UNIVERSITY OF CALIFORNIA, IRVINE

On the Mathematics of Slow Light

DISSertation

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DOCTOR OF PHILOSOPHY

in Mathematics

by

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2011
DEDICATION

To Jayme
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ABSTRACT OF THE DISSERTATION

On the Mathematics of Slow Light

By

Aaron Thomas Welters

Doctor of Philosophy in Mathematics

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Professor Aleksandr Figotin, Chair

This thesis develops a mathematical framework based on spectral perturbation theory for the analysis of slow light and the slow wave regime for lossless one-dimensional photonic crystals.

Electrodynamics of lossless one-dimensional photonic crystals incorporating general bianisotropic and dispersive materials are considered. The time-harmonic Maxwell’s equations governing the electromagnetic wave propagation through such periodic structures is shown to reduce to a canonical system of period differential-algebraic equations (DAEs) depending holomorphically on frequency which we call Maxwell’s DAEs. In this context, a definition of slow light is given.

We give a detailed perturbation analysis for degenerate eigenvalues of non-self-adjoint matrices. A generic condition is considered and its consequences are studied. We prove the generic condition implies the degenerate eigenvalue of the unperturbed matrix under consideration has a single Jordan block in its Jordan normal form corresponding to that eigenvalue. We find explicit recursive formulas to calculate the perturbation expansions of the splitting eigenvalues and their eigenvectors. The coefficients up to the second order for these expansions are given.
An exposition on the spectral theory and spectral perturbation theory for holomorphic matrix functions is given. This serves as a mathematical background for the tools and theory used in this thesis.

We give a model for studying slow light based on canonical equations which are any canonical system of periodic differential (canonical ODEs) or differential-algebraic (canonical DAEs) equations depending holomorphically on a spectral parameter referred to as the frequency. For canonical ODEs, we prove formulas connecting the dispersion relation, energy flux, and energy density in our model to the monodromy matrix of the canonical ODEs. We prove one of the main results of this thesis relating the existence of (non-Bloch) Floquet solutions of the canonical ODEs and the occurrence of degenerate eigenvalues with nondiagonalizable Jordan normal form of the monodromy matrix to the band structure of the dispersion relation near spectral stationary points. For canonical DAEs, of which it is shown Maxwell’s DAEs are an example, a theorem is given which shows that the model for canonical DAEs, including the Floquet, spectral, and perturbation theory, is reduced to the model for canonical ODEs.
Chapter 1

Introduction

1.1 Motivation

Technology has progressed to the point at which we can now fabricate materials with dimensions on the nanoscale. To give you a perspective on how small this is consider that one nanometer (nm) is equal to one billionth of a meter (m), the width of a human hair is about $10^5$ nm, and the wavelength of visible light lies in the approximate range of 400–700 nm. As a result, there has emerged a new class of materials (such as metamaterials [11, 54]), which allow the possibility of novel electromagnetic properties and effects. For example, bianisotropic crystals can be used to observe optical singularities [6], photonic crystals [27] can be used to slow down light [16], negative index materials can be used to create a perfect lens [50], and metamaterials derived from transformation optics can be used to induce electromagnetic invisibility [21, 22, 40, 51]. The novel electromagnetic properties and effects that metamaterials such as photonic crystals have, is a fascinating and productive area of research for not only physicists but for mathematicians as well (see for example [32]). In particular, research on slow light has received considerable attention recently. With its numerous appli-
cations such as solar cell efficiency enhancement [10] or antenna miniaturization [49, 59, 62], there is a growing need for applied and theoretical studies on slow light phenomena.

Although there are several methods for slowing down light including electromagnetically induced transparency (EIT) and photonic crystals, it is the latter that is becoming of particular interest to mathematicians. A major reason for this is the mathematics of photonic crystals has been developed significantly over the last 20 years and this gives us a rigorous mathematical setting in which to begin studying problems involving slow light. Moreover, many of the results on slow wave phenomenon in photonic crystals are expected to have analogies with wave propagation in general linear dispersive media. Thus in order to facilitate further applied and theoretical studies on slow light, to better understand slow light from a mathematical perspective, and to explore analogies with slow waves in periodic media, this thesis considers a general nondissipative but dispersive model for wave propagation using an important class of periodic differential and differential-algebraic equations (DAEs) called canonical equations [31]. As we will show in the next section, this model is general enough to include electromagnetic waves governed by the time-harmonic Maxwell’s equations for lossless one-dimensional photonic crystals whose constituent layers can be any combination of isotropic, anisotropic, or bianisotropic materials with or without material dispersion (i.e., frequency-dependent response of materials). This makes our work particularly significant in the study of slow light since metamaterials are widening the range of potential photonic crystals that can be fabricated and so a model like ours that has the ability to analysis slow light phenomena for a broad range of photonic crystals is in need.
1.2 Electrodynamics of Lossless One-Dimensional Photonic Crystals

1.2.1 Time-Harmonic Maxwell’s Equations

Electromagnetic waves will be governed by the time-harmonic Maxwell’s equations \( (e^{-i\omega t} \text{ convention}, \omega \neq 0) \) without sources in Gaussian units and Cartesian coordinates (with respect to the standard orthonormal basis vectors \( e_1, e_2, e_3 \) of \( \mathbb{R}^3 \)). These equations may be written in the \( 2 \times 2 \) block operator matrix form (see [5])

\[
\begin{bmatrix}
0 & \nabla \times \\
- \nabla \times & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{E}(\mathbf{r}) \\
\mathbf{H}(\mathbf{r})
\end{bmatrix}
= -i \frac{\omega}{c}
\begin{bmatrix}
\mathbf{D}(\mathbf{r}) \\
\mathbf{B}(\mathbf{r})
\end{bmatrix}
\]  

where \( c \) is the speed of light in a vacuum, \( \mathbf{r} := (x_1, x_2, x_3) \) are the spatial variables and the electric field \( \mathbf{E} \), magnetic field \( \mathbf{H} \), electric induction \( \mathbf{D} \), and magnetic induction \( \mathbf{B} \) take values in \( \mathbb{C}^3 \). Here \( \nabla \times \) denotes the curl operator on these fields and it is given by the matrix operator

\[
\nabla \times := \begin{bmatrix}
0 & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\
\frac{\partial}{\partial x_3} & 0 & -\frac{\partial}{\partial x_1} \\
-\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0
\end{bmatrix}.
\]

(1.2)

The linear constitutive relations in \( 2 \times 2 \) block matrix form are

\[
\begin{bmatrix}
\mathbf{D}(\mathbf{r}) \\
\mathbf{B}(\mathbf{r})
\end{bmatrix}
= \mathbf{C}(x_3, \omega)
\begin{bmatrix}
\mathbf{E}(\mathbf{r}) \\
\mathbf{H}(\mathbf{r})
\end{bmatrix}, \quad \mathbf{C}(x_3, \omega) = \begin{bmatrix}
\varepsilon(x_3, \omega) & \xi(x_3, \omega) \\
\zeta(x_3, \omega) & \mu(x_3, \omega)
\end{bmatrix},
\]

(1.3)

where \( \varepsilon, \mu, \xi, \zeta \) are \( 3 \times 3 \) matrix-valued functions representing the electric permittivity, magnetic permeability, and magnetoelectric coupling tensors, respectively. Here we have tacitly
assumed materials which are plane parallel layers normal to the \( x_3 \)-axis with each layer consisting of a homogeneous material whose frequency dependency is implicitly indicated.

### 1.2.2 Lossless 1-D Photonic Crystals

We are considering lossless one-dimensional photonic crystals with dispersive materials and hence there exists a \( d > 0 \) and an open connected set \( \Omega \subseteq \mathbb{C} \), the frequency domain, with \( \Omega_{\mathbb{R}} := \Omega \cap \mathbb{R} \neq \emptyset \) such that

(i) \( C(x_3, \omega) = C(x_3 + d, \omega) \), for every \( \omega \in \Omega \) and a.e. \( x_3 \).

(ii) \( C(x_3, \omega)^* = C(x_3, \omega) \), for every \( \omega \in \Omega_{\mathbb{R}} \) and a.e. \( x_3 \).

(iii) \( C(\cdot, \omega) \in M_6(L^2(\mathbb{T})), C \in \mathcal{O}(\Omega, M_6(L^2(\mathbb{T}))) \).

### 1.2.3 Maxwell’s Equations as Canonical Equations

As in [16], we seek field solutions of the form

\[
\begin{bmatrix}
E(r) \\
H(r)
\end{bmatrix} = e^{i k_\perp \cdot r_\perp} \begin{bmatrix}
E(x_3) \\
H(x_3)
\end{bmatrix},
\]

where \( r_\perp = (x_1, x_2, 0) \), \( k_\perp = (k_1, k_2, 0) \), and \( k_1, k_2 \in \mathbb{R} \). Hence solutions to the time-harmonic Maxwell’s equations in (1.1) with the field representations (1.4) are solutions to the canonical equations, which we call Maxwell’s DAEs,

\[
\mathcal{G} y'(x_3) = V(x_3, \omega)y(x_3),
\]

\(^1\)See section 4.5 for notation.
where \( y(x_3) \) is a \( 6 \times 1 \) column vector, \( \mathcal{G} \) is a \( 6 \times 6 \) singular matrix, and \( V \), the Hamiltonian, is a \( 6 \times 6 \) matrix-valued function having the block representations

\[
y(x_3) = \begin{bmatrix} \mathbf{E}(x_3) \\ \mathbf{H}(x_3) \end{bmatrix}, \quad \mathcal{G} = i \begin{bmatrix} \mathbf{e}_3 \times \\ -\mathbf{e}_3 \times \end{bmatrix}, \quad V(x_3, \omega) = \frac{\omega}{c} \mathbf{C}(x_3, \omega) + \begin{bmatrix} \mathbf{k} \times \\ -\mathbf{k} \times \end{bmatrix}.
\]

(1.6)

Here \( \mathbf{e}_3 \times \) and \( \mathbf{k} \times \) are the matrices

\[
\mathbf{e}_3 \times := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{k} \times := \begin{bmatrix} 0 & 0 & k_2 \\ 0 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}.
\]

(1.7)

In particular, they are a canonical system of differential-algebraic equations (DAEs) with periodic coefficients that depend holomorphically on the frequency \( \omega \) where the leading matrix coefficient \( \mathcal{G} \in \mathcal{M}_6(\mathbb{C}) \) and the matrix-valued function \( V : \mathbb{R} \times \Omega \to \mathcal{M}_6(\mathbb{C}) \) have the properties:

(i) \( \det(\mathcal{G}) = 0 \), \( \mathcal{G}^* = -\mathcal{G} \)

(ii) \( V(x_3, \omega)^* = V(x_3, \omega) \), for each \( \omega \in \Omega_\mathbb{R} \) and a.e. \( x_3 \in \mathbb{R} \)

(iii) \( V(x_3 + d, \omega) = V(x_3, \omega) \), for each \( \omega \in \Omega \) and a.e. \( x_3 \in \mathbb{R} \)

(iv) \( V \in \mathcal{O}(\Omega, \mathcal{M}_6(\mathcal{L}^2(\mathbb{T}))) \) as a function of frequency.

In Chapter 4 we will consider these types of equations further.
1.2.4 Definition of Slow Light

In this context, Bloch solutions of Maxwell’s DAEs, i.e. solutions of the canonical equations in (1.5) satisfying

\[ y(x_3 + d) = e^{ikd} y(x_3), \]

for some \( k \in \mathbb{C} \), give rise to the (axial) dispersion relation

\[ \omega = \omega(k). \]

For points \((k_0, \omega_0) \in \mathbb{R}^2\) on the graph of this dispersion relation, i.e., the Bloch variety \( \mathcal{B} \), a Bloch solution \( y \) with wavenumber-frequency pair \((k_0, \omega_0)\) is said to be propagating with its group velocity

\[ \frac{d\omega}{dk} \bigg|_{(k_0, \omega_0)}. \]

Under certain reservations [7, 8, 27, 63], its group velocity equals its energy velocity, i.e., the ratio of its averaged energy flux to its energy density,

\[ \frac{d\omega}{dk} \bigg|_{(k_0, \omega_0)} = \frac{1}{d} \int_0^d \langle i\mathcal{G} y(x_3), y(x_3) \rangle dx_3 = \frac{1}{d} \int_0^d \langle V_\omega(x_3, \omega_0)y(x_3), y(x_3) \rangle dx_3, \]

with its energy flux and its energy density are (up to multiplication by a constant) the functions of the space variable \( x_3 \) given by

\[ S = \langle i\mathcal{G} y, y \rangle, \]

\[ U = \langle V_\omega(\cdot, \omega_0)y, y \rangle, \]
respectively, where $V_\omega$ denotes the derivative of the Hamiltonian $V$ with respect to frequency in the $M_N(L^2(\mathbb{T}))$ norm and $\langle \ , \ \rangle$ is the standard inner product for $\mathbb{C}^6$.

We now give an intuitive definition of slow light for lossless one-dimensional photonic crystals. In Chapter 4 we give a more precise definition using canonical equations. We will treat the (axial) dispersion relation $\omega = \omega(k)$ as a multi-valued function of the wavenumber $k$. We will use the notation $\left. \frac{d\omega}{dk} \right|_{(k_0,\omega_0)} = 0$ to mean $(k_0,\omega_0)$ is a stationary point on the graph of the dispersion relation $\omega = \omega(k)$.

**Definition 1 (Slow Light)** If $(k_0,\omega_0) \in \mathbb{R}^2$ is a stationary point on the graph of the dispersion relation $\omega = \omega(k)$, i.e.,

$$\left. \frac{d\omega}{dk} \right|_{(k_0,\omega_0)} = 0,$$

(1.14)

then any propagating Bloch solution of Maxwell’s DAEs (1.5) with wavenumber-frequency pair $(k,\omega)$ satisfying $0 < ||(k,\omega) - (k_0,\omega_0)|| \ll 1$ with its group velocity $\left. \frac{d\omega}{dk} \right|_{(k,\omega)}$ satisfying $|\left. \frac{d\omega}{dk} \right|_{(k,\omega)}| \ll c$ is called a **slow wave** or **slow light**.

