Section 2-6: Input Impedance

**Problem 2.17** At an operating frequency of 300 MHz, a lossless 50-Ω air-spaced transmission line 2.5 m in length is terminated with an impedance \( Z_L = (60 + j20) \) Ω. Find the input impedance.

**Solution:** Given a lossless transmission line, \( Z_0 = 50 \) Ω, \( f = 300 \) MHz, \( l = 2.5 \) m, and \( Z_L = (60 + j20) \) Ω. Since the line is air filled, \( u_p = c \) and therefore, from Eq. (2.38),

\[
\beta = \frac{\omega}{u_p} = \frac{2\pi \times 300 \times 10^6}{3 \times 10^8} = 2\pi \text{ rad/m.}
\]

Since the line is lossless, Eq. (2.69) is valid:

\[
Z_n = Z_0 \left( \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)} \right) = 50 \left( \frac{(60 + j20) + j50 \tan(2\pi \text{ rad/m} \times 2.5 \text{ m})}{50 + j(60 + j20) \tan(2\pi \text{ rad/m} \times 2.5 \text{ m})} \right)
\]

\[
= 50 \left( \frac{(60 + j20) + j50 \times 0}{50 + j(60 + j20) \times 0} \right) = 50(60 + j20) \Omega.
\]

**Problem 2.18** A lossless transmission line of electrical length \( l = 0.35\lambda \) is terminated in a load impedance as shown in Fig. 2-38 (P2.18). Find \( \Gamma, S, \) and \( Z_n \).

![Figure P2.18: Loaded transmission line.](image)

**Solution:** From Eq. (2.49a),

\[
\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(60 + j30) - 100}{(60 + j30) + 100} = 0.307 e^{j132.5^\circ}.
\]

From Eq. (2.59),

\[
S = \frac{|1 + |\Gamma||}{1 - |\Gamma||} = \frac{1 + 0.307}{1 - 0.307} = 1.89.
\]
\[ Z_{\text{in}} = Z_0 \left( \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l} \right) \]

From Eq. (2.63)

\[ Z_{\text{in}} = Z_0 \left( \frac{(60 + j30) + j100 \tan \left( \frac{2\pi \text{rad}}{\lambda} 0.35\lambda \right)}{100 + j(60 + j30) \tan \left( \frac{2\pi \text{rad}}{\lambda} 0.35\lambda \right)} \right) = (64.8 - j38.3) \Omega. \]

**Problem 2.20** Show that at the position where the magnitude of the voltage on the line is a maximum the input impedance is purely real.

**Solution:** From Eq. (2.56), \( l_{\text{max}} = (\theta + 2\pi)/2 \beta \), so from Eq. (2.61), using polar representation for \( \Gamma \),

\[ Z_{\text{in}}(l_{\text{max}}) = Z_0 \left( \frac{1 + |\Gamma| e^{j\theta} e^{-j2\beta l_{\text{max}}}}{1 - |\Gamma| e^{j\theta} e^{-j2\beta l_{\text{max}}}} \right) = Z_0 \left( \frac{1 + |\Gamma| e^{j\theta} e^{-j(\theta + 2\pi)}}{1 - |\Gamma| e^{j\theta} e^{-j(\theta + 2\pi)}} \right) = Z_0 \left( \frac{1 + |\Gamma|}{1 - |\Gamma|} \right), \]

which is real, provided \( Z_0 \) is real.

**Problem 2.21** A voltage generator with \( v_g(t) = 5 \cos(2\pi \times 10^9 t) \) V and internal impedance \( Z_g = 50 \Omega \) is connected to a 50-\( \Omega \) lossless air-spaced transmission line. The line length is 5 cm and it is terminated in a load with impedance \( Z_L = (100 - j100) \Omega \). Find

(a) \( \Gamma \) at the load.

(b) \( Z_{\text{in}} \) at the input to the transmission line.

(c) the input voltage \( \hat{V}_i \) and input current \( \hat{I}_i \).
Section 2-7: Special Cases

Problem 2.24 At an operating frequency of 200 MHz, it is desired to use a section of a lossless 50-Ω transmission line terminated in a short circuit to construct an equivalent load with reactance \( X = 25 \Omega \). If the phase velocity of the line is 0.75c, what is the shortest possible line length that would exhibit the desired reactance at its input?

