \quad v(x) = \sqrt{x} \sin(\mu \log x).
\]  

(18)

We consider the operators \( L^\alpha, \alpha \in [0, \pi) \), having the domains,

\[
D(L^\alpha) := \left\{ f \in L_2(0, 1) \mid f \text{ and } f' \text{ absolutely continuous on compact subsets of } (0, 1] \right. \\
\left. \tau f \in L_2(0, 1), \ f(1) = 0, \text{ and } R_0^\alpha(f) := (Sf)_1(0) \cos \alpha + (Sf)_2(0) \sin \alpha = 0 \right\}
\]  

(19)

\(^{40}\)The function \( \tilde{m}_0^2(\lambda) (\alpha = 0) \) is the \( m(\lambda) \)-function associated with Rellich’s ‘ausgezeichnete Randaufgabe’ (cf. [22, p.354-55]) since \( v(x) \) and \( u(x) \) are, respectively, principal and nonprincipal solutions for \( \lambda = 0 \).

\(^{41}\)This example does not seem to have been worked out in detail in the existing literature. Compare [31, p.81-86]; [35, p.577, line 11]; [18, p.128, Section 3.12.4]; [1, p. 472-475]; [17, p.260, Problem 16]; [38, p.202, Section 53]; [9, p.518, Exercise 4.2]; and [6, p.70-73, Section 7.10.4]. We note, however, that this potential has been treated by K. Meetz [19, p.702-03], in the range \((0, \infty)\) where one has the limit point case at \( \infty \) and the limit circle case at 0. Meetz obtains a formula for the Jost function associated with each possible boundary condition at 0 (cf. [9, p.702, Equa. 4.22]) which is somewhat analogous to our function \( w^\alpha (\lambda) \) in (7.21). Our theory, however, does not apply to the problem considered by Meetz because of the occurrence of the limit point case at \( \infty \).
If \( \phi_{1,\lambda} (x) \), \( \theta_{1,\lambda} (x) \) is the fundamental system of (2.8) defined by the initial conditions in (11) at \( c = 1 \), then as in (12) we have,

\[
\phi_{1,\lambda} (x) = \frac{i \pi \sqrt{x}}{2 \sinh (\mu \pi)} \left[ J_{i\mu} (xs) J_{-i\mu} (s) - J_{-i\mu} (xs) J_{i\mu} (s) \right]
\]

and

\[
\theta_{1,\lambda} (x) = \frac{s i \pi \sqrt{x}}{2 \sinh (\mu \pi)} \left[ J_{i\mu} (xs) J'_{-i\mu} (s) - J_{-i\mu} (xs) J'_{i\mu} (s) \right] + \frac{\phi_{1,\lambda} (x)}{2} \tag{20}
\]

where \( s := \sqrt{\lambda} \). According to the theory of Section 2 the spectra of the operators \( L^\alpha, \alpha \in [0, \pi) \), are the zeros of

\[
w^\alpha (\lambda) := - \left( S \phi_{1,\lambda} \right)_1 (0) \cos \alpha - \left( S \phi_{1,\lambda} \right)_2 (0) \sin \alpha. \tag{21}
\]

Making use of the behavior of the Bessel functions \( J_{i\mu} (z) \) and \( J_{-i\mu} (z) \) near \( z = 0 \) as before (cf. [35, p.43, line 5]), the quantities in (21) can be computed and are found to be,

\[
\left( S \phi_{1,\lambda} \right)_1 (0) = \frac{\sqrt{\mu} \pi}{2 \sinh (\mu \pi)} \left[ \left( \frac{z}{2} \right)^{i\mu} J_{-i\mu} (s) - \left( \frac{z}{2} \right)^{-i\mu} J_{i\mu} (s) \right] \Gamma (1 + i\mu) \Gamma (1 - i\mu)
\]

\[
\left( S \phi_{1,\lambda} \right)_2 (0) = \frac{\sqrt{\mu} \pi}{2 \sinh (\mu \pi)} \left[ - \left( \frac{z}{2} \right)^{i\mu} J_{-i\mu} (s) - \left( \frac{z}{2} \right)^{-i\mu} J_{i\mu} (s) \right] \Gamma (1 + i\mu) \Gamma (1 - i\mu)
\]