If $(k_0,\omega_0) \in \mathbb{R}^2$ is a stationary point on the graph of the dispersion relation $\omega = \omega(k)$ then an open ball $B((k_0,\omega_0),r)$ in $\mathbb{C}^2$ with $0 < r \ll 1$ is called the **slow wave regime**.

### 1.3 Main Results

The objective of this dissertation is to begin developing a mathematical framework based on spectral perturbation theory for the analysis of slow light and the slow wave regime for lossless one-dimensional photonic crystals.

In the following we describe the main contributions of this thesis which are contained in
Chapters 2 and 4.

Perturbation Analysis of Degenerate Eigenvalues from a Jordan block

A fundamental problem in the perturbation theory for non-self-adjoint matrices with a degenerate spectrum is the determination of the perturbed eigenvalues and eigenvectors. Formulas for the higher order terms of these perturbation expansions are often needed in problems which require an accurate asymptotic analysis.

For example, my advisor A. Figotin and his colleague, I. Vitebskiy, considered scattering problems involving slow light in one-dimensional semi-infinite photonic crystals [3, 12–16]. They found that only in the case of the frozen mode regime could incident light enter a photonic crystal with little reflection and be converted into a slow wave. This frozen mode regime was found to correspond to a stationary inflection point of the dispersion relation and a $3 \times 3$ Jordan block in the Jordan normal form of the unit cell transfer matrix – the monodromy matrix of the reduced Maxwell’s equations given in [16, §5] or [16, p. 332, (180)] which are canonical ODEs although not in canonical form. In this setting, the eigenpairs of the monodromy matrix corresponded to Bloch waves and their Floquet multipliers. Thus in order for them to rigorously prove the physical results and provide a better understanding of the very essence of the frozen mode regime, they needed an asymptotic analytic description of the frozen mode regime which required a sophisticated mathematical framework based on the spectral perturbation theory of a Jordan block. Unfortunately, at the time when [16] was written such a theory did not exist and hence this was a big motivating factor for Chapter 2 of this thesis.

\[\text{The contents of this chapter also appear in [61]. Copyright } \copyright \text{ 2011 Society for Industrial and Applied Mathematics. Reprinted with permission. All rights reserved.}\]
In Chapter 2 of this thesis we develop the spectral perturbation theory of a Jordan block and address the following:

1. Determine a generic condition that allows a constructive spectral perturbation analysis for non-self-adjoint matrices with degenerate eigenvalues.

2. What connection is there between that condition and the local Jordan normal form of the matrix perturbation corresponding to a perturbed eigenvalue?

3. Does there exist explicit recursive formulas to determine the perturbed eigenvalues and eigenvectors for non-selfadjoint perturbations of matrices with degenerate eigenvalues?

The statement of my main results regarding those three issues are contained in Theorem 1 and Theorem 2 of this thesis which are Theorem 2.1 and Theorem 3.1 from [61]. In particular, I have developed a constructive perturbation theory for non-self-adjoint matrices with degenerate eigenvalues and found explicit recursive formulas to calculate the perturbation expansions of the splitting eigenvalues and their eigenvectors, under a generic condition.

**Canonical Equations: A Model for Studying Slow Light**

We established above that the study of slow light, i.e., slow waves for lossless one-dimensional photonic crystals is reduced to the study of Maxwell’s DAEs (1.5) near stationary points of the dispersion relation, i.e., in the slow wave regime. As Maxwell’s DAEs are canonical equations, it will be beneficial to use canonical equations to study slow wave propagation. By analogy with that physical model, we get a general mathematical model for wave propagation in periodic structures in which we can study the dispersion relation, band structure, spectral stationary points, slow waves, and the slow wave regime. This is exactly what we do in Chapter 4. The contents of this chapter is part of the paper [60] currently in preparation.
One of the major contribution in Chapter 4 of this thesis is Theorem 48. Combined with Theorem 35, Theorem 47, and Theorem 50, it is our answer to the question:

1. How are the analytic properties of the dispersion relation and the degree of flatness of spectral bands near stationary points related to (non-Bloch) Floquet solutions and the occurrence of degenerate eigenvalues and a nondiagonalizable Jordan normal form for the monodromy matrix of the canonical equations.

To answer this question and to proof Theorem 48 we need some deep results in the spectral perturbation theory for holomorphic matrix functions. We give an exposition of this theory in Chapter 3.

1.4 Overview

This thesis is organized in the following manner.

Chapter 2 concerns the perturbation analysis of non-self-adjoint matrices with degenerate eigenvalues. A generic condition is considered and its consequences are studied. It is shown that the generic condition implies the degenerate eigenvalue of the unperturbed matrix under consideration has a single Jordan block in its Jordan normal form corresponding to the degenerate eigenvalue. Explicit recursive formulas are given to calculate the perturbation expansions of the splitting eigenvalues and their eigenvectors. The coefficients up to the second order for these expansions are conveniently listed for quick reference.

Chapter 3 is an exposition on the spectral theory and spectral perturbation theory for holomorphic matrix functions. Its content is the required background material needed in Chapter 4 to prove the main result for that chapter, namely, Theorem 48.

Chapter 4 formulates and studies a model for slow wave propagation in period structures
using canonical equations. Both canonical ODEs and canonical DAEs are considered. Electromagnetic wave propagation in lossless one dimensional photonic crystals was discussed in section 1.2 as a example for the need to include DAEs into the model. Motivation for the definitions in the model was given by a consideration of their relevance in the example.

Next, in section 4.2, canonical ODEs are considered. Results are given on energy flux and energy density. Then in section 4.2.3 an important theorem, Theorem 47, is given on points of definite type for canonical ODEs. In particular, the result justifies the use of the term definite type and states that to each point of definite type of the canonical ODEs there exists a neighborhood of such points. This is followed up immediately in the next section, section 4.2.4, with the main result of the chapter, Theorem 48, pertaining to the perturbation theory for canonical ODEs near points of definite type. As a corollary we give a result that connects the generic condition of Chapter 2 to points of definite type and the Jordan normal form of the monodromy matrix of the canonical ODEs.

Next, in section 4.3, canonical DAEs are considered. A theorem is stated that tells us the theory of canonical DAEs is reduced to the study of canonical ODEs including the Floquet, spectral, and perturbation theory.

Finally, section 4.4 gives the proofs of all the statements in this chapter. The use of the spectral perturbation theory of holomorphic matrix functions is use to prove the main result of the chapter, Theorem 48.

All of the work done here was completed under the guidance and supervision of Professor Aleksandr Figotin.
Chapter 2

Perturbation Analysis of Degenerate Eigenvalues from a Jordan block\textsuperscript{1}

2.1 Introduction

Consider an analytic square matrix $A(\varepsilon)$ and its unperturbed matrix $A(0)$ with a degenerate eigenvalue $\lambda_0$. A fundamental problem in the analytic perturbation theory of non-selfadjoint matrices is the determination of the perturbed eigenvalues near $\lambda_0$ along with their corresponding eigenvectors of the matrix $A(\varepsilon)$ near $\varepsilon = 0$. More specifically, let $A(\varepsilon)$ be a matrix-valued function having a range in $M_n(\mathbb{C})$, the set of $n \times n$ matrices with complex entries, such that its matrix elements are analytic functions of $\varepsilon$ in a neighborhood of the origin. Let $\lambda_0$ be an eigenvalue of the matrix $A(0)$ with algebraic multiplicity $m \geq 1$. Then in this situation, it is well known [4, §6.1.7], [28, §II.1.8] that for sufficiently small $\varepsilon$ all the perturbed eigenvalues near $\lambda_0$, called the $\lambda_0$-group, and their corresponding eigenvectors may be represented as a collection of convergent Puiseux series, i.e., convergent Taylor series in

\textsuperscript{1}The contents of this chapter also appear in [61]. Copyright © 2011 Society for Industrial and Applied Mathematics. Reprinted with permission. All rights reserved.
a fractional power of $\varepsilon$. What is not well known, however, is how we compute these Puiseux series when $A(\varepsilon)$ is a non-selfadjoint analytic perturbation and $\lambda_0$ is a defective eigenvalue of $A(0)$. There are sources on the subject like [4, §7.4], [41], [47], [57, §32], and [58] but it was found that there lacked explicit formulas, recursive or otherwise, to compute the series coefficients beyond the first order terms. Thus the fundamental problem that this paper addresses is to find explicit recursive formulas to determine the Puiseux series coefficients for the $\lambda_0$-group and their eigenvectors.

This problem is of applied and theoretic importance, for example, in studying the spectral properties of dispersive media such as photonic crystals. In particular, this is especially true in the study of slow light [16, 49, 62], where the characteristic equation, $\det (\lambda I - A(\varepsilon)) = 0$, represents implicitly the dispersion relation for Bloch waves in the periodic crystal. In this setting $\varepsilon$ represents a small change in frequency, $A(\varepsilon)$ is the Transfer matrix of a unit cell, and its eigenpairs, $(\lambda(\varepsilon), x(\varepsilon))$, correspond to the Bloch waves. From a practical and theoretical point of view, condition (2.1) on the dispersion relation or its equivalent formulation in Theorem 1. (i) of this paper regarding the group velocity for this setting, arises naturally in the study of slow light where the Jordan normal form of the unperturbed Transfer matrix, $A(0)$, and the perturbation expansions of the eigenpairs of the Transfer matrix play a central role in the analysis of slow light waves.

**Main Results**

In this paper under the *generic condition*,

$$\left. \frac{\partial}{\partial \varepsilon} \det (\lambda I - A(\varepsilon)) \right|_{(\varepsilon, \lambda) = (0, \lambda_0)} \neq 0,$$

we show that $\lambda_0$ is a non-derogatory eigenvalue of $A(0)$ and the fundamental problem mentioned above can be solved. In particular, we prove Theorem 1 and Theorem 2 which together
state that when condition (2.1) is true then the Jordan normal form of $A(0)$ corresponding to the eigenvalue $\lambda_0$ consists of a single $m \times m$ Jordan block, the $\lambda_0$-group and their corresponding eigenvectors can each be represented by a single convergent Puiseux series whose branches are given by

$$\lambda_h(\varepsilon) = \lambda_0 + \sum_{k=1}^{\infty} \alpha_k \left( \zeta^k \varepsilon^{\frac{1}{m}} \right)^k$$

$$x_h(\varepsilon) = \beta_0 + \sum_{k=1}^{\infty} \beta_k \left( \zeta^k \varepsilon^{\frac{1}{m}} \right)^k$$

for $h = 0, \ldots, m - 1$ and any fixed branch of $\varepsilon^{\frac{1}{m}}$, where $\zeta = e^{\frac{2\pi i}{m}}$, $\{\alpha_k\}_{k=1}^{\infty} \subseteq \mathbb{C}$, $\{\beta_k\}_{k=0}^{\infty} \subseteq \mathbb{C}^n$, $\alpha_1 \neq 0$, and $\beta_0$ is an eigenvector of $A(0)$ corresponding to the eigenvalue $\lambda_0$. More importantly though, Theorem 2 gives explicit recursive formulas that allows us to determine the Puiseux series coefficients, $\{\alpha_k\}_{k=1}^{\infty}$ and $\{\beta_k\}_{k=0}^{\infty}$, from just the derivatives of $A(\varepsilon)$ at $\varepsilon = 0$. Using these recursive formulas, we compute the leading Puiseux series coefficients up to the second order and list them in Corollary 4.

The key to all of our results is the study of the characteristic equation for the analytic matrix $A(\varepsilon)$ under the generic condition (2.1). By an application of the implicit function theorem, we are able to derive the functional relation between the eigenvalues and the perturbation parameter. This leads to the implication that the Jordan normal form of the unperturbed matrix $A(0)$ corresponding to the eigenvalue $\lambda_0$ is a single $m \times m$ Jordan block. From this, we are able to use the method of undetermined coefficients along with a careful combinatorial analysis to get explicit recursive formulas for determining the Puiseux series coefficients.

We want to take a moment here to show how the results of this paper can be used to determine the Puiseux series coefficients up to the second order for the case in which the non-derogatory eigenvalue $\lambda_0$ has algebraic multiplicity $m \geq 2$. We start by putting $A(0)$
into the Jordan normal form \cite[§6.5: The Jordan Theorem]{35}

\[ U^{-1} A(0) U = \begin{bmatrix} J_m(\lambda_0) \\ W_0 \end{bmatrix}, \quad (2.2) \]

where (see notations at end of §1) \( J_m(\lambda_0) \) is an \( m \times m \) Jordan block corresponding to the eigenvalue \( \lambda_0 \) and \( W_0 \) is the Jordan normal form for the rest of the spectrum. Next, define the vectors \( u_1, \ldots, u_m \), as the first \( m \) columns of the matrix \( U \),

\[ u_i := U e_i, \quad 1 \leq i \leq m \quad (2.3) \]

(These vectors have the properties that \( u_1 \) is an eigenvector of \( A(0) \) corresponding to the eigenvalue \( \lambda_0 \), they form a Jordan chain with generator \( u_m \), and are a basis for the algebraic eigenspace of \( A(0) \) corresponding to the eigenvalue \( \lambda_0 \)). We then partition the matrix \( U^{-1} A'(0) U \) conformally to the blocks \( J_m(\lambda_0) \) and \( W_0 \) of the matrix \( U^{-1} A(0) U \) as such

\[ U^{-1} A'(0) U = \begin{bmatrix} * & * & * & \cdots & * & * & \cdots & * \\
* & * & * & \cdots & * & * & \cdots & * \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
* & * & * & \cdots & * & \cdots & * & * \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
* & * & * & \cdots & * & \cdots & * & * \end{bmatrix}. \quad (2.4) \]

Now, by Theorem 1 and Theorem 2, it follows that

\[ a_{m,1} = -\frac{\partial}{\partial \lambda_0} \left. \det (\lambda I - A(\varepsilon)) \right|_{(\varepsilon,\lambda) = (0,\lambda_0)} \left( \frac{\partial^m}{\partial \lambda_0^m} \det (\lambda I - A(\varepsilon)) \right|_{(\varepsilon,\lambda) = (0,\lambda_0)} \right)^{-1} \quad (2.5) \]
And hence the generic condition is true if and only if \( a_{m,1} \neq 0 \). This gives us an alternative method to determine whether the generic condition (2.1) is true or not.