Solution:

\[
\beta = \frac{\omega}{u_p} = \frac{(2\pi \text{ rad/cycle}) \times (200 \times 10^6 \text{ cycle/s})}{0.75 \times (3 \times 10^8 \text{ m/s})} = 5.59 \text{ rad/m}.
\]

On a lossless short-circuited transmission line, the input impedance is always purely imaginary; i.e., \( Z_{\text{in}}^{\text{sc}} = jX_{\text{in}}^{\text{sc}} \). Solving Eq. (2.68) for the line length,

\[
l = \frac{1}{\beta} \tan^{-1} \left( \frac{X_{\text{in}}^{\text{sc}}}{Z_0} \right) = \frac{1}{5.59 \text{ rad/m}} \tan^{-1} \left( \frac{25 \Omega}{50 \Omega} \right) = \frac{(0.464 + n\pi) \text{ rad}}{5.59 \text{ rad/m}},
\]

for which the smallest positive solution is 8.3 cm (with \( n = 0 \)).

Problem 2.25 A lossless transmission line is terminated in a short circuit. How long (in wavelengths) should the line be in order for it to appear as an open circuit at its input terminals?

Solution: From Eq. (2.68), \( Z_{\text{in}}^{\text{sc}} = jZ_0 \tan \beta l \). If \( \beta l = (\pi/2 + n\pi) \), then \( Z_{\text{in}}^{\text{sc}} = j\infty \Omega \). Hence,

\[
l = \frac{\lambda}{2\pi} \left( \frac{\pi}{2} + n\pi \right) = \frac{\lambda}{4} + \frac{n\lambda}{2}.
\]

This is evident from Figure 2.15(d).

Problem 2.26 The input impedance of a 31-cm-long lossless transmission line of unknown characteristic impedance was measured at 1 MHz. With the line terminated in a short circuit, the measurement yielded an input impedance equivalent to an inductor with inductance of 0.128 \( \mu \text{H} \), and when the line was open circuited, the measurement yielded an input impedance equivalent to a capacitor with capacitance of 20 pF. Find \( Z_0 \) of the line, the phase velocity, and the relative permittivity of the insulating material.

Solution: Now \( \omega = 2\pi f = 6.28 \times 10^6 \text{ rad/s} \), so

\[
Z_{\text{in}}^{\text{sc}} = j\omega L = j2\pi \times 10^6 \times 0.128 \times 10^{-6} = j0.804 \Omega
\]
and $Z_{in}^{\infty} = 1/j\omega C = 1/(j2\pi \times 10^6 \times 20 \times 10^{-12}) = -j8000 \Omega$.

From Eq. (2.74), $Z_0 = \sqrt{Z_{in}^{\infty}Z_{in}^{\infty}} = \sqrt{(j0.804 \Omega)(-j8000 \Omega)} = 80 \Omega$. Using Eq. (2.75),

$$u_p = \frac{\omega}{\beta} = \frac{\omega}{\tan^{-1}\sqrt{-Z_{in}^{\infty}/Z_{in}^{\infty}}} = \frac{6.28 \times 10^6 \times 0.31}{\tan^{-1}\left(\pm\sqrt{j0.804/(-j8000)}\right)} = \frac{1.95 \times 10^6}{(\pm 0.01 + n\pi)} \text{ m/s},$$

where $n \geq 0$ for the plus sign and $n \geq 1$ for the minus sign. For $n = 0$, $u_p = 1.94 \times 10^8 \text{ m/s} = 0.65c$ and $\varepsilon_r = (c/u_p)^2 = 1/0.65^2 = 2.4$. For other values of $n$, $u_p$ is very slow and $\varepsilon_r$ is unreasonably high.

**Problem 2.27** A 60-\Omega resistive load is preceded by a $\lambda/4$ section of a 50-\Omega lossless line, which itself is preceded by another $\lambda/4$ section of a 100-\Omega line. What is the input impedance?