The fact that \( \left( S \phi_{1,\lambda} \right)_1 (0) \) and \( \left( S \phi_{1,\lambda} \right)_2 (0) \) are entire functions of \( \lambda \) (and real-valued for real \( \lambda \)) can be verified by computing the McLaurin expansions of the above quantities which are found to be,

\[
\left( S \phi_{1,\lambda} \right)_1 (0) = \frac{1}{\sqrt{\mu}} \sum_{m=1}^{\infty} a_m \lambda^m
\]

where \( a_m := \frac{(-1)^{m+1}}{m! 2^{2m}} \left[ \frac{\text{Im} \left( \prod_{n=1}^{m} (n + i\mu) \right)}{\prod_{n=1}^{m} (n^2 + \mu^2)} \right] \]

and

\[
\left( S \phi_{1,\lambda} \right)_2 (0) = \frac{1}{\sqrt{\mu}} \left[ -1 + \sum_{m=1}^{\infty} b_m \lambda^m \right]
\]

where \( b_m := \frac{(-1)^{m+1}}{m! 2^{2m}} \left[ \frac{\text{Re} \left( \prod_{n=1}^{m} (n + i\mu) \right)}{\prod_{n=1}^{m} (n^2 + \mu^2)} \right] \tag{23}
\]

The spectrum of the operator \( L^\alpha \) is then the zeros of the function \( w^\alpha (\lambda) \) in (21).
Analogous to example 1 we observe that
\[ \lim_{b \to 0} \left( \frac{u(b) \cos \beta + u'(b) \sin \beta}{v(b) \cos \beta + v'(b) \sin \beta} \right) \]
does not exist for any fixed \( \beta \in [0, \pi) \). We note, however, that there are sequences of the form \((b_n, \beta)\) with \( \beta \) fixed for which
\[ F(b_n, \beta) := \frac{u(b_n) \cos \beta + u'(b_n) \sin \beta}{v(b_n) \cos \beta + v'(b_n) \sin \beta} \]
does have a limit as \( b_n \to 0 \). For if \( b^{(\theta)}_n \) is defined by,
\[ b^{(\theta)}_n := \exp \left( \frac{\theta - 2n\pi}{\mu} \right), \quad \theta \in [0, \pi), \quad n = 1, 2, \ldots, \quad (24) \]
it will be observed that
\[ \text{For } \beta \neq 0, \quad \lim_{n \to \infty} F\left(b^{(\theta)}_n, \beta\right) = \frac{1}{2} \cos \theta - \frac{\mu}{2} \sin \theta \quad (25) \]
and
\[ \text{for } \beta = 0, \quad \lim_{n \to \infty} F\left(b^{(\theta)}_n, 0\right) = \begin{cases} \cot \theta, & \theta \in (0, \pi) \\ \infty, & \theta = 0. \end{cases} \]
It follows from (the left-handed analogue of) Corollary (6.2)' that the sequences \((b^{(\theta)}_n, \beta)\) are admissible in (6.15)' for each \( \beta \in [0, \pi) \) and that
\[ \lim_{n \to \infty} \ell^{(\beta)}_{b^{(\theta)}_n}(\lambda) = m_2^{\alpha^*}(\lambda) \]
where \( \alpha^* \) is determined by \( \theta \) via (6.15)', that is,
\[ \alpha^* := \text{Arc} \cot \left( \lim_{n \to \infty} F\left(b^{(\theta)}_n, \beta\right) \right). \quad (26) \]