Let’s now assume that \( a_{m,1} \neq 0 \) and hence that the generic condition is true. Define \( f(\varepsilon, \lambda) := \det(\lambda I - A(\varepsilon)) \). Then by Theorem 2 and Corollary 4 there is exactly one convergent Puiseux series for the perturbed eigenvalues near \( \lambda_0 \) and one for their corresponding eigenvectors whose branches are given by

\[
\lambda_h(\varepsilon) = \lambda_0 + \alpha_1 \left( \zeta^h \varepsilon^{\frac{1}{m}} \right) + \alpha_2 \left( \zeta^h \varepsilon^{\frac{1}{m}} \right)^2 + \sum_{k=3}^{\infty} \alpha_k \left( \zeta^h \varepsilon^{\frac{1}{m}} \right)^k \tag{2.6}
\]

\[
x_h(\varepsilon) = x_0 + \beta_1 \left( \zeta^h \varepsilon^{\frac{1}{m}} \right) + \beta_2 \left( \zeta^h \varepsilon^{\frac{1}{m}} \right)^2 + \sum_{k=3}^{\infty} \beta_k \left( \zeta^h \varepsilon^{\frac{1}{m}} \right)^k \tag{2.7}
\]

for \( h = 0, \ldots, m - 1 \) and any fixed branch of \( \varepsilon^{\frac{1}{m}} \), where \( \zeta = e^{\frac{2\pi i}{m}} \). Furthermore, the series coefficients up to second order may be given by

\[
\alpha_1 = a_{m,1}^{-1/m} = \left( -\frac{\partial f}{\partial \varepsilon}(0, \lambda_0) \right)^{1/m} \neq 0, \tag{2.8}
\]

\[
\alpha_2 = \frac{a_{m-1,1} + a_{m,2}}{m\alpha_1^{m-2}} = \frac{-\left( \alpha_1^{m+1} \frac{1}{(m+1)!} \frac{\partial^{m+1} f}{\partial \lambda^{m+1}}(0, \lambda_0) + \alpha_1 \frac{\partial^2 f}{\partial \lambda^2}(0, \lambda_0) \right)}{m\alpha_1^{m-1} \left( \frac{1}{m!} \frac{\partial f}{\partial \lambda^1}(0, \lambda_0) \right)}, \tag{2.9}
\]

\[
\beta_0 = u_1, \quad \beta_1 = \alpha_1 u_2, \quad \beta_2 = \begin{cases} -\Lambda A'(0)u_1 + \alpha_2 u_2, & \text{if } m = 2 \\ \alpha_2 u_2 + \alpha_1^2 u_3, & \text{if } m > 2 \end{cases} \tag{2.10}
\]

for any choice of the \( m \)th root of \( a_{m,1} \) and where \( \Lambda \) is given in (2.15).

The explicit recursive formulas for computing higher order terms, \( \alpha_k, \beta_k \), are given by (2.24) and (2.25) in Theorem 2. The steps which should be used to determine these higher order terms are discussed in Remark 1 and an example showing how to calculating \( \alpha_3, \beta_3 \) using these steps, when \( m \geq 3 \), is provided.
Example

The following example may help to give a better idea of these results. Consider

\[
A(\varepsilon) := \begin{bmatrix}
-\frac{1}{2} & 1 & \frac{1}{2} \\
\frac{1}{2} & 0 & -\frac{1}{2} \\
-1 & 1 & 1
\end{bmatrix} + \varepsilon \begin{bmatrix}
2 & 0 & -1 \\
2 & 0 & -1 \\
1 & 0 & 0
\end{bmatrix}.
\]  
\tag{2.11}

Here \(\lambda_0 = 0\) is a non-derogatory eigenvalue of \(A(0)\) of algebraic multiplicity \(m = 2\). We put \(A(0)\) into the Jordan normal form

\[
U^{-1}A(0)U = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1/2
\end{bmatrix}, \quad U = \begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}, \quad U^{-1} = \begin{bmatrix}
1 & -1 & 0 \\
-1 & 1 & 1 \\
1 & 0 & -1
\end{bmatrix},
\]

so that \(W_0 = 1/2\). We next define the vectors \(u_1, u_2\), as the first two columns of the matrix \(U\),

\[
u_1 := \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}, \quad u_2 := \begin{bmatrix}
1 \\
1
\end{bmatrix}.
\]

Next we partition the matrix \(U^{-1}A'(0)U\) conformally to the blocks \(J_m(\lambda_0)\) and \(W_0\) of the matrix \(U^{-1}A(0)U\) as such

\[
U^{-1}A'(0)U = \begin{bmatrix}
1 & -1 & 0 \\
-1 & 1 & 1 \\
1 & 0 & -1
\end{bmatrix} \begin{bmatrix}
2 & 0 & -1 \\
2 & 0 & -1 \\
1 & 0 & 0
\end{bmatrix} \begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix} = \begin{bmatrix}
0 & * & * \\
1 & 1 & * \\
* & * & *
\end{bmatrix}.
\]
Here \( a_{2,1} = 1, a_{1,1} = 0, \) and \( a_{2,2} = 1. \) Then

\[
1 = a_{2,1} = -\frac{\frac{\partial}{\partial \varepsilon} \det (\lambda I - A(\varepsilon))}{\frac{1}{2!} \det (\lambda I - A(\varepsilon))} \bigg|_{(\varepsilon, \lambda) = (0, \lambda_0)},
\]

implying that the generic condition (2.1) is true. Define \( f(\varepsilon, \lambda) := \det (\lambda I - A(\varepsilon)) = \lambda^3 - 2\lambda^2 \varepsilon - \frac{1}{2} \lambda^2 + \lambda \varepsilon^2 - \frac{1}{2} \lambda \varepsilon + \varepsilon^2 + \frac{1}{2} \varepsilon. \) Then there is exactly one convergent Puiseux series for the perturbed eigenvalues near \( \lambda_0 = 0 \) and one for their corresponding eigenvectors whose branches are given by

\[
\lambda_h(\varepsilon) = \lambda_0 + \alpha_1 \left( -1 \right)^h \varepsilon^{\frac{1}{2}} + \alpha_2 \left( -1 \right)^h \varepsilon^{\frac{3}{2}} + \sum_{k=3}^{\infty} \alpha_k \left( -1 \right)^h \varepsilon^{\frac{k}{2}}
\]

\[
x_h(\varepsilon) = \beta_0 + \beta_1 \left( -1 \right)^h \varepsilon^{\frac{1}{2}} + \beta_2 \left( -1 \right)^h \varepsilon^{\frac{3}{2}} + \sum_{k=3}^{\infty} \beta_k \left( -1 \right)^h \varepsilon^{\frac{k}{2}}
\]

for \( h = 0, 1 \) and any fixed branch of \( \varepsilon^{\frac{1}{2}}. \) Furthermore, the series coefficients up to second order may be given by

\[
\alpha_1 = 1 = \sqrt{1} = \sqrt{a_{2,1}} = \sqrt{-\frac{\partial f}{\partial \varepsilon}(0, \lambda_0)} \neq 0,
\]

\[
\alpha_2 = \frac{1}{2} = \frac{a_{1,1} + a_{2,2}}{2} = -\frac{\partial^2 f}{\partial \varepsilon^2}(0, \lambda_0) + \alpha_1 \frac{\partial f}{\partial \lambda}(0, \lambda_0) \bigg|_{(\varepsilon, \lambda) = (0, \lambda_0)},
\]

\[
\beta_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \beta_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \beta_2 = -\Lambda A'(0) u_1 + \alpha_2 u_2
\]

by choosing the positive square root of \( a_{2,1} = 1 \) and where \( \Lambda \) is given in (2.15). Here

\[
\Lambda = U \begin{bmatrix} J_m(0)^* \\ (W_0 - \lambda_0 I_{n-m})^{-1} \end{bmatrix} U^{-1}
\]
\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1/(2)\end{bmatrix}
\begin{bmatrix}
1 & -1 & 0 \\
-1 & 1 & 1 \\
1 & 0 & -1
\end{bmatrix}
= \begin{bmatrix}
3 & -1 & -2 \\
3 & -1 & -2 \\
1 & -1 & 0
\end{bmatrix}
\]

\[
\beta_2 = -\Lambda A'(0)u_1 + \alpha_2 u_2
\]

Now compare this to the actual perturbed eigenvalues of our example (2.11) near \(\lambda_0 = 0\) and their corresponding eigenvectors

\[
\lambda_h(\varepsilon) = \frac{1}{2} \varepsilon + (-1)^h \frac{1}{2} \varepsilon^{\frac{1}{2}} (\varepsilon + 4)^{\frac{1}{2}}
\]

\[
= \left((-1)^h \varepsilon^{\frac{1}{2}}\right) + \frac{1}{2} \left((-1)^h \varepsilon^{\frac{1}{2}}\right)^2 + \sum_{k=3}^{\infty} \alpha_k \left((-1)^h \varepsilon^{\frac{1}{2}}\right)^k
\]

\[
x_h(\varepsilon) = \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix} + \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} \lambda_h(\varepsilon)
\]

\[
= \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix} + \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} \left((-1)^h \varepsilon^{\frac{1}{2}}\right) + \frac{1}{2} \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} \left((-1)^h \varepsilon^{\frac{1}{2}}\right)^2 + \sum_{k=3}^{\infty} \beta_k \left((-1)^h \varepsilon^{\frac{1}{2}}\right)^k
\]

for \(h = 0, 1\) and any fixed branch of \(\varepsilon^{\frac{1}{2}}\). We see that indeed our formulas for the Puiseux series coefficients are correct up to the second order.
Comparison to Known Results

There is a fairly large amount of literature on eigenpair perturbation expansions for analytic perturbations of non-selfadjoint matrices with degenerate eigenvalues (e.g. [1, 2, 4, 9, 23, 26, 28, 33, 34, 37, 41, 43, 46–48, 53, 55–58]). However, most of the literature (e.g. [1, 2, 23, 33, 34, 37, 41, 43, 46–48, 55, 56]) contains results only on the first order expansions of the Puiseux series or considers higher order terms only in the case of simple or semisimple eigenvalues. For those works that do address higher order terms for defective eigenvalues (e.g. [4, 9, 26, 28, 53, 57, 58]), it was found that there did not exist explicit recursive formulas for all the Puiseux coefficients when the matrix perturbations were non-linear. One of the purposes and achievements of this paper are the explicit recursive formulas (2.23)–(2.25) in Theorem 2 which give all the higher order terms in the important case of degenerate eigenvalues which are non-derogatory, that is, the case in which a degenerate eigenvalue of the unperturbed matrix has a single Jordan block for its corresponding Jordan structure. Our theorem generalizes and extends the results of [4, pp. 315–317, (4.96) & (4.97)], [57, pp. 415–418], and [58, pp. 17–20] to non-linear analytic matrix perturbations and makes explicit the recursive formulas for calculating the perturbed eigenpair Puiseux expansions. Furthermore, in Proposition 8 we give an explicit recursive formula for calculating the polynomials \( \{r_l\}_{l \in \mathbb{N}} \). These polynomials must be calculated in order to determine the higher order terms in the eigenpair Puiseux series expansions (see (2.25) in Theorem 2 and Remark 1). These polynomials appear in [4, p. 315, (4.95)], [57, p. 414, (32.24)], and [58, p. 19, (34)] under different notation (compare with Proposition 2.41.ii) but no method is given to calculate them. As such, Proposition 8 is an important contribution in the explicit recursive calculation of the higher order terms in the eigenpair Puiseux series expansions.

Another purpose of this paper is to give, in the case of degenerate non-derogatory eigenvalues, an easily accessible and quickly referenced list of first and second order terms for the Puiseux series expansions of the perturbed eigenpairs. When the generic condition (2.1) is
satisfied, Corollary 4 gives this list. Now for first order terms there are quite a few papers on formulas for determining them, see for example [48] which gives a good survey of first order perturbation theory. But for second order terms, it was difficult to find any results in the literature similar to and as explicit as Corollary 4 for the case of degenerate non-derogatory eigenvalues with arbitrary algebraic multiplicity and non-linear analytic perturbations. Results comparable to ours can be found in [4, p. 316], [57, pp. 415–418], [58, pp. 17-20], and [53, pp. 37–38, 50–54, 125–128], although it should be noted that in [57, p. 417] the formula for the second order term of the perturbed eigenvalues contains a misprint.

Overview

Section 2 deals with the generic condition (2.1). We give conditions that are equivalent to the generic condition in Theorem 1. In §3 we give the main results of this paper in Theorem 2, on the determination of the Puiseux series with the explicit recursive formulas for calculating the series coefficients. As a corollary we give the exact leading order terms, up to the second order, for the Puiseux series coefficients. Section 4 contains the proofs of the results in §2 and §3.