Solution: The input impedance of the $\lambda/4$ section of line closest to the load is found from Eq. (2.77):

$$Z_\text{in} = \frac{Z_0^2}{Z_L} = \frac{50^2}{60} = 41.7 \Omega.$$

The input impedance of the line section closest to the load can be considered as the load impedance of the next section of the line. By reapplying Eq. (2.77), the next section of $\lambda/4$ line is taken into account:

$$Z_\text{in} = \frac{Z_0^2}{Z_L} = \frac{100^2}{41.7} = 240 \Omega.$$

**Problem 2.28** A 100-MHz FM broadcast station uses a 300-\Omega transmission line between the transmitter and a tower-mounted half-wave dipole antenna. The antenna impedance is 73 \Omega. You are asked to design a quarter-wave transformer to match the antenna to the line.

(a) Determine the electrical length and characteristic impedance of the quarter-wave section.

(b) If the quarter-wave section is a two-wire line with $d = 2.5$ cm, and the spacing between the wires is made of polystyrene with $\varepsilon_r = 2.6$, determine the physical length of the quarter-wave section and the radius of the two wire conductors.
Figure P2.32: Antenna configuration for Problem 2.32.

\[ P_{\text{in}} = \frac{1}{2} \Re \{ \overline{h} V^* \} = \frac{1}{2} \Re \{ \overline{h} Z_{\text{in}} Z^*_{\text{in}} \} = \frac{(2.86)^2 \times 37.5}{2} = 153.37 \text{ (W)}. \]

This is divided equally between the two antennas. Hence, each antenna receives \( \frac{153.37}{2} = 76.68 \text{ (W)} \).

**Problem 2.33.** For the circuit shown in Fig. 2-42 (P2.33), calculate the average incident power, the average reflected power, and the average power transmitted into the infinite 100-Ω line. The λ/2 line is lossless and the infinitely long line is slightly lossy. (Hint: The input impedance of an infinitely long line is equal to its characteristic impedance so long as \( \alpha \neq 0 \).)

**Solution:** Considering the semi-infinite transmission line as equivalent to a load (since all power sent down the line is lost to the rest of the circuit), \( Z_L = Z_I = 100 \Omega \). Since the feed line is λ/2 in length, Eq. (2.76) gives \( Z_{\text{in}} = Z_L = 100 \Omega \) and \( \beta l = (2\pi/\lambda)(\lambda/2) = \pi \), so \( e^{\pm j\beta l} = -1 \). From Eq. (2.49a),

\[ \Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{100 - 50}{100 + 50} = \frac{1}{3}. \]
Also, converting the generator to a phasor gives \( \tilde{V}_g = 2e^{j\theta^o} \) (V). Plugging all these results into Eq. (2.66),

\[
V_0^+ = \left( \frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}} \right) \left( \frac{1}{e^{j\beta l} + 1} \frac{1}{e^{-j\beta l}} \right) = \left( 2 \times 100 \right) \left( \frac{1}{50 + 100} \right) \left( \frac{1}{(-1) + \frac{1}{2}(-1)} \right)
= 1e^{j180^o} = -1 \text{ (V)}.
\]

From Eqs. (2.84), (2.85), and (2.86),

\[
P_{AV} = \frac{|V_0^+|^2}{2Z_0} = \frac{|1e^{j180^o}|^2}{2 \times 50} = 10.0 \text{ mW},
\]

\[
P_{AV} = -|\Gamma|^2 \frac{P_{AV}}{3} = -\left( \frac{1}{3} \right)^2 \times 10 \text{ mW} = -1.1 \text{ mW},
\]

\[
P_{AV} = P_{AV} = P_{AV} + P_{AV} = 10.0 \text{ mW} - 1.1 \text{ mW} = 8.9 \text{ mW}.
\]

**Problem 2.34** An antenna with a load impedance \( Z_L = (75 + j25) \Omega \) is connected to a transmitter through a 50-\( \Omega \) lossless transmission line. If under matched conditions (50-\( \Omega \) load), the transmitter can deliver 10 W to the load, how much power does it deliver to the antenna? Assume \( Z_g = Z_0 \).
CHAPTER 2

