On Tensor Products, Vector Spaces, and Kronecker Products

We begin with the definition of the tensor product.

**Definition 1** Let $V$ and $W$ be vector spaces over a field $F$ with bases $\beta$ and $\gamma$, respectively. Then the tensor product $V \otimes W = \text{span} \{ v \otimes w : v \in V \text{ and } w \in W \}$ is a vector space over $F$ with the tensor properties:

\[
(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w,
\]

\[
v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2,
\]

\[
(cv) \otimes w = v \otimes (cw)
\]

for every $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, and $c \in F$. The scalar multiplication property is

\[
c(v \otimes w) = (cv) \otimes w = v \otimes (cw)
\]

for every $v \in V$, $w \in W$, and $c \in F$. Moreover, $\beta \otimes \gamma = \{ v \otimes w : v \in \beta \text{ and } w \in \gamma \}$ is a basis for $V \otimes W$.

**Remark 2** The definition of $V \otimes W$ is fine for our purposes, but is not as precise as a mathematician would like (for us it is enough to know such a vector space exists). More precisely (see [DF04]), it is the freest group with the group operation being addition on pairs $(v, w)$ then take as $v \otimes w$ the coset of $(v, w)$ which is equivalence class from moding out by the smallest subgroup so that the tensor properties above are true for all the equivalence classes. Then the vector addition and scalar multiplication is the group addition and scalar multiplication for the cosets.

Our goal is to try to characterize the tensor product and describe some of its various properties especially for finite-dimensional vector spaces.

**Proposition 3** Let $V$ and $W$ be a finite dimensional vector space over the field $F$ with basis $\beta = \{ v_1, \ldots, v_n \}$ and $\gamma = \{ w_1, \ldots, w_m \}$, respectively. Then $V \otimes W$ is a vector space of $F$ of dimension $nm$ with basis $v_i \otimes w_j$, $1 \leq i \leq n$ and $1 \leq j \leq m$.

**Proof.** Prove this proposition. ■

The following proposition characterizes tensor products in terms of products of linear functionals (which can be used as another path to rigorously defining tensor products).
Proposition 4 Let $V$ and $W$ be finite-dimensional vector spaces over a field $F$. Then on the vector space $\mathcal{F}(V \times W, F)$ of all functions from the vector space $V \times W$ to the field $F$, the subspace $Z := \text{span}\{f \times g : f \in V^* \text{ and } g \in W^*\}$, where $(f \times g)[(v,w)] = f(v)g(w)$, is isomorphic to $V \otimes W$. Furthermore, if $\beta = \{v_1, \ldots, v_n\}$ and $\gamma = \{w_1, \ldots, w_m\}$ are bases for $V$ and $W$, respectively, and $\beta^* = \{f_1, \ldots, f_n\}$ and $\gamma^* = \{g_1, \ldots, g_m\}$ are the corresponding dual bases for the dual spaces $V^*$ and $W^*$ (i.e., $f_i(v_j) = \delta_{ij}$ and $g_k(w_l) = \delta_{kl}$, where $\delta$ denotes the Kronecker delta function), then the linear transformation $\Psi : V \otimes W \to Z$ defined by

$$\Psi(v_i \otimes w_j) = f_i \times g_j, \quad 1 \leq i \leq n \text{ and } 1 \leq j \leq m,$$

and extended by linearity, is an isomorphism. Moreover, if $\phi_V : V \to V^*$ and $\phi_W : W \to W^*$ are the isomorphisms satisfying

$$\phi_V \left( \sum_{i=1}^{n} a_i v_i \right) = \sum_{i=1}^{n} a_i f_i \text{ and } \phi_W \left( \sum_{j=1}^{m} b_j w_i \right) = \sum_{j=1}^{m} b_j g_i,$$

then

$$\Psi(v \otimes w) = \phi_V(v) \times \phi_W(w) \text{ for all } (v, w) \in V \times W.$$