Notation

Let $M_n(\mathbb{C})$ be the set of all $n \times n$ matrices with complex entries and $\mathbb{C}^n$ the set of all $n \times 1$ column vectors with complex entries. For $a \in \mathbb{C}$, $A \in M_n(\mathbb{C})$, and $x = [a_i]_{i=1}^n \in \mathbb{C}^n$ we denote by $a^*$, $A^*$, and $x^*$, the complex conjugate of $a$, the conjugate transpose of $A$, and the $1 \times n$ row vector $x^* := [a_{1,1}^* \ldots a_{n,1}^*]$. For $x, y \in \mathbb{C}^n$ we let $\langle x, y \rangle := x^*y$ be the standard inner product. The matrix $I \in M_n(\mathbb{C})$ is the identity matrix and its $j$th column is $e_j \in \mathbb{C}^n$. The matrix $I_{n-m}$ is the $(n - m) \times (n - m)$ identity matrix. Define an $m \times m$ Jordan block.
with eigenvalue $\lambda$ to be

$$J_m(\lambda) := \begin{bmatrix}
\lambda & 1 \\
. & . \\
. & . \\
. & . \\
& 1 \\
& & \lambda
\end{bmatrix}.$$ 

When the matrix $A(\varepsilon) \in M_n(\mathbb{C})$ is analytic at $\varepsilon = 0$ we define $A'(0) := \frac{dA}{d\varepsilon}(0)$ and $A_k := \frac{1}{k!} \frac{d^k A}{d\varepsilon^k}(0)$. Let $\zeta := e^{i \frac{2\pi}{m}}$.

### 2.2 The Generic Condition and its Consequences

The following theorem, which is proved in §4, gives conditions which are equivalent to the generic one (2.1).

**Theorem 1** Let $A(\varepsilon)$ be a matrix-valued function having a range in $M_n(\mathbb{C})$ such that its matrix elements are analytic functions of $\varepsilon$ in a neighborhood of the origin. Let $\lambda_0$ be an eigenvalue of the unperturbed matrix $A(0)$ and denote by $m$ its algebraic multiplicity. Then the following statements are equivalent:

(i) The characteristic polynomial $\det(\lambda I - A(\varepsilon))$ has a simple zero with respect to $\varepsilon$ at $\lambda = \lambda_0$ and $\varepsilon = 0$, i.e.,

$$\frac{\partial}{\partial \varepsilon} \det(\lambda I - A(\varepsilon)) \bigg|_{(\varepsilon, \lambda)=(0, \lambda_0)} \neq 0.$$ 

(ii) The characteristic equation, $\det(\lambda I - A(\varepsilon)) = 0$, has a unique solution, $\varepsilon(\lambda)$, in a neighborhood of $\lambda = \lambda_0$ with $\varepsilon(\lambda_0) = 0$. This solution is an analytic function with a
zero of order \( m \) at \( \lambda = \lambda_0 \), i.e.,

\[
\frac{d^0 \varepsilon (\lambda)}{d \lambda^0} \bigg|_{\lambda = \lambda_0} = \cdots = \frac{d^{m-1} \varepsilon (\lambda)}{d \lambda^{m-1}} \bigg|_{\lambda = \lambda_0} = 0, \quad \frac{d^m \varepsilon (\lambda)}{d \lambda^m} \bigg|_{\lambda = \lambda_0} \neq 0.
\]

(iii) There exists a convergent Puiseux series whose branches are given by

\[
\lambda_h (\varepsilon) = \lambda_0 + \alpha_1 \zeta^h \varepsilon^{\frac{1}{m}} + \sum_{k=2}^{\infty} \alpha_k \left( \zeta^h \varepsilon^{\frac{1}{m}} \right)^k,
\]

for \( h = 0, \ldots, m-1 \) and any fixed branch of \( \varepsilon^{\frac{1}{m}} \), where \( \zeta = e^{\frac{2\pi i}{m}} \), such that the values of the branches give all the solutions of the characteristic equation, for sufficiently small \( \varepsilon \) and \( \lambda \) sufficiently near \( \lambda_0 \). Furthermore, the first order term is nonzero, i.e.,

\[
\alpha_1 \neq 0.
\]

(iv) The Jordan normal form of \( A(0) \) corresponding to the eigenvalue \( \lambda_0 \) consists of a single \( m \times m \) Jordan block and there exists an eigenvector \( u_0 \) of \( A(0) \) corresponding to the eigenvalue \( \lambda_0 \) and an eigenvector \( v_0 \) of \( A(0)^* \) corresponding to the eigenvalue \( \lambda_0^* \) such that

\[
\langle v_0, A'(0)u_0 \rangle \neq 0.
\]

2.3 Explicit Recursive Formulas for Calculating the Perturbed Spectrum

This section contains the main results of this paper presented below in Theorem 2. To begin we give some preliminaries that are needed to set up the theorem. Suppose that \( A(\varepsilon) \) is a matrix-valued function having a range in \( M_n(\mathbb{C}) \) with matrix elements that are analytic.
functions of \( \varepsilon \) in a neighborhood of the origin and \( \lambda_0 \) is an eigenvalue of the unperturbed matrix \( A(0) \) with algebraic multiplicity \( m \). Assume that the generic condition

\[
\frac{\partial}{\partial \varepsilon} \det (\lambda I - A(\varepsilon)) \bigg|_{(\varepsilon, \lambda) = (0, \lambda_0)} \neq 0,
\]

is true.

Now, by these assumptions, we may appeal to Theorem 1.(iv) and conclude that the Jordan canonical form of \( A(0) \) has only one \( m \times m \) Jordan block associated with \( \lambda_0 \). Hence there exists a invertible matrix \( U \in \mathbb{C}^{n \times n} \) such that

\[
U^{-1} A(0) U = \begin{bmatrix}
J_m(\lambda_0) \\
W_0
\end{bmatrix}, \quad (2.12)
\]

where \( W_0 \) is a \( (n - m) \times (n - m) \) matrix such that \( \lambda_0 \) is not one of its eigenvalues [35, §6.5: The Jordan Theorem].

We define the vectors \( u_1, \ldots, u_m, v_1, \ldots, v_m \in \mathbb{C}^n \) as the first \( m \) columns of the matrix \( U \) and \((U^{-1})^*\), respectively, i.e.,

\[
u_i := U e_i, \quad 1 \leq i \leq m,
\]

\[
v_i := (U^{-1})^* e_i, \quad 1 \leq i \leq m.
\]

And define the matrix \( \Lambda \in M_n(\mathbb{C}) \) by

\[
\Lambda := U \begin{bmatrix}
J_m(0)^* \\
(W_0 - \lambda_0 I_{n-m})^{-1}
\end{bmatrix} U^{-1}, \quad (2.15)
\]

where \((W_0 - \lambda_0 I_{n-m})^{-1}\) exists since \( \lambda_0 \) is not an eigenvalue of \( W_0 \)(for the important properties of the matrix \( \Lambda \) see Section 2.5).
Next, we introduce the polynomials $p_{j,i} = p_{j,i}(\alpha_1, \ldots, \alpha_{j-i+1})$ in $\alpha_1, \ldots, \alpha_{j-i+1}$, for $j \geq i \geq 0$, as the expressions

$$p_{0,0} := 1, \quad p_{j,0} := 0, \text{ for } j > 0,$$

$$p_{j,i}(\alpha_1, \ldots, \alpha_{j-i+1}) := \sum_{s_1 + \cdots + s_i = j} \prod_{\ell=1}^i \alpha_{s_{\ell}}, \text{ for } j \geq i > 0$$

and the polynomials $r_l = r_l(\alpha_1, \ldots, \alpha_l)$ in $\alpha_1, \ldots, \alpha_l$, for $l \geq 1$, as the expressions

$$r_1 := 0, \quad r_l(\alpha_1, \ldots, \alpha_l) := \sum_{s_1 + \cdots + s_m = m+l} \prod_{\ell=1}^m \alpha_{s_{\ell}}, \text{ for } l > 1$$

(see Section 2.5 for more details on these polynomials including recursive formulas for their calculation).

With these preliminaries we can now state the main results of this paper. Proofs of these results are contained in the next section.

**Theorem 2** Let $A(\varepsilon)$ be a matrix-valued function having a range in $M_n(\mathbb{C})$ such that its matrix elements are analytic functions of $\varepsilon$ in a neighborhood of the origin. Let $\lambda_0$ be an eigenvalue of the unperturbed matrix $A(0)$ and denote by $m$ its algebraic multiplicity. Suppose that the generic condition

$$\left. \frac{\partial}{\partial \varepsilon} \det (\lambda I - A(\varepsilon)) \right|_{(\varepsilon, \lambda)=(0, \lambda_0)} \neq 0,$$

(2.18)

is true. Then there is exactly one convergent Puiseux series for the $\lambda_0$-group and one for their corresponding eigenvectors whose branches are given by

$$\lambda_h(\varepsilon) = \lambda_0 + \sum_{k=1}^{\infty} \alpha_k \left( \zeta^{h} \frac{\varepsilon}{m} \right)^k$$

(2.19)
\[ x_h(\varepsilon) = \beta_0 + \sum_{k=1}^{\infty} \beta_k \left( \zeta^h \varepsilon^{\frac{1}{m}} \right)^k \]  

(2.20)

for \( h = 0, \ldots, m - 1 \) and any fixed branch of \( \varepsilon^{\frac{1}{m}} \), where \( \zeta = e^{\frac{2\pi}{m}} i \) with

\[ \alpha^m_1 = \langle v_m, A_1 u_1 \rangle = -\frac{\partial}{\partial \varepsilon} \det(\lambda I - A(\varepsilon)) \big|_{(\varepsilon, \lambda) = (0, 0)} (\frac{\partial^m}{\partial \varepsilon^m} \det(\lambda - A(\varepsilon))) \big|_{(\varepsilon, \lambda) = (0, 0)} \neq 0 \]

(Here \( A_1 \) denotes \( \frac{dA}{d\varepsilon}(0) \) and the vectors \( u_1 \) and \( v_m \) are defined in (2.13) and (2.14)). Furthermore, we can choose

\[ \alpha_1 = \langle v_m, A_1 u_1 \rangle^{1/m}, \]  

(2.21)

for any fixed \( m \)th root of \( \langle v_m, A_1 u_1 \rangle \) and the eigenvectors to satisfy the normalization conditions

\[ \langle v_1, x_h(\varepsilon) \rangle = 1, \quad h = 0, \ldots, m - 1. \]  

(2.22)

Consequently, under these conditions \( \alpha_1, \alpha_2, \ldots \) and \( \beta_0, \beta_1, \ldots \) are uniquely determined and are given by the recursive formulas

\[ \alpha_1 = \langle v_m, A_1 u_1 \rangle^{1/m} = \left( -\frac{\partial}{\partial \varepsilon} \det(\lambda I - A(\varepsilon)) \big|_{(\varepsilon, \lambda) = (0, 0)} \right) \left( \frac{\partial^m}{\partial \varepsilon^m} \det(\lambda - A(\varepsilon))) \big|_{(\varepsilon, \lambda) = (0, 0)} \right)^{1/m} \]  

(2.23)

\[ \alpha_s = \frac{\sum_{i=0}^{\min\{s, m\} - 1} \sum_{j=i}^{s-1} p_{j, i} \left\langle v_{m-i}, \sum_{k=1}^{m+s-1-j} A_k \beta_{m+s-1-j-km} \right\rangle}{m^{m-1}} \]  

(2.24)

\[ \beta_s = \left\{ \begin{array}{ll}
\sum_{i=0}^{s} p_{s,i} u_{i+1}, & \text{if } 0 \leq s \leq m - 1 \\
\sum_{i=0}^{m-1} p_{s,i} u_{i+1} - \sum_{j=0}^{s-m} \sum_{k=0}^{j} \sum_{l=1}^{\left\lfloor \frac{s-m}{j} \right\rfloor} p_{j,k} A_l \beta_{s-j-lm}, & \text{if } s \geq m
\end{array} \right. \]  

(2.25)

where \( u_i \) and \( v_i \) are the vectors defined in (2.13) and (2.14), \( p_{j,i} \) and \( r_l \) are the polynomials defined in (2.16) and (2.17), \( \lfloor \rfloor \) denotes the floor function, \( A_k \) denotes the matrix \( \frac{1}{k!} \frac{d^k A}{d\varepsilon^k}(0) \), and \( \Lambda \) is the matrix defined in (2.15).
Corollary 3 The calculation of the $k$th order terms, $\alpha_k$ and $\beta_k$, requires only the matrices $A_0, \ldots, A_{\left\lfloor \frac{m+k-1}{m} \right\rfloor}$.

Corollary 4 The coefficients of those Puiseux series up to second order are given by

$$\alpha_1 = \left( -\frac{\partial f}{\partial \varepsilon} (0, \lambda_0) \frac{1}{m!} \frac{\partial^m f}{\partial \varepsilon^m} (0, \lambda_0) \right)^{1/m} = \langle v_m, A_1 u_1 \rangle^{1/m},$$

$$\alpha_2 = \begin{cases} \left( \frac{\partial f}{\partial \varepsilon} (0, \lambda_0) \frac{1}{m!} \frac{\partial^m f}{\partial \varepsilon^m} (0, \lambda_0) \right)^{1/m} \left( \frac{\partial f}{\partial \varepsilon} (0, \lambda_0) \right), & \text{if } m = 1 \\ -\left( \frac{\partial f}{\partial \varepsilon} (0, \lambda_0) \frac{1}{m!} \frac{\partial^m f}{\partial \varepsilon^m} (0, \lambda_0) \right)^{1/m} \left( \frac{\partial f}{\partial \varepsilon} (0, \lambda_0) \right), & \text{if } m > 1 \end{cases},$$

$$\beta_0 = u_1,$$

$$\beta_1 = \begin{cases} -\Lambda A_1 u_1, & \text{if } m = 1 \\ \alpha_1 u_2, & \text{if } m > 1 \end{cases},$$

$$\beta_2 = \begin{cases} (-\Lambda A_2 + (\Lambda A_1)^2 - \alpha_1 \Lambda^2 A_1) u_1, & \text{if } m = 1 \\ -\Lambda A_1 u_1 + \alpha_2 u_2, & \text{if } m = 2 \\ \alpha_2 u_2 + \alpha_1^2 u_3, & \text{if } m > 2 \end{cases}.$$
arrow diagram:

\[(2.26) \rightarrow r_k \rightarrow \alpha_{k+1} \rightarrow \{p_{k+1,i}\}_{i=0}^{k+1} \rightarrow \beta_{k+1}. \quad (2.27)\]

After we have followed these steps we not only will have calculated \(\alpha_{k+1}, \beta_{k+1}\) but we will now know

\[A_0, \ldots, A\left[\frac{m+k+1}{m}\right], \{r_j\}^{k}_{j=1}, \{\alpha_j\}^{k+1}_{j=1}, \{\beta_j\}^{k+1}_{j=0}, \{p_{j,i}\}^{k+1}_{j=0} \quad (2.28)\]

as well. But these are the terms in (2.26) for \(k+1\) and so we may repeat the steps indicated above to calculate \(\alpha_{k+2}, \beta_{k+2}\).