Solution: From Eqs. (2.66) and (2.61),
\[
V_0^+ = \left( \frac{V_g Z_{in}}{Z_g + Z_{in}} \right) \left( \frac{1}{e^{j\beta l} + e^{-j\beta l}} \right)
\]
\[
= \frac{V_g Z_0 \left( (1 + e^{-j2\beta l})/(1 - e^{-j2\beta l}) \right)}{Z_0 + Z_0 \left( (1 + e^{-j2\beta l})/(1 - e^{-j2\beta l}) \right)} \left( \frac{e^{-j\beta l}}{1 + e^{-j2\beta l}} \right)
\]
\[
= \frac{V_g e^{-j\beta l}}{(1 - e^{-j2\beta l})/(1 + e^{-j2\beta l})}
\]
\[
= \frac{1}{2} V_g e^{-j\beta l}.
\]

Thus, in Eq. (2.86),
\[
P_{av} = \frac{|V_0^+|^2}{2Z_0} (1 - |\Gamma|^2) = \frac{1}{2} \frac{|V_g e^{-j\beta l}|^2}{Z_0} (1 - |\Gamma|^2) = \frac{|V'_{g+}|^2}{8Z_0} (1 - |\Gamma|^2).
\]

Under the matched condition, |\Gamma| = 0 and \( P_L = 10 \text{ W} \), so \( |V'_{g+}|^2/8Z_0 = 10 \text{ W} \).

When \( Z_L = (75 + j25) \Omega \), from Eq. (2.49a),
\[
\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(75 + j25) \Omega - 50 \Omega}{(75 + j25) \Omega + 50 \Omega} = 0.277e^{33.6^\circ},
\]
so \( P_{av} = 10 \text{ W} (1 - |\Gamma|^2) = 10 \text{ W} (1 - 0.277^2) = 9.23 \text{ W} \).

Section 2-9: Smith Chart

Problem 2.35 Use the Smith chart to find the reflection coefficient corresponding to a load impedance:
(a) \( Z_L = 3Z_0 \),
(b) \( Z_L = (2 - 2j)Z_0 \),
(c) \( Z_L = -2jZ_0 \),
(d) \( Z_L = 0 \) (short circuit).

Solution: Refer to Fig. P2.35.
(a) Point A is \( z_L = 3 + j0 \). \( \Gamma = 0.5e^{0^\circ} \)
(b) Point B is \( z_L = 2 - j2 \). \( \Gamma = 0.62e^{-29.7^\circ} \)
(c) Point C is \( z_L = 0 - j2 \). \( \Gamma = 1.0e^{-33.1^\circ} \)
(d) Point D is \( z_L = 0 + j0 \). \( \Gamma = 1.0e^{180.0^\circ} \)
Problem 2.36 Use the Smith chart to find the normalized load impedance corresponding to a reflection coefficient:
(a) $\Gamma = 0.5$,
(b) $\Gamma = 0.5 \angle 60^\circ$,
(c) $\Gamma = -1$,
(d) $\Gamma = 0.3 \angle -30^\circ$,
(e) $\Gamma = 0$,
(f) $\Gamma = j$.

Solution: Refer to Fig. P2.36.
(a) Point $A'$ is $\Gamma = 0.5$ at $Z_L = 3 + j0$.
(b) Point $B'$ is $\Gamma = 0.5e^{j60^\circ}$ at $Z_L = 1 + j1.15$.
(c) Point $C'$ is $\Gamma = -1$ at $Z_L = 0 + j0$.
(d) Point $D'$ is $\Gamma = 0.3e^{-j30^\circ}$ at $Z_L = 1.60 - j0.53$.
(e) Point $E'$ is $\Gamma = 0$ at $Z_L = 1 + j0$.
(f) Point $F'$ is $\Gamma = j$ at $Z_L = 0 + j1$.