Proof. Prove this proposition. $\blacksquare$

Definition 5 Given two finite-dimensional vector spaces $V$ and $W$ with ordered bases $\beta = \{v_1, \ldots, v_n\}$ and $\gamma = \{w_1, \ldots, w_m\}$, respectively, the ordered basis $\beta \otimes \gamma$ for $V \otimes W$ will be given the lexicographical ordering, i.e., $v_i \otimes w_j$ comes before $v_k \otimes w_l$ in the ordering if and only if $(i, j) \leq (k, l)$, where $\leq$ is the lexicographical ordering (i.e., $(i, j) \leq (k, l)$ if and only if either $i < k$ or $(i = k$ and $j \leq l$).

Definition 6 The (right) Kronecker product $A \otimes B$ between an $r \times n$ matrix $A = [a_{ij}]$ and an $s \times m$ matrix $B = [b_{kl}]$ over a field $F$ to be the $rs \times nm$ matrix which in block matrix is defined as

$$A \otimes B := \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{s1}B & \cdots & a_{sn}B \end{bmatrix}.$$

Problem 7 Compute the Kronecker product $A \otimes B$ of

\begin{itemize}
  \item[(a)] $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, \quad $B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$;
  \item[(b)] $A = \begin{bmatrix} 1 & 2 \end{bmatrix}$, \quad $B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$;
  \item[(c)] $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, \quad $B = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$;
  \item[(d)] $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, \quad $B = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$;
  \item[(e)] $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, \quad $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$.
\end{itemize}
The following proposition connects linear transformations on finite-dimensional vector spaces to Kronecker product of matrices.

**Proposition 8** Let $T : V \to X$ and $U : W \to Y$ be linear transformations on finite-dimensional vector spaces $V$, $X$, $W$, and $Y$. Then there exists a unique linear transformation $T \otimes U : V \otimes W \to X \otimes Y$ (called the tensor product of $T$ and $U$) satisfying

$$(T \otimes U)(v \otimes w) = (Tv) \otimes (Uw)$$

for all pairs $(v, w) \in V \times W$.

Furthermore, if $\beta_V = \{v_1, \ldots, v_n\}$, $\beta_X = \{x_1, \ldots, x_r\}$, $\beta_W = \{w_1, \ldots, w_m\}$, and $\beta_Y = \{y_1, \ldots, y_s\}$ are ordered bases for $V$, $X$, $W$, and $Y$, respectively, then

$$[T \otimes U]^{\beta_X \otimes \beta_Y} = [T]^{\beta_X} \otimes [U]^{\beta_Y}.$$

**Proof.** Prove this proposition. ■

The following connection between Kronecker product of matrices is finally revealed by the next corollary.

**Corollary 9** Let $F$ be a field, $A \in M_{r \times n}(F)$ and $B \in M_{s \times m}(F)$. Then $A \otimes B \in M_{rs \times nm}(F)$ and, in terms of the vector spaces $V = F^n$, $X = F^r$, $W = F^m$, $Y = F^s$ with their standard ordered bases $\beta_V$, $\beta_X$, $\beta_W$, and $\beta_Y$, respectively, we have

$$[L_A \otimes L_B]^{\beta_X \otimes \beta_Y} = A \otimes B,$$

$$(L_A \otimes L_B)(v \otimes w) = (Av) \otimes (Bw)$$

for all $v \in V$ and all $w \in W$, where the linear transformations $L_A : V \to X$ and $L_B : W \to Y$ are left multiplication by $A$ and $B$, respectively.

**Proof.** Prove this corollary. ■

**Remark 10** Below are some more references for the interested reader on the Kronecker and Tensor product. For instance, we have not discussed the connection to the vectorization operation, Kronecker product, and inner products which is very important or the usage of the Kronecker product in solving linear matrix equations which is even more important. I have thus introduced you to only the tip of the iceberg and encourage you to dive into the subject.

**References**