It is in this way we see how all the higher order terms can be calculated using the results of this paper.

**Example**

In order to illustrate these steps we give the following example which recursively calculates the third order terms for \(m \geq 3\).

The goal is to determine \(\alpha_3, \beta_3\). To do this we follow the steps indicated in the above remark with \(k = 2\). The first step is to collect the terms in (2.26). Assuming \(A_0, A_1\) are known then by (2.16), (2.17), Corollary 4, and Proposition 7 we have

\[
A_0, A_1, r_1 = 0, \alpha_1 = \langle v_m, A_1 u_1 \rangle^{1/m}, \alpha_2 = \frac{\langle v_m-1, A_1 u_1 \rangle + \langle v_m, A_1 u_2 \rangle}{m\alpha_1}, \beta_0 = u_1, \beta_1 = \alpha_1 u_2, \beta_2 = \alpha_2 u_2 + \alpha_1^2 u_3, \]

\[
p_{0,0} = 1, p_{1,0} = 0, p_{1,1} = \alpha_1, p_{2,0} = 0, p_{2,1} = \alpha_2, p_{2,2} = \alpha_1^2. \]

The next step is to determine \(r_2\) using the recursive formula for the \(r_l\)’s given in (2.45). We
find that
\[
    r_2 = \frac{1}{2\alpha_1} \sum_{j=1}^{1} [(3 - j)m - (m + j)]\alpha_{3-j}r_j + \frac{m}{2}\alpha_1^{m-2} \sum_{j=1}^{1} [(3 - j)m - (m + j)]\alpha_{3-j}\alpha_{j+1}
\]
\[
= \frac{m(m-1)}{2}\alpha_1^{m-2}\alpha_2^2.
\]

Now, since \(r_2\) is determined, we can use the recursive formula in (2.24) for the \(\alpha_i\)'s to calculate \(\alpha_3\). In doing so we find that
\[
\alpha_3 = -r_2 + \sum_{i=0}^{\min\{3,m\}-1} \sum_{j=1}^{2} p_{j,i} \left< \sum_{k=1}^{\left\lfloor \frac{m+2-j}{m} \right\rfloor} A_k\beta_{m+2-j-km} \right>
\]
\[
= -r_2 + \frac{p_{2,1}\left< v_{m-1}, A_1\beta_0 \right> + p_{0,0}\left< v_m, A_1\beta_2 \right>}{m\alpha_1^{m-1}}
\]
\[
+ \frac{p_{2,2}\left< v_{m-2}, A_1\beta_0 \right> + p_{1,1}\left< v_{m-1}, A_1\beta_1 \right>}{m\alpha_1^{m-1}}
\]
\[
+ \frac{\alpha_1^2\left< v_{m-2}, A_1u_1 \right> + \alpha_1\left< v_{m-1}, A_1u_2 \right>}{m\alpha_1^{m-1}}
\]
\[
= \left( \frac{3 - m}{2} \right) \alpha_1^{-1}\alpha_2^2 + \frac{\left< v_{m-2}, A_1u_1 \right> + \left< v_{m-1}, A_1u_2 \right> + \left< v_m, A_1u_3 \right>}{m\alpha_1^{m-3}}.
\]

Next, since \(\alpha_3\) is determined, we can use (2.44) to calculate \(\{p_{3,i}\}_{i=0}^{3}\). In this case though it suffices to use Proposition 7 and in doing so we find that
\[
p_{3,0} = 0, p_{3,1} = \alpha_3, p_{3,2} = 2\alpha_1\alpha_2, p_{3,3} = \alpha_1^3.
\]

Finally, we can compute \(\beta_3\) using the recursive formula in (2.25) for the \(\beta_i\)'s. In doing so we
find that

\[ \beta_3 = \begin{cases} 
\sum_{i=0}^{\frac{m-1}{3}} p_{3,i} u_{i+1}, & \text{if } m > 3 \\
\sum_{i=0}^{\frac{m-3}{3}} p_{3,i} u_{i+1} - \sum_{j=0}^{\frac{m}{3}} \sum_{k=0}^{\frac{j-1}{3}} \sum_{l=1}^{\frac{3-j}{3}} p_{j,k} \Lambda A_{i-1} \beta_{3-j-lm}, & \text{if } m = 3 \\
p_{3,1} u_2 + p_{3,2} u_3 + p_{3,3} u_4, & \text{if } m > 3 \\
2 \sum_{i=0}^{\frac{m-2}{3}} p_{3,i} u_{i+1} - 3 A_1 \beta_0, & \text{if } m = 3 \\
\alpha_3 u_2 + 2 \alpha_1 \alpha_3 u_3 + \alpha_4^3 u_4, & \text{if } m > 3 \\
\alpha_3 u_2 + 2 \alpha_1 \alpha_3 u_3 - 3 A_1 u_1, & \text{if } m = 3. 
\end{cases} \]

This completes the calculation of the third order terms, \( \alpha_3, \beta_3 \), when \( m \geq 3 \).

### 2.4 Proofs

This section contains the proofs of the results of this paper. We begin by proving Theorem 1 of §2 on conditions equivalent to the generic condition. We next follow this up with the proof of the main result of this paper Theorem 2. We finish by proving the Corollaries 3 and 4.

#### Proof of Theorem 1

To prove this theorem we will prove the chain of statements (i)\( \Rightarrow \) (ii)\( \Rightarrow \) (iii)\( \Rightarrow \) (iv)\( \Rightarrow \) (i).

We begin by proving (i)\( \Rightarrow \) (ii). Define \( f(\varepsilon, \lambda) := \det(\lambda I - A(\varepsilon)) \) and suppose (i) is true. Then \( f \) is an analytic function of \( (\varepsilon, \lambda) \) near \((0, \lambda_0)\) since the matrix elements of \( A(\varepsilon) \) are analytic functions of \( \varepsilon \) in a neighborhood of the origin and the determinant of a matrix is a polynomial in its matrix elements. Also we have \( f(0, \lambda_0) = 0 \) and \( \frac{\partial f}{\partial \varepsilon}(0, \lambda_0) \neq 0 \).
Hence by the holomorphic implicit function theorem [30, §1.4 Theorem 1.4.11] there exists a unique solution, \( \varepsilon (\lambda) \), in a neighborhood of \( \lambda = \lambda_0 \) with \( \varepsilon (\lambda_0) = 0 \) to the equation \( f (\varepsilon, \lambda) = 0 \), which is analytic at \( \lambda = \lambda_0 \). We now show that \( \varepsilon (\lambda) \) has a zero there of order \( m \) at \( \lambda = \lambda_0 \). First, the properties of \( \varepsilon (\lambda) \) imply there exists \( \varepsilon_q \neq 0 \) and \( q \in \mathbb{N} \) such that \( \varepsilon (\lambda) = \varepsilon_q (\lambda - \lambda_0)^q + O \left( (\lambda - \lambda_0)^{q+1} \right) \), for \( |\lambda - \lambda_0| << 1 \). Next, by hypothesis \( \lambda_0 \) is an eigenvalue of \( A (0) \) of algebraic multiplicity \( m \) hence \( \frac{\partial^i f}{\partial \lambda^i} (0, \lambda_0) = 0 \) for \( 0 \leq i \leq m - 1 \) but \( \frac{\partial^m f}{\partial \lambda^m} (0, \lambda_0) \neq 0 \). Combining this with the fact that \( f (0, \lambda_0) = 0 \) and \( \frac{\partial f}{\partial \varepsilon} (0, \lambda_0) = 0 \) we have

\[
f (\varepsilon, \lambda) = a_{10} \varepsilon + a_{0m} (\lambda - \lambda_0)^m + \sum_{i+j \geq 2, i,j \in \mathbb{N}} a_{ij} \varepsilon^i (\lambda - \lambda_0)^j \tag{2.29}
\]

for \( |\varepsilon| + |\lambda - \lambda_0| << 1 \), where \( a_{10} = \frac{\partial f}{\partial \varepsilon} (0, \lambda_0) \neq 0 \) and \( a_{0m} = \frac{1}{m!} \frac{\partial^m f}{\partial \lambda^m} (0, \lambda_0) \neq 0 \). Then using the expansions of \( f (\varepsilon, \lambda) \) and \( \varepsilon (\lambda) \) together with the identity \( f (\varepsilon (\lambda), \lambda) = 0 \) for \( |\lambda - \lambda_0| << 1 \), we find that \( q = m \) and

\[
\varepsilon_m = \frac{a_{0m}}{a_{10}} = -\frac{1}{m!} \frac{\partial^m \det (\lambda I - A (\varepsilon))}{\partial \lambda^m} \bigg|_{(\lambda, \varepsilon) = (\lambda_0, 0)} \frac{\partial \det (\lambda I - A (\varepsilon))}{\partial \varepsilon} \bigg|_{(\lambda, \varepsilon) = (\lambda_0, 0)}. \tag{2.30}
\]

Therefore we conclude that \( \varepsilon (\lambda) \) has a zero of order \( m \) at \( \lambda = \lambda_0 \), which proves (ii).

Next, we prove (ii)⇒(iii). Suppose (ii) is true. The first part of proving (iii) involves inverting \( \varepsilon (\lambda) \) near \( \varepsilon = 0 \) and \( \lambda = \lambda_0 \). To do this we expand \( \varepsilon (\lambda) \) in a power series about \( \lambda = \lambda_0 \) and find that \( \varepsilon (\lambda) = g(\lambda)^m \) where

\[
g(\lambda) = (\lambda - \lambda_0) \left( \varepsilon_m + \sum_{k=m+1}^{\infty} \varepsilon_k (\lambda - \lambda_0)^{k-m} \right)^{1/m}
\]

and we are taking any fixed branch of the \( m \)th root that is analytic at \( \varepsilon_m \). Notice that, for \( \lambda \) in a small enough neighborhood of \( \lambda_0 \), \( g \) is an analytic function, \( g (\lambda_0) = 0 \), and \( \frac{dg}{d\lambda} (\lambda_0) = \varepsilon_m^{1/m} \neq 0 \). This implies, by the inverse function theorem for analytic functions, that for \( \lambda \) in a small enough neighborhood of \( \lambda_0 \) the analytic function \( g (\lambda) \) has an analytic
inverse $g^{-1}(\varepsilon)$ in a neighborhood of $\varepsilon = 0$ with $g^{-1}(0) = \lambda_0$. Define a multivalued function $\lambda(\varepsilon)$, for sufficiently small $\varepsilon$, by $\lambda(\varepsilon) := g^{-1}\left(\varepsilon^{\frac{1}{m}}\right)$ where by $\varepsilon^{\frac{1}{m}}$ we mean all branches of the $m$th root of $\varepsilon$. We know that $g^{-1}$ is analytic at $\varepsilon = 0$ so that for sufficiently small $\varepsilon$ the multivalued function $\lambda(\varepsilon)$ is a Puiseux series. And since $\frac{dg^{-1}}{d\varepsilon}(0) = \left[\frac{dg}{d\lambda}(\lambda_0)\right]^{-1} \neq 0$ we have an expansion

$$\lambda(\varepsilon) = g^{-1}\left(\varepsilon^{\frac{1}{m}}\right) = \lambda_0 + \alpha_1 \varepsilon^{\frac{1}{m}} + \sum_{k=2}^{\infty} \alpha_k \left(\varepsilon^{\frac{1}{m}}\right)^k.$$ 

Now suppose for fixed $\lambda$ sufficiently near $\lambda_0$ and for sufficiently small $\varepsilon$ that $\det(\lambda I - A(\varepsilon)) = 0$. We want to show this implies $\lambda = \lambda(\varepsilon)$ for one of the branches of the $m$th root. We know by hypothesis we must have $\varepsilon = \varepsilon(\lambda)$. But as we know this implies that $\varepsilon = \varepsilon(\lambda) = g(\lambda)^m$ hence for some branch of the $m$th root, $b_m(\cdot)$, we have $b_m(\varepsilon) = b_m(g(\lambda)^m) = g(\lambda)$. But $\lambda$ is near enough to $\lambda_0$ and $\varepsilon$ is sufficiently small that we may apply the $g^{-1}$ to both sides yielding $\lambda = g^{-1}(g(\lambda)) = g^{-1}(b_m(\varepsilon)) = \lambda(\varepsilon)$, as desired. Furthermore, all the $m$ branches $\lambda_h(\varepsilon)$, $h = 0, \ldots, m-1$ of $\lambda(\varepsilon)$ are given by taking all branches of the $m$th root of $\varepsilon$ so that

$$\lambda_h(\varepsilon) = \lambda_0 + \alpha_1 \zeta^h \varepsilon^{\frac{1}{m}} + \sum_{k=2}^{\infty} \alpha_k \left(\zeta^h \varepsilon^{\frac{1}{m}}\right)^k$$

for any fixed branch of $\varepsilon^{\frac{1}{m}}$, where $\zeta = e^{\frac{2\pi i}{m}}$ and

$$\alpha_1 = \frac{dg^{-1}}{d\varepsilon}(0) = \left[\frac{dg}{d\lambda}(\lambda_0)\right]^{-1} = \varepsilon^{-1/m} \neq 0,$$ (2.31)

which proves (iii).