**Problem 2.37** On a lossless transmission line terminated in a load $Z_L = 100 \ \Omega$, the standing-wave ratio was measured to be $2.5$. Use the Smith chart to find the two possible values of $Z_0$. 
Solution: Refer to Fig. P2.37. $S = 2.5$ is at point $L_1$ and the constant SWR circle is shown. $z_L$ is real at only two places on the SWR circle, at $L_1$, where $z_L = S = 2.5$, and $L_2$, where $z_L = 1/S = 0.4$. So $Z_{01} = z_L/z_{L1} = 100 \, \Omega/2.5 = 40 \, \Omega$ and $Z_{02} = z_L/z_{L2} = 100 \, \Omega/0.4 = 250 \, \Omega$.

Figure P2.37: Solution of Problem 2.37.

Problem 2.38 A lossless 50-Ω transmission line is terminated in a load with $Z_L = (50 + j25) \, \Omega$. Use the Smith chart to find the following:
(a) the reflection coefficient $\Gamma$,
(b) the standing-wave ratio,
(c) the input impedance at 0.35λ from the load,
(d) the input admittance at 0.35λ from the load,
(e) the shortest line length for which the input impedance is purely resistive,
(f) the position of the first voltage maximum from the load.

Solution: Refer to Fig. P2.38. The normalized impedance

\[ Z_L = \frac{(50 + j25) \, \Omega}{50 \, \Omega} = 1 + j0.5 \]

is at point Z-LOAD.

(a) \( \Gamma = 0.24e^{j76.8^\circ} \) The angle of the reflection coefficient is read of that scale at the point \( \theta_r \).
(b) At the point \( SWR \): \( S = 1.64 \).

(c) \( Z_{in} \) is 0.350\( \lambda \) from the load, which is at 0.144\( \lambda \) on the wavelengths to generator scale. So point \( Z-IN \) is at 0.144\( \lambda \) + 0.350\( \lambda \) = 0.494\( \lambda \) on the WTG scale. At point \( Z-IN \):

\[
Z_{in} = z_{in}Z_0 = (0.61 - j0.022) \times 50 \Omega = (30.5 - j1.09) \Omega.
\]

(d) At the point on the SWR circle opposite \( Z-IN \),

\[
Y_{in} = \frac{y_{in}}{Z_0} = \frac{(1.64 + j0.06)}{50 \Omega} = (32.7 + j1.17) \text{ mS}.
\]

(e) Traveling from the point \( Z-LOAD \) in the direction of the generator (clockwise), the SWR circle crosses the \( x_L = 0 \) line first at the point \( SWR \). To travel from \( Z-LOAD \) to \( SWR \) one must travel 0.250\( \lambda \) - 0.144\( \lambda \) = 0.106\( \lambda \). (Readings are on the wavelengths to generator scale.) So the shortest line length would be 0.106\( \lambda \).

(f) The voltage max occurs at point \( SWR \). From the previous part, this occurs at \( z = -0.106\lambda \).

---

**Problem 2.39**  A lossless 50-\( \Omega \) transmission line is terminated in a short circuit. Use the Smith chart to find

(a) the input impedance at a distance 2.3\( \lambda \) from the load,
(b) the distance from the load at which the input admittance is \( Y_{in} = -j0.04 \) S.

**Solution:** Refer to Fig. P2.39.

(a) For a short, \( z_{in} = 0 + j0 \). This is point \( Z-SHORT \) and is at 0.000\( \lambda \) on the WTG scale. Since a lossless line repeats every \( \lambda /2 \), traveling 2.3\( \lambda \) toward the generator is equivalent to traveling 0.3\( \lambda \) toward the generator. This point is at \( A : Z-IN \), and

\[
Z_{in} = z_{in}Z_0 = (0 - j3.08) \times 50 \Omega = -j154 \Omega.
\]

(b) The admittance of a short is at point \( Y-SHORT \) and is at 0.250\( \lambda \) on the WTG scale:

\[
y_{in} = Y_{in}Z_0 = -j0.04 \text{ S} \times 50 \Omega = -j2,
\]

which is point \( B : Y-IN \) and is at 0.324\( \lambda \) on the WTG scale. Therefore, the line length is 0.324\( \lambda \) - 0.250\( \lambda \) = 0.074\( \lambda \). Any integer half wavelengths farther is also valid.