9 The Weyl Spaces of M.H. Stone

In his treatise on Linear Transformations in Hilbert Space [29], M.H. Stone gives boundary conditions at the singular point as follows:
Let \( \phi^\alpha_{\alpha,i}(x) \) be the function defined in (2.9) for \( \lambda = i \), and put
\[ y_{\alpha,\theta}(x) := \frac{\exp(i\theta/2) \phi^\alpha_{\alpha,i}(x) - \exp(-i\theta/2) \phi^\alpha_{\alpha,i}(x) \bar{\phi}^\alpha_{\alpha,i}(x)}{2i} \quad (1) \]
for \( \theta \in [0, 2\pi) \). We take the boundary condition at \( c \) to be
\[ W_c(f, y_{\alpha,\theta}) = 0, \quad (2) \]
and the corresponding Weyl Space is then,

$$U_{8}^{\alpha,\theta} := \{ f \in D(L_1) \mid R_{0}^{\alpha}(f) = 0, W_c(f, y_{a,\theta}) = 0 \}. \quad (3)$$

Stone proves (cf. [29, p. 475, Theorem 10.17]) that for fixed $\alpha \in [0, \pi)$, the collection of all Weyl Spaces arising from imposition of a boundary condition at $c$ is precisely $\{U_{8}^{\alpha,\theta} \mid \theta \in [0, 2\pi)\}$. In the following theorem we establish a one-to-one correspondence between $\theta$ and $\gamma$ under which

$$U_{2}^{\alpha,\gamma} = U_{8}^{\alpha,\theta}.$$ 

It follows (from the theorem of Stone) that any self-adjoint extension of $L_0$ defined by separated boundary conditions at $a$ and $c$ necessarily has as its domain of definition one of the spaces $U_{2}^{\alpha,\gamma}$ for some $\alpha \in [0, \pi)$ and $\gamma \in [0, \pi)$.

**Theorem 7.** Fix $\alpha \in [0, \pi)$. The relation

$$e^{i\theta} = \frac{w_{\alpha,\gamma}^{\alpha,\gamma}(i)}{w_{\alpha,\gamma}^{\alpha,\gamma}(i)} \quad (4)$$

defines a one-to-one correspondence between the parameters $\theta \in [0, 2\pi)$ and $\gamma \in [0, \pi)$ such that

$$U_{2}^{\alpha,\gamma} = U_{8}^{\alpha,\theta}.$$ 

Expressing $\gamma$ as a function of $\theta$ we have

$$\gamma(\theta) := \text{Arc} \cot \left( -\frac{\text{Im} \left[ \exp(i\theta/2) \left( S_{\phi_{a,i}}^{\alpha} \right) \right]}{\text{Im} \left[ \exp(i\theta/2) \left( S_{\phi_{a,i}}^{\alpha} \right) \right]} \right) \quad (5)$$

which is equivalent to (4).

Moreover, the relation connecting the boundary functionals (2) and (3.7) is,

$$W_{c}(f, y_{a,\theta}) = K^{\gamma(\theta)} R_{c}^{\gamma(\theta)}(f) \quad \text{for all} \quad f \in D(L_1)$$

with $K^{\gamma(\theta)} \neq 0$ for all $\theta$, and where in fact,

$$K^{\gamma(\theta)} := \begin{cases} 
-\frac{\text{Im}[\exp(i\theta/2) (S_{\phi_{a,i}}^{\alpha})]}{\sin \gamma(\theta)}, & \text{if} \ \gamma(\theta) \neq 0, \pi/2 \\
\text{Im} \left[ \exp(i\theta/2) \left( S_{\phi_{a,i}}^{\alpha} \right) \right], & \text{if} \ \gamma(\theta) = 0 \\
-\text{Im} \left[ \exp(i\theta/2) \left( S_{\phi_{a,i}}^{\alpha} \right) \right], & \text{if} \ \gamma(\theta) = \pi/2
\end{cases} \quad (6)$$