Next, we prove (iii)$\Rightarrow$(iv). Suppose (iii) is true. Define the function $y(\varepsilon) := \lambda_0(\varepsilon^m)$. Then $y$ is analytic at $\varepsilon = 0$ and $\frac{dy}{d\varepsilon}(0) = \lambda_1 \neq 0$. Also we have for $\varepsilon$ sufficiently small $\det(y(\varepsilon) I - A(\varepsilon^m)) = 0$. Consider the inverse of $y(\varepsilon)$, $y^{-1}(\lambda)$. It has the property that $0 = \det(y(y^{-1}(\lambda)) I - A([y^{-1}(\lambda)]^m)) = \det(\lambda I - A([y^{-1}(\lambda)]^m))$ with $y^{-1}(\lambda_0) = 0$, $\frac{dy^{-1}}{d\lambda}(\lambda_0) = \alpha_1^{-1}$. Define $g(\lambda) := [y^{-1}(\lambda)]^m$. Then $g$ has a zero of order $m$ at $\lambda_0$ and $\det(\lambda I - A(g(\lambda))) = 0$ for $\lambda$ in a neighborhood of $\lambda_0$. 32
Now we consider the analytic matrix $A(g(\lambda)) - \lambda I$ in a neighborhood of $\lambda = \lambda_0$ with the constant eigenvalue 0. Because 0 is an analytic eigenvalue of it then there exists an analytic eigenvector, $x(\lambda)$, of $A(g(\lambda)) - \lambda I$ corresponding to the eigenvalue 0 in a neighborhood of $\lambda_0$ such that $x(\lambda_0) \neq 0$. Hence for $\lambda$ near $\lambda_0$ we have

$$0 = (A(g(\lambda)) - \lambda I) x(\lambda)$$

$$= (A(0) - \lambda_0 I) x(\lambda_0) + \left( (A(0) - \lambda_0 I) \frac{dx}{d\lambda}(\lambda_0) - x(\lambda_0) \right) (\lambda - \lambda_0) + \cdots$$

$$+ \left( (A(0) - \lambda_0 I) \frac{d^{m-1}x}{d\lambda^{m-1}}(\lambda_0) - \frac{d^{m-2}x}{d\lambda^{m-2}}(\lambda_0) \right) (\lambda - \lambda_0)^{m-1}$$

$$+ \left( (A(0) - \lambda_0 I) \frac{d^m x}{d\lambda^m}(\lambda_0) - \frac{d^{m-1}x}{d\lambda^{m-1}}(\lambda_0) + \alpha_1^{-1} A'(0) x(\lambda_0) \right) (\lambda - \lambda_0)^m$$

$$+ O((\lambda - \lambda_0)^{m+1}).$$

This implies that

$$(A(0) - \lambda_0 I) x(\lambda_0) = 0, (A(0) - \lambda_0 I) \frac{d^j x}{d\lambda^j}(\lambda_0) = \frac{d^{j-1} x}{d\lambda^{j-1}}(\lambda_0), \text{ for } j = 1, \ldots, m - 1,$$

$$(A(0) - \lambda_0 I) \frac{d^m x}{d\lambda^m}(\lambda_0) = \frac{d^{m-1} x}{d\lambda^{m-1}}(\lambda_0) - \alpha_1^{-1} A'(0) x(\lambda_0). \quad (2.32)$$

The first $m$ equations imply that $x(\lambda_0)$, $\frac{dx}{d\lambda}(\lambda_0)$, $\ldots$, $\frac{d^{m-1}x}{d\lambda^{m-1}}(\lambda_0)$ is a Jordan chain of length $m$ generated by $\frac{d^{m-1}x}{d\lambda^{m-1}}(\lambda_0)$. Since the algebraic multiplicity of $\lambda_0$ for $A(0)$ is $m$ this implies that the there is a single $m \times m$ Jordan block corresponding to the eigenvalue $\lambda_0$ where we can take $x(\lambda_0)$, $\frac{dx}{d\lambda}(\lambda_0)$, $\ldots$, $\frac{d^{m-1}x}{d\lambda^{m-1}}(\lambda_0)$ as a Jordan basis. It follows from basic properties of Jordan chains that there exists an eigenvector $v$ of $A(0)^*$ corresponding to the eigenvalue $\lambda_0^*$ such that $\langle v, \frac{d^{m-1}x}{d\lambda^{m-1}}(\lambda_0) \rangle = 1$. Hence

$$0 = \left( (A(0) - \lambda_0 I)^* v, \frac{d^m x}{d\lambda^m}(\lambda_0) \right) \overset{(2.32)}{=} 1 - \alpha_1^{-1} \langle v, A'(0) x(\lambda_0) \rangle$$

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implying that $\langle v, \frac{dA}{d\varepsilon} (0) x (\lambda_0) \rangle = \alpha_1^m \neq 0$. Therefore we have shown that the Jordan normal form of $A (0)$ corresponding to the eigenvalue $\lambda_0$ consists of a single $m \times m$ Jordan block and there exists an eigenvector $u$ of $A (0)$ corresponding to the eigenvalue $\lambda_0$ and an eigenvector $v$ of $A (0)^*$ corresponding to the eigenvalue $\lambda_0^*$ such that $\langle v, A'(0) u \rangle \neq 0$. This proves (iv).

Finally, we show (iv)$\Rightarrow$(i). Suppose (iv) is true. We begin by noting that since

$$\det (\lambda_0 I - A (\varepsilon)) = (-1)^n \det ((A (0) - \lambda_0 I) + A'(0) \varepsilon) + o (\varepsilon)$$

it suffices to show that

$$S_{n-1} := \frac{d}{d\varepsilon} \det ((A (0) - \lambda_0 I) + A'(0) \varepsilon) \bigg|_{\varepsilon=0} \neq 0. \quad (2.33)$$

We will use the result from [24, Theorem 2.16] to prove (2.33). Let $A (0) - \lambda_0 I = Y \Sigma X^*$ be a singular-value decomposition of the matrix $A (0) - \lambda_0 I$ where $X$, $Y$ are unitary matrices and $\Sigma = \text{diag} (\sigma_1, \ldots, \sigma_{n-1}, \sigma_n)$ with $\sigma_1 \geq \ldots \geq \sigma_{n-1} \geq \sigma_n \geq 0$ (see [35, §5.7, Theorem 2]). Now since the Jordan normal form of $A (0)$ corresponding to the eigenvalue $\lambda_0$ consists of a single Jordan block this implies that rank of $A (0) - \lambda_0 I$ is $n - 1$. This implies that $\sigma_1 \geq \ldots \geq \sigma_{n-1} > \sigma_n = 0$, $u = X e_n$ is an eigenvalue of $A (0)$ corresponding to the eigenvalue $\lambda_0$, $v = Y e_n$ is an eigenvalue of $A (0)$ corresponding to the eigenvalue $\lambda_0^*$, and there exist nonzero constants $c_1, c_2$ such that $u = c_1 u_0$ and $v = c_2 v_0$.

Now using the result of [24, Theorem 2.16] for (2.33) we find that

$$S_{n-1} = \det (Y X^*) \sum_{1 \leq i_1 < \ldots < i_{n-1} \leq n} \sigma_{i_1} \cdots \sigma_{i_{n-1}} \det \left( (Y^* A'(0) X)_{i_1 \ldots i_{n-1}} \right),$$

where $(Y^* A'(0) X)_{i_1 \ldots i_{n-1}}$ is the matrix obtained from $Y^* A'(0) X$ by removing rows and
columns \( i_1 \ldots i_{n-1} \). But since \( \sigma_n = 0 \) and

\[
(Y^*A'(0)X)_{i_1 \ldots i_{n-1}} = e_n^*Y^*A'(0)Xe_n = (v, A'(0)u) = c_2^*c_1 (v_0, A'(0)u_0) \neq 0
\]

then \( S_{n-1} = \det (Y^*) \prod_{j=1}^{n-1} \sigma_j c_2^*c_1 (v_0, A'(0)u_0) \neq 0 \). This completes the proof. \( \blacksquare \)

**Proof of Theorem 2**

We begin by noting that our hypotheses imply that statements (ii), (iii), and (iv) of Theorem 1 are true. In particular, statement (iii) implies that there is exactly one convergent Puiseux series for the \( \lambda_0 \)-group whose branches are given by

\[
\lambda_h (\varepsilon) = \lambda_0 + \alpha_1 \zeta^h \varepsilon \frac{1}{m} + \sum_{k=2}^{\infty} \alpha_k \left( \zeta^h \varepsilon \frac{1}{m} \right)^k,
\]

for \( h = 0, \ldots, m-1 \) and any fixed branch of \( \varepsilon \frac{1}{m} \), where \( \zeta = e^{\frac{2\pi}{m}i} \) and \( \alpha_1 \neq 0 \). Then by well known results [4, §6.1.7, Theorem 2], [28, §II.1.8] there exists a convergent Puiseux series for the corresponding eigenvectors whose branches are given by

\[
x_h (\varepsilon) = \beta_0 + \sum_{k=1}^{\infty} \beta_k \left( \zeta^h \varepsilon \frac{1}{m} \right)^k,
\]

for \( h = 0, \ldots, m-1 \), where \( \beta_0 \) is an eigenvector of \( A_0 = A(0) \) corresponding to the eigenvalue \( \lambda_0 \). Now if we examine the proof of (ii)⇒(iii) in Theorem 1 we see by equation (2.31) that \( \alpha_1^m = \varepsilon_m^{-1} \), where \( \varepsilon_m \) is given in equation (2.30) in the proof of (i)⇒(iii) for Theorem 1. Thus we can conclude that

\[
\alpha_1^m = -\frac{\frac{\partial}{\partial \varepsilon} \det (\lambda I - A(\varepsilon)) \mid_{(\varepsilon, \lambda) = (0, \lambda_0)}}{\left( \frac{\partial^m}{\partial \varepsilon^m} \det (\lambda I - A(\varepsilon)) \mid_{(\varepsilon, \lambda) = (0, \lambda_0)} \right) m!} \neq 0. \tag{2.34}
\]
Choose any $m$th root of $\langle v_m, A_1 u_1 \rangle$ and denote it by $\langle v_m, A_1 u_1 \rangle^{1/m}$. By (2.34) we can just reindexing the Puiseux series (2.19) and (2.20) and assume that

$$
\alpha_1 = \left( -\frac{\partial}{\partial \varepsilon} \det (\lambda I - A(\varepsilon)) \big|_{(\varepsilon, \lambda)=(0, \lambda_0)} \right)^{1/m}.
$$

Next, we wish to prove that we can choose the perturbed eigenvectors (2.20) to satisfy the normalization conditions (2.22). But this follows by Theorem 1 (iv) and the fact $\beta_0$ is an eigenvector of $A(0)$ corresponding to the eigenvalue $\lambda_0$ since then $\langle v_1, \beta_0 \rangle \neq 0$ and so we may take $\frac{x_h(\varepsilon)}{\langle v_1, x_h(\varepsilon) \rangle}$, for $h = 0, \ldots, m - 1$, to be the perturbed eigenvectors in (2.20) that satisfy the normalization conditions (2.22).

Now we are ready to begin showing that $\{\alpha_s\}_{s=1}^\infty, \{\beta_s\}_{s=0}^\infty$ are given by the recursive formulas (2.23)-(2.25). The first key step is proving the following:

$$(A_0 - \lambda_0 I) \beta_s = -\sum_{k=1}^s \left( A_{\frac{k}{m}} - \alpha_k I \right) \beta_{s-k}, \text{ for } s \geq 1, \quad (2.35)$$

$$\beta_0 = u_1, \beta_s = \Lambda (A_0 - \lambda_0 I) \beta_s, \text{ for } s \geq 1, \quad (2.36)$$

where we define $A_{\frac{k}{m}} := 0$, if $\frac{k}{m} \notin \mathbb{N}$.

The first equality holds since in a neighborhood of the origin

$$0 = (A(\varepsilon) - \lambda_0(\varepsilon) I) x_0(\varepsilon) = \sum_{s=0}^\infty \left( \sum_{k=0}^s \left( A_{\frac{k}{m}} - \alpha_k I \right) \beta_{s-k} \right) \varepsilon^\frac{s}{m}.$$

The second equality will be proven once we show $\beta_0 = u_1$ and $\beta_s \in S := \text{span}\{U e_i | 2 \leq i \leq n\}$, for $s \geq 1$, where $U$ is the matrix from (2.12). This will prove (2.36) because $\Lambda (A_0 - \lambda_0 I)$ acts as the identity on $S$ by Proposition 6.i. But these follow from the facts that
\[ S = \{ x \in \mathbb{C}^n \mid \langle v_1, x \rangle = 0 \} \] and the normalization conditions \((2.22)\) imply that \(\langle v_1, \beta_0 \rangle = 1\) and \(\langle v_1, \beta_s \rangle = 0, \text{ for } s \geq 1.\)

The next key step in this proof is the following lemma:

**Lemma 5** For all \(s \geq 0\) the following identity holds

\[
(A_0 - \lambda_0 I) \beta_s = \begin{cases} 
\sum_{i=0}^{s} p_{s,i} u_i, & \text{for } 0 \leq s \leq m - 1 \\
\sum_{i=0}^{m} p_{s,i} u_i - \sum_{j=0}^{s-m} \sum_{k=0}^{j} \sum_{l=1}^{\lfloor \frac{s-j}{m} \rfloor} p_{j,k} A_k^l \beta_{s-j-lm}, & \text{for } s \geq m
\end{cases}
\]  

(2.37)

where we define \(u_0 := 0\).

**Proof.** The proof is by induction on \(s\). The statement is true for \(s = 0\) since \(p_{0,0} u_0 = 0 = (A_0 - \lambda_0 I) \beta_0\). Now suppose it was true for all \(r\) with \(0 \leq r \leq s\) for some nonnegative integer \(s\). We will show the statement is true for \(s + 1\) as well.

Suppose \(s + 1 \leq m - 1\) then \((A_0 - \lambda_0 I) \beta_r = \sum_{i=0}^{r} p_{r,i} u_i\) for \(0 \leq r \leq s\) and we must show that \((A_0 - \lambda_0 I) \beta_{s+1} = \sum_{i=0}^{s+1} p_{s+1,i} u_i\). Well, for \(1 \leq r \leq s\),

\[
\beta_r \overset{(2.36)}{=} \Lambda (A_0 - \lambda_0 I) \beta_r = \sum_{i=0}^{r} p_{r,i} \Lambda u_i \overset{(2.40)}{=} \sum_{i=1}^{r} p_{r,i} u_{i+1}.
\]  

(2.38)

Hence the statement is true if \(s + 1 \leq m - 1\) since

\[
(A_0 - \lambda_0 I) \beta_{s+1} \overset{(2.35)}{=} -\sum_{k=1}^{s+1} \left( A_k^m - \alpha_k I \right) \beta_{s+1-k} \overset{(2.38)}{=} \sum_{k=1}^{s+1} \sum_{i=0}^{s+1-k} \alpha_k p_{s+1-k,i} u_{i+1}
\]

\[
= \sum_{i=0}^{s+1} \sum_{k=1}^{s+1-i} \alpha_k p_{s+1-k,i} u_{i+1} \overset{(2.43)}{=} \sum_{i=0}^{s+1} p_{s+1,i} u_i.
\]  

(2.46)

Now suppose that \(s + 1 \geq m\). The proof is similar to what we just proved. By the induction
hypothesis \((2.37)\) is true for \(1 \leq r \leq s\) and \(\beta_r^{(2.36)} \Lambda (A_0 - \lambda_0 I) \beta_r \) thus

\[
\beta_r^{(2.40)} = \begin{cases} 
\sum_{i=0}^{r} p_{r,i} u_{i+1}, & \text{for } 0 \leq r \leq m - 1 \\
\sum_{i=0}^{m-1} p_{r,i} u_{i+1} - \sum_{j=0}^{r-m} \sum_{k=0}^{j} \sum_{l=1}^{\left\lfloor \frac{j}{m} \right\rfloor} p_{j,k} A^{k+1} A_l^{r-j-lm}, & \text{for } r \geq m.
\end{cases}
\] 
\((2.39)\)

Hence we have

\[
(A_0 - \lambda_0 I) \beta_{s+1}^{(2.35)} = \sum_{k=1}^{s+1} \left( A_{k/m} - \alpha_k I \right) \beta_{s+1-k}
\]
\((2.39)\)  

\[
= -\sum_{l=1}^{\left\lfloor \frac{s+1}{m} \right\rfloor} A_l \beta_{s+1-lm} + \sum_{k=1}^{s+1-m} \sum_{i=0}^{m-1} \alpha_k p_{s+1-k,i} u_{i+1}
\]

\[
- \sum_{k=1}^{s+1-m} \sum_{j=0}^{s+1-k-m} \sum_{i=0}^{\left\lfloor \frac{s+1-k-j}{m} \right\rfloor} \alpha_k p_{j,i} A^{i+1} A_l \beta_{s+1-k-j-lm}
\]

\[
+ \sum_{k>s+1-m}^{s+1} \sum_{i=0}^{s+1-k} \alpha_k p_{s+1-k,i} u_{i+1}
\]
\((2.46)\)  

\[
= -\sum_{l=1}^{\left\lfloor \frac{s+1}{m} \right\rfloor} A_l \beta_{s+1-lm} + \sum_{i=0}^{m-1} \left( \sum_{k=1}^{s+1-i} \alpha_k p_{s+1-k,i} \right) u_{i+1}
\]

\[
- \sum_{k=1}^{s+1-m} \sum_{j=0}^{s+1-k-m} \sum_{i=0}^{\left\lfloor \frac{s+1-k-j}{m} \right\rfloor} \alpha_k p_{j,i} A^{i+1} A_l \beta_{s+1-k-j-lm}
\]
\((2.43)\)  

\[
- \sum_{k=1}^{s+1-m} \sum_{j=0}^{s+1-k-m} \sum_{i=0}^{\left\lfloor \frac{s+1-k-j}{m} \right\rfloor} \alpha_k p_{j,i} A^{i+1} A_l \beta_{s+1-k-j-lm}.
\]
\(2.43\)

Now let \(a_{k,j,i} := \sum_{l=1}^{\left\lfloor \frac{s+1-k-j}{m} \right\rfloor} \alpha_k p_{j,i} A^{i+1} A_l \beta_{s+1-k-j-lm}\). Then using the sum identity

\[
\sum_{k=1}^{s+1-m} \sum_{j=0}^{s+1-k-m} \sum_{i=0}^{\left\lfloor \frac{s+1-k-j}{m} \right\rfloor} a_{k,j,i} = \sum_{j=0}^{s-m} \sum_{k=1}^{s+1-j-m} a_{k,j,i} = \sum_{j=0}^{s-m} \sum_{i=0}^{s+1-j-m} \sum_{k=1}^{s+1-j-m} a_{k,j,i} = \sum_{k=1}^{s+1-m} \sum_{j=0}^{s+1-j-m} \sum_{i=0}^{s+1-j-m} a_{k,j,i}
\]
\((2.48)\)  

\[
\sum_{k=1}^{s+1-m} \sum_{j=0}^{s+1-j-m} \sum_{i=0}^{q-i+1} a_{k,j,i} = \sum_{k=1}^{s+1-m} \sum_{j=0}^{s+1-m} \sum_{i=0}^{q-i+1} a_{k,j,i}
\]
\((2.49)\)  

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we can concluded that

\[(A_0 - \lambda_0 I) \beta_{s+1} = - \sum_{l=1}^{s+1} A_l \beta_{s+1-lm} + \sum_{i=0}^{m} p_{s+1,i} u_i \]

\[= - \sum_{i=0}^{s-m} \sum_{q=i+1}^{s-m} \sum_{k=1}^{s+1-q} \alpha_k p_{q-k,i} A^i A_l \beta_{s+1-q-lm} \]

\[= - \sum_{i=0}^{s-m} \sum_{q=i+1}^{s-m} \sum_{l=1}^{\frac{s+1-q}{m}} \left( \sum_{k=1}^{s+1-q} \alpha_k p_{q-k,i} A^i A_l \right) \beta_{s+1-q-lm} \]

\[(2.43) = - \sum_{i=0}^{s-m} \sum_{q=i+1}^{s-m} \sum_{l=1}^{\frac{s+1-q}{m}} \left( \sum_{k=1}^{s+1-q} \alpha_k p_{q-k,i} A^i A_l \right) \beta_{s+1-q-lm} \]

\[(2.47) = - \sum_{i=0}^{s-m} \sum_{q=i+1}^{s-m} \sum_{l=1}^{\frac{s+1-q}{m}} \left( \sum_{k=1}^{s+1-q} \alpha_k p_{q-k,i} A^i A_l \right) \beta_{s+1-q-lm} \]

\[= \sum_{i=0}^{m} p_{s+1,i} u_i - \sum_{j=0}^{m} \sum_{k=0}^{j} \sum_{l=1}^{\frac{s+1-j}{m}} \left( \sum_{k=1}^{s+1-j} \alpha_k p_{q-k,i} A^i A_l \right) \beta_{s+1-j-lm} \]

But this is the statement we needed to prove for \(s + 1 \geq m\). Therefore by induction the statement (2.37) is true for all \(s \geq 0\) and the lemma is proved. 

The lemma above is the key to prove the recursive formulas for \(\alpha_s\) and \(\beta_s\) as given by (2.23)-(2.25). First we prove that \(\beta_s\) is given by (2.25). For \(s = 0\) we have already shown \(\beta_0 = u_1 = p_{0,0} u_1\). So suppose \(s \geq 1\). Then by (2.36) and (2.37) we find that

\[\beta_s \begin{cases} 
\sum_{i=0}^{s} p_{s,i} u_{i+1}, \text{ if } 0 \leq s \leq m - 1 \\
\sum_{i=0}^{m-1} p_{s,i} u_{i+1} - \sum_{j=0}^{s-m} \sum_{k=0}^{j} \sum_{l=1}^{\frac{s+1-j}{m}} p_{j,k} A^k A_l \beta_{s-j-lm}, \text{ if } s \geq m. 
\end{cases}\]

This proves that \(\beta_s\) is given by (2.25).
Next we will prove that $\alpha_s$ is given by (2.23) and (2.24). We start with $s = 1$ and prove $\alpha_1$ is given by (2.23). First, $(A_0 - \lambda_0 I)^* v_m = 0$ and $\langle v_m, u_i \rangle = \delta_{m,i}$ hence

$$0 = \langle v_m, (A_0 - \lambda_0 I) \beta_m \rangle \overset{(2.37)}{=} \left\langle v_m, \sum_{i=0}^{m} p_{m,i} u_i - A_1 u_1 \right\rangle \overset{(2.42)}{=} \alpha_1^m - \langle v_m, A_1 u_1 \rangle$$

so that $\alpha_1^m = \langle v_m, A_1 u_1 \rangle$. This and identity (2.34) imply that formula (2.23) is true.

Finally, suppose that $s \geq 2$. Then $(A_0 - \lambda_0 I)^* v_m = 0$ and $\langle v_m, u_i \rangle = \delta_{m,i}$ implies

$$0 = \langle v_m, (A_0 - \lambda_0 I) \beta_{m+s-1} \rangle$$

$$\overset{(2.37)}{=} \left\langle v_m, \sum_{i=0}^{m} p_{m+s-1,i} u_i - \sum_{j=0}^{s-1} \sum_{k=0}^{j} \sum_{l=1}^{m+j-i} p_{j,k} \Lambda_{k,l} A_{l} \beta_{m+s-1-j-lm} \right\rangle$$

$$\overset{(2.48)}{=} p_{m+s-1,m} - \sum_{k=0}^{s-1} \sum_{j=0}^{s-1} p_{j,k} \left( \Lambda^* \right)^{k} v_m, \sum_{l=1}^{m+j-i} A_{l} \beta_{m+s-1-j-lm}$$

$$\overset{(2.41)}{=} r_{s-1} + m \alpha_1^{m-1} \alpha_s + \sum_{i=0}^{s-1} \sum_{j=i}^{s-1} p_{j,i} \left( \Lambda^* \right)^{i} v_m, \sum_{k=1}^{m+j-i} A_{k} \beta_{m+s-1-j-km}$$

Therefore with this equality, the fact $\alpha_1 \neq 0$, and Proposition 6.iii, we can solve for $\alpha_s$ and we will find that it is given by (2.24). This completes the proof.

**Proof of Corollaries 3 and 4**

Both corollaries follow almost trivially now. To prove Corollary 3, we just examine the recursive formulas (2.23)-(2.25) in Theorem 2 to see that $\alpha_k, \beta_k$ requires only $A_0, \ldots, A_{\left\lfloor \frac{m+k-1}{m} \right\rfloor}$. To prove Corollary 4, we use Proposition 7 to show that

$$p_{0,0} = 1, p_{1,0} = p_{2,0} = 0, p_{1,1} = \alpha_1, p_{2,1} = \alpha_2, p_{2,2} = \alpha_1^2$$
and then from this and (2.23)-(2.25) we get the desired result for \( \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2 \) in terms of \( A_0, A_1, A_2 \). The last part to prove is the formula for \( \alpha_2 \) in terms of \( f(\varepsilon, \lambda) \) and its partial derivatives. But the formula follows from the series representation of \( f(\varepsilon, \lambda) \) in (2.29) and \( \lambda_0(\varepsilon) \) in (2.19) since, for \( \varepsilon \) in a neighborhood of the origin,

\[
0 = f(\varepsilon, \lambda_0(\varepsilon)) \\
= (a_{10} + a_{0m}p_{m,m}) \varepsilon \\
+ \begin{cases} 
(a_{01}p_{2,1} + a_{02}p_{2,2} + a_{11}p_{1,1} + a_{20}p_{00}) \varepsilon^2, & \text{for } m = 1 \\
(a_{0m}p_{m+1,m} + a_{0m+1}p_{m+1,m+1} + a_{11}p_{1,1}) \varepsilon^{\frac{m+1}{m}}, & \text{for } m > 1 
\end{cases} \\
+ O\left(\frac{\varepsilon^{m+2}}{m}\right)
\]

which together with Proposition 7 implies the formula for \( \alpha_2 \). ■

2.5 Auxiliary Results

Properties of the Matrix \( \Lambda \)

The fundamental properties of the matrix \( \Lambda \) defined in (2.15) which are needed in this paper are given in the following proposition:

**Proposition 6**

(i) We have \( \Lambda (A_0 - \lambda_0 I) U e_1 = 0, \Lambda (A_0 - \lambda_0 I) U e_i = U e_i, \) for \( 2 \leq i \leq n \).