**Proof.** As in Theorem 3 the boundary functionals in (2) and (19) are equivalent if and only if

$$W_{c}(\phi_{c,i}, y_{a,\theta}) = 0. \quad (7)$$
Making use of (3.22), (3.19) and (3.23) it is readily shown that this equation is equivalent to
\[ e^{i\theta} = \frac{w^{\alpha,\gamma} (i)}{w^{\alpha,\gamma} (i)} = \frac{(S\phi_{a,i})_1^\alpha (c) \cos \gamma + (S\phi_{a,i})_2^\alpha (c) \sin \gamma}{(S\phi_{a,i})_1^\alpha (c) \cos \gamma + (S\phi_{a,i})_2^\alpha (c) \sin \gamma}. \] (8)

In order to show that this equation defines a one-to-one correspondence between \( \theta \) and \( \gamma \) we consider the linear fractional transformation,
\[ B(z) := e^{i\nu} \frac{z - \xi}{z - \xi}', \] (9)
where \( \nu \) and \( \xi \) are defined by,
\[ \xi := -\frac{(S\phi_{a,i})_2^\alpha (c)}{(S\phi_{s,i})_1^\alpha (c)} \quad \text{and} \quad e^{i\nu} := \frac{(S\phi_{a,i})_1^\alpha (c)}{(S\phi_{s,i})_1^\alpha (c)}. \] (10)

It will be observed that \( \xi \) is the point in the \( z \)-plane which maps onto the center of the limit circle, \( C_{c,i}^\alpha \), under the linear fractional transformation \( T_{c,i}^\alpha (z) \) defined in (6.4). Since the real \( z \)-axis maps onto the circle \( C_{c,i}^\alpha \), itself, it clear that \( \text{Im} \xi \neq 0 \). Now since \( \text{Im} \xi \neq 0 \) it follows that (9) defines an invertible linear fractional transformation which maps the real \( z \)-axis onto the unit circle (cf. Titchmarsh [33, p.193, Section 6.23]). From (8) and (9) it follows that the equation,
\[ e^{i\theta} = B(\cot \gamma) \] (11)
defines a one-to-one correspondence between \( \theta \) and \( \gamma \) under which (7) holds. Under this one-to-one correspondence the boundary functionals in (2) and (7) are equivalent. The relation (5) expressing \( \gamma \) in terms of \( \theta \) can be obtained by solving (11) for \( \gamma \) in terms of \( \theta \). Alternatively, it will be noted that Theorem 4 applies since the functions \( y_{\alpha,\theta} (x) \) are real-valued. Hence the formula (5) can also be obtained by putting \( w_c (x) = -y_{\alpha,\theta} (x) \) in (5.5). The formula for \( K^{\gamma(\nu)} \) in (6) is then obtained by putting \( w_c (x) = -y_{\alpha,\theta} (x) \) in (5.6).

Remark 8.1 Since the function \( y_{\alpha,\theta} (x) \) in (1) depends on \( \alpha \) as well as on \( \theta \), the \( \theta \)-parametrization of the boundary conditions in (2) is dependent on the choice of \( \alpha \). The \( \gamma \)-parametrization of (3.7) by contrast does not depend on \( \alpha \). If \( \alpha \) is varied in (3) the \( \theta \)-labeling of the boundary conditions at \( c \) changes. Values of \( \theta \) corresponding to a fixed choice of \( \gamma \) are, however, identified by the relation (4).

A different proof that \( \{ U_{a,\gamma}^\alpha : \gamma \in [0, \pi) \} \) is complete collection of Weyl Spaces arising from separated boundary conditions at \( a \) and \( c \) can be given as
From Green’s Formula and (3.5) we have for all \( f, g \in D(L_1) \):

\[
(\tau f, g) - (f, \tau g) = - \left\{ f(a) \overline{g'(a)} - f'(a) \overline{g(a)} \right\} + \left\{ (Sf)_1(c) (Sg)_2(c) - (Sf)_2(c) (Sg)_1(c) \right\}
\]

If \( L \) is an arbitrary self-adjoint extension of \( L_0 \) defined by separated boundary conditions it is evident from symmetry considerations that one must be a boundary condition at \( a \) and one at \( c \). If \( b_a(\cdot) \) and \( b_c(\cdot) \) are the associated boundary functionals they must by representable (cf.

\[
\begin{align*}
b_a(f) &= z_1 f(a) + z_2 f'(a) \\
b_c(f) &= z_3 (Sf)_1(c) + z_4 (Sf)_2(c)
\end{align*}
\]

where \( |z_1|^2 + |z_2|^2 > 0 \) and \( |z_3|^2 + |z_4|^2 > 0 \). (13)

From (12) the symmetry of \( L \) requires that

\[
\begin{align*}
f(a) \overline{g'(a)} - f'(a) \overline{g(a)} &= 0 \\
(Sf)_1(c) \overline{(Sg)_2(c)} - (Sf)_2(c) \overline{(Sg)_1(c)} &= 0
\end{align*}
\]

(14)

for all \( f, g \) in the domain of \( L \), from which it is easily deduced that the boundary functionals \( b_a(\cdot) \) and \( b_c(\cdot) \) defining \( L \) can be expressed in the form,

\[
\begin{align*}
b_a(f) &= K_1 \left[ \cos \alpha f(a) + \sin \alpha f'(a) \right] \\
b_c(f) &= K_2 \left[ \cos \gamma (Sf)_1(c) + \sin \gamma (Sf)_2(c) \right]
\end{align*}
\]

(15)

where \( K_1, K_2 \in \mathbb{C}, K_1 \neq 0, K_2 \neq 0 \) and \( \alpha, \gamma \in [0, \pi) \). Since \( K_1 \neq 0 \) and \( K_2 \neq 0 \) in this representation it follows that the domain of \( L \) is \( U_2^{\alpha, \gamma} \) for suitable \( \alpha, \gamma \in [0, \pi) \).

---

42Dunford and Schwartz [5, p.1306-1309, Theorem 30 and Corollary 31]. The present argument represents only the substance of Corollary 31. In this corollary the general form of (separated) boundary conditions which are of the ‘symmetric type’ ([5, p.1235, Definition 25]) is deduced from results obtained in [5, Theorem 30, p.1306] (that is, from (8.12)). The more critical information that an arbitrary self-adjoint extension of \( L_0 \) necessarily arises from 2 linearly independent boundary conditions which must be of the symmetric type relies on the theory of the Deficiency Index as given in [5, Chapter XII, Section 4, cf. Theorem 30, p. 1238].

43From this equation it is clear that the boundary functionals in (3.4) are in ‘normal form’ of [5, p. 1306].

44In [5, p.1309], it is deduced that the boundary functionals ‘must have the form’ \( \alpha_1 f(a) + \alpha_2 f'(a) \) and \( \beta_1 (Sf)_1(c) + \beta_2 (Sf)_2(c) \) where \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) are real and \( \alpha_1^2 + \alpha_2^2 \neq 0, \beta_1^2 + \beta_2^2 \neq 0 \), and that the separated set of boundary conditions is therefore ‘necessarily real’. As we have seen in Section 5 the Weyl-boundary-functionals in (5.2) are not of the real type (Definition 3 (i)). They are, however, ‘necessarily equivalent’ (by Theorem 5) to the boundary functionals \( \mathcal{R}^2(\cdot) \) which are of the real type. (A proof that the class of all Weyl-boundary conditions associated with \( \lambda = i \) represent a complete set of all possible boundary conditions was given by F. Rellich in [23, p.60]).
10 Bibliography

References

Theorie der Linearen Operatoren im Hilbert Raum.
Akademie Verlag, Berlin 1968 (fünfte Auflage)

Complex Analysis.

Abstract Symmetric Boundary Conditions.
Trans. Amer. Math Soc. 45, 369-442 (1939)

Theory of Ordinary Differential Equations.

Linear Operators, II.
Interscience, New York 1964 (2nd. Printing)

Higher Transcendental Functions, Vol. II.
(Bateman Manuscript Project)
McGraw-Hill, New York 1953

[7] Friedrichs, K.
Über die ausgezeichnete Randbedingung in der
Spektraltheorie der halbbeschränkten gewöhnlichen
Differentialoperatoren zweiter Ordnung.

[8] Friedrichs, K.
Criteria for Discrete Spectra.

[9] Hartman, P.
Ordinary Differential Equations.
John Wiley and Sons, New York 1964
    An Oscillation Theorem for Continuous Spectra.

    Differential Operators of Mathematical Physics.
    Addison-Wesley, Reading, Mass. 1967
    (English Translation)

[12] Heywood, P.
    On the Asymptotic Distribution of Eigenvalues.

[13] Hille, E.
    Lectures on Ordinary Differential Equations.
    Addison-Wesley, Reading, Mass. 1969

[14] John, F.
    Ordinary Differential Equations
    Lecture Notes, New York University (1964-65)

    Lectures Notes, Aarhus Universitet, Aarhus, Denmark (1962-63)

[16] Kodaira, K.
    The Eigenvalue Problem for Ordinary Differential
    Equations of the Second Order and Heisenberg’s
    Theory of S-Matrices.
    Amer. Jour. Math. 71, 921-945 (1949)

[17] Levinson, N.
    A Simplified Proof of the Expansion Theorem for
    Singular Second-order Linear Differential Equations.
    Duke Math. J. 18, 57-71 (1951)

[18] Magnus, W., Oberhettinger, F., and Soni, R.P.
    Formulas and Theorems for the Special Functions of
    Mathematical Physics.
[19] Meetz, K.
Singular Potentials in Nonrelativistic Quantum Mechanics.

[20] Neumark, M.A.
Lineare Differential Operatoren.
Akademie Verlag, Berlin 1960

[21] Pleijel, A.
Some Remarks about the Limit Point and Limit Circle Theory.
Arkiv för Matematik Bd. 7, nr. 41, 543-550 (1968)

[22] Rellich, F.
Halbbeschränkte gewöhnliche Differentialoperatoren zweiter Ordnung.
Math. Ann. 122, 343-368 (1951)

[23] Rellich, F.
Lecture Notes, New York University (1951)

[24] Sears, D.B.
On the Solutions of a Linear Second-order Differential Equation which are of Integrable Square.

[25] Sears, D.B.
On the Spectrum of a certain Differential Equation.
J. London Math. Soc. 26, 205-210 (1951)

[26] Sears, D.B.
Integral Transforms and Eigenfunction Theory.

[27] Sears, D.B., and Titchmarsh, E.C.
Some Eigenfunction Formulae.
Quart. J. Math. Oxford (2) 1, 165-175 (1950)

[28] Schneider, A.
Eine Bemerkung zum Weyl-Stoneschen Eigenwertproblem.
Arch. Math. 17, 352-358 (1966)
[29] Stone, M.H.
Linear Transformations in Hilbert Space and their Applications to Analysis.

Eigenfunction Expansions Associated with Second-order Differential Equations, I.

[31] Titchmarsh, E.C.
Eigenfunction Expansions Associated with Second-order Differential Equations, I.

[32] Titchmarsh, E.C.
On Expansions in Eigenfunctions, IV.

[33] Titchmarsh, E.C.
The Theory of Functions.

[34] Walter, J.
Bemerkungen zu dem Grenzpunktfallkriterium von N. Levinson

[35] Watson, G.N.
Theory of Bessel Functions.
Cambridge at the University Press 1944 (Second Edition)

[36] Weidmann, J.
[37] Weyl, H.
Über gewöhnliche Differentialgleichungen mit Singuläritäten und die zugehörige Entwicklung willkürlicher Funktionen.
Math. Ann. 68, 220-269 (1910)

[38] Yosida, K.
Lectures on Differential and Integral Equations.
Interscience, New York 1960