(ii) For \( 1 \leq i \leq m - 1 \) we have

\[
\Lambda u_m = 0, \Lambda u_i = u_{i+1} \tag{2.40}
\]

(iii) \( \Lambda^* v_1 = 0, \) and \( \Lambda^* v_i = v_{i-1}, \) for \( 2 \leq i \leq m \).
Proof. i. Using the fact \( J_m (0)^* J_m (0) = \text{diag}[0, I_{m-1}] \), (2.12), and (2.15) we find by block multiplication that \( U^{-1} \Lambda (A_0 - \lambda_0 I) U = \text{diag}[0, I_{n-1}] \). This implies the result.

ii. \& iii. The results follow from the definition of \( u_i, v_i \) in (2.13), (2.14) and the fact

\[
(U^{-1} \Lambda U)^* = U^* \Lambda^* (U^{-1})^* = \begin{bmatrix} J_m (0) \\ \text{diag}[(W_0 - \lambda_0 I_{n-m})^{-1}]^* \end{bmatrix}.
\]

Properties of the Polynomials in the Recursive Formulas

This appendix contains two propositions. The first proposition gives fundamental identities that help to characterize the polynomials \( \{p_{j,i}\}_{j=i}^{\infty} \) and \( \{r_l\}_{l \in \mathbb{N}} \) in (2.16) and (2.17). The second proposition gives explicit recursive formulas to calculate these polynomials.

We may assume \( \sum_{j=1}^{\infty} \alpha_j z^j \) is a convergent Taylor series and \( \alpha_1 \neq 0 \).

Proposition 7 The polynomials \( \{p_{j,i}\}_{j=i}^{\infty} \) and \( \{r_l\}_{l \in \mathbb{N}} \) have the following properties:

(i) \( \sum_{j=i}^{\infty} p_{j,i} z^j = \left( \sum_{j=1}^{\infty} \alpha_j z^j \right)^i, \) for \( j \geq i \geq 0. \)

(ii) For \( l \geq 1 \) we have

\[
r_l = p_{m+l,m} - m \alpha_1^{m-1} \alpha_{l+1}.
\]

(iii) \( p_{j,1} = \alpha_j, \) for \( j \geq 1. \)

(iv) For \( j \geq 0 \) we have

\[
p_{j,j} = \alpha_1^j.
\]
(v) \( p_{j+1,j} = j\alpha_1^{j-1}\alpha_2 \), for \( j > 0 \).

(vi) For \( j \geq i > 0 \) we have

\[
\sum_{q=1}^{j-i+1} \alpha_q p_{j-q,i-1} = p_{j,i}.
\] (2.43)

Proof. i. For \( i \geq 0 \),

\[
\left( \sum_{j=1}^{\infty} \alpha_j z^j \right)^i = \sum_{s_1=1}^{\infty} \cdots \sum_{s_i=1}^{\infty} \left( \prod_{q=1}^{i} \alpha_{s_q} \right) z^{s_1+\cdots+s_i} = \sum_{j=i}^{\infty} p_{j,i} z^j.
\]

ii. Let \( l \geq 1 \). Then by definition of \( p_{m+l,m} \) we have

\[
p_{m+l,m} = \sum_{s_1+\cdots+s_m=m+l} \prod_{q=1}^{m} \alpha_{s_q} + \sum_{1 \leq s_e \leq l+1} \prod_{q=1}^{m} \alpha_{s_q} \text{ where } s_e = l+1
\]

\[
\sum_{1 \leq s_e \leq l+1} \prod_{q=1}^{m} \alpha_{s_q} \text{ and } s_e = l+1\]

\[
= m\alpha_1^{m-1}\alpha_{l+1} + r_l.
\]

iii. For \( j \geq 1 \) we have

\[
p_{j,1} = \sum_{1 \leq s_e \leq j} \prod_{q=1}^{1} \alpha_{s_q} = \alpha_j.
\]

iv. For \( j \geq 0 \), \( p_{0,0} = 1 \) and

\[
p_{j,j} = \sum_{1 \leq s_e \leq 1} \prod_{q=1}^{j} \alpha_{s_q} = \prod_{q=1}^{j} \alpha_1 = \alpha_1^j.
\]

v. For \( j > 0 \),

\[
p_{j+1,j} = \sum_{1 \leq s_e \leq j+1} \prod_{q=1}^{j} \alpha_{s_q} = \sum_{q=1}^{j} \alpha_1^{j-1}\alpha_2 = j\alpha_1^{j-1}\alpha_2.
\]

vi. It follows by

\[
\sum_{j=i}^{\infty} p_{j,i} z^j = \sum_{j=i}^{\infty} \sum_{j=1}^{\infty} p_{j,i-1} z^j = \sum_{j=i}^{\infty} \left( \sum_{q=1}^{j-i+1} p_{j-q,i-1} z^j \right) = \left( \sum_{q=1}^{\infty} p_{j-q,i-1} p_{q,1} \right) z^j.
\]

This next proposition gives explicit recursive formulas to calculate the polynomials \( \{p_{j,i}\}_{j=i}^{\infty} \) and \( \{r_l\}_{l \in \mathbb{N}} \).

**Proposition 8** For each \( i \geq 0 \), the sequence of polynomials, \( \{p_{j,i}\}_{j=i}^{\infty} \), is given by the recur-
sive formula

\[ p_{i,i} = \alpha_i^i, \quad p_{j,i} = \frac{1}{(j-i)\alpha_1} \sum_{k=i}^{j-1} [(j + 1 - k)i - k]\alpha_{j+1-k}p_{k,i}, \text{ for } j > i. \quad (2.44) \]

Furthermore, the polynomials \( \{r_l\}_{l \in \mathbb{N}} \) are given by the recursive formula:

\[ r_1 = 0, \quad r_l = \frac{1}{l\alpha_1} \sum_{j=1}^{l-1} [(l + 1 - j)m - (m + j)]\alpha_{l+1-j}r_j + \frac{m}{l} \alpha_1^{m-2} \sum_{j=1}^{l-1} [(l + 1 - j)m - (m + j)]\alpha_{l+1-j}\alpha_{j+1}, \text{ for } l > 1. \quad (2.45) \]

**Proof.** We begin by showing (2.44) is true. For \( i = 0 \), (2.44) follows from the definition of the \( p_{j,0} \). If \( i > 0 \) then by (2.42) and [20, (1.1) & (3.2)] it follows that (2.44) is true. Let's now prove (2.45). From (2.41) and (2.44), we have

\[ r_l = p_{m+l,m} - m\alpha_1^{m-1}\alpha_{l+1} \]

\[ \quad = \frac{1}{l\alpha_1} \sum_{k=m}^{m+l-1} [(m + l + 1 - k)m - k]\alpha_{m+l+1-k}p_{k,m} - m\alpha_1^{m-1}\alpha_{l+1} \]

\[ \quad \overset{(2.42)}{=} \frac{1}{l\alpha_1} \sum_{k=m+1}^{m+l-1} [(m + l + 1 - k)m - k]\alpha_{m+l+1-k}p_{k,m} \]

\[ \quad = \frac{1}{l\alpha_1} \sum_{j=1}^{l-1} [(l + 1 - j)m - (m + j)]\alpha_{l+1-j}p_{m+j,m} \]

\[ \quad \overset{(2.41)}{=} \frac{1}{l\alpha_1} \sum_{j=1}^{l-1} [(l + 1 - j)m - (m + j)]\alpha_{l+1-j} (r_j + m\alpha_1^{m-1}\alpha_{j+1}) \]

\[ \quad = \frac{1}{l\alpha_1} \sum_{j=1}^{l-1} [(l + 1 - j)m - (m + j)]\alpha_{l+1-j}r_j + \frac{m}{l} \alpha_1^{m-2} \sum_{j=1}^{l-1} [(l + 1 - j)m - (m + j)]\alpha_{l+1-j}\alpha_{j+1}, \]

for \( l > 1 \). This completes the proof. \( \blacksquare \)
Sum Identities

These double sum identities are used in the proof of Theorem 3.1

\[
\begin{align*}
\sum_{x=c}^{d} \sum_{y=0}^{d-x} a_{x,y} &= \sum_{y=0}^{d-c} \sum_{x=c}^{d-y} a_{x,y}, & (2.46) \\
\sum_{x=0}^{d-1} \sum_{y=x+1}^{d} a_{x,y} &= \sum_{y=1}^{d-1} \sum_{x=0}^{y-1} a_{x,y}, & (2.47) \\
\sum_{x=0}^{d} \sum_{y=0}^{d-x} a_{x,y} &= \sum_{y=0}^{d} \sum_{x=0}^{d-y} a_{x,y}, & (2.48) \\
\sum_{y=c}^{d-1} \sum_{x=1}^{d-y} a_{x,y} &= \sum_{q=c+1}^{d} \sum_{x=1}^{d-q} a_{x,q-x}. & (2.49)
\end{align*}
\]
Chapter 3

Spectral Perturbation Theory for Holomorphic Matrix Functions

In this section we consider holomorphic matrix-valued functions of one and two variables. A holomorphic matrix-valued function of one variable will be referred to as a holomorphic matrix function. Our goal is to give some basic facts on the spectral theory and spectral perturbation theory for holomorphic matrix functions that will be needed in this thesis. The exposition we give here is drawn from many sources, in particular, [44], [29, Appendix A], and [19] for the spectral theory and [4, Appendix A], [23], and [34] for the spectral perturbation theory.

3.1 On Holomorphic Matrix-Valued Functions of One Variable

We introduce the following notation and convention for this section:
1. \( \mathcal{O}(\lambda_0) := \bigcup_{r>0} \mathcal{O}(B(\lambda_0, r), \mathbb{C}) \) – the set of scalar functions holomorphic at \( \lambda_0 \in \mathbb{C} \).

2. \( \mathcal{O}_{m \times n}(\lambda_0) := \bigcup_{r>0} \mathcal{O}(B(\lambda_0, r), M_{m,n}(\mathbb{C})) \) – the set of \( m \times n \) matrix-valued functions holomorphic at \( \lambda_0 \in \mathbb{C} \).

3. As convention we identify \( \mathcal{O}_{1 \times 1}(\lambda_0) \) with \( \mathcal{O}(\lambda_0) \).

4. If \( f_1, f_2 \in \mathcal{O}_{m \times n}(\lambda_0) \) then we use the notation \( f_1 = f_2 \) to mean there exists an \( r > 0 \) such that \( f_1, f_2 \in \mathcal{O}(B(\lambda_0, r), M_{m,n}(\mathbb{C})) \) and \( f_1(\lambda) = f_2(\lambda) \) for \( |\lambda - \lambda_0| < r \).

5. As is known, if \( f \in \mathcal{O}_{m \times n}(\lambda_0) \) then \( f(\lambda) \) is analytic at \( \lambda = \lambda_0 \), i.e., \( f(\lambda) \) is infinitely differentiable at \( \lambda = \lambda_0 \) and, denoting its \( j \)th derivative at \( \lambda = \lambda_0 \) by \( f^{(j)}(\lambda_0) \), there exists an \( r > 0 \) such that

\[
    f(\lambda) = \sum_{j=0}^{\infty} \frac{1}{j!} f^{(j)}(\lambda_0)(\lambda - \lambda_0)^j, \quad |\lambda - \lambda_0| < r,
\]

where the power series on the right converges absolutely in the \( M_{m,n}(\mathbb{C}) \) norm to \( f(\lambda) \) for \( |\lambda - \lambda_0| < r \).

We begin with the definition of the spectrum of a holomorphic matrix function and the important notion of spectral equivalence for holomorphic matrix functions.

**Definition 2** If \( F \in \mathcal{O}(U, M_n(\mathbb{C})) \), where \( U \) is an open connected set in \( \mathbb{C} \), then the **spectrum** of \( F \), denoted by \( \sigma(F) \), is the subset of \( U \) such that the matrix \( F(\lambda) \) is not invertible, i.e.,

\[
    \sigma(F) := \{ \lambda \in U \mid \det F(\lambda) = 0 \}. \tag{3.1}
\]

A point \( \lambda_0 \in U \) is called an **eigenvalue** of \( F \) provided \( \lambda_0 \in \sigma(F) \).

**Definition 3** Two holomorphic matrix functions \( F, G \in \mathcal{O}_{n \times n}(\lambda_0) \) are called **equivalent**
at $\lambda_0$ provided there exists $N, M \in \mathcal{O}_{n \times n}(\lambda_0)$ and an $r > 0$ such that $N(\lambda_0), M(\lambda_0)$ are invertible matrices and

$$F(\lambda) = N(\lambda)G(\lambda)M(\lambda),$$  \hfill (3.2)

for every $\lambda \in B(\lambda_0, r)$.

For the local spectral theory of holomorphic matrix functions the follows theorem plays a central role:

**Theorem 9 (Local Smith Form)** Let $F \in \mathcal{O}_{n \times n}(\lambda_0)$ with $\lambda_0 \in \sigma(F)$. Then $F$ is equivalent at $\lambda_0$ to a unique diagonal matrix $G \in \mathcal{O}_{n \times n}(\lambda_0)$ with

$$G(\lambda) = \text{diag}\{0, \ldots, 0, (\lambda - \lambda_0)^{m_\mu+1}, \ldots, (\lambda - \lambda_0)^{m_g}, 1, \ldots, 1\},$$  \hfill (3.3)

where $0 \leq \mu \leq g \leq n$ and the number of zeros down the diagonal is $\mu$, when $\mu = g$ there are no $(\lambda - \lambda_0)$ terms appearing in the diagonal, when $\mu < g$ we have $m_{\mu+1} \geq \ldots \geq m_g \geq 1$ with $\{m_{\mu+j}\}_{j=1}^{g} \subseteq \mathbb{N}$, and when $g = n$ there are no ones appearing in the diagonal.

**Proof.** This existence is proven in [29, pp. 414–415, Theorem A.6.1] and uniqueness follows from [19, p. 106, Theorem 4.3.1]. \hfill \blacksquare

**Definition 4** The matrix $G$ in (3.3) is called the *local Smith form* of $F$ corresponding to the eigenvalue $\lambda_0$.

Now its obvious from the theorem that

$$g = \dim(\ker(F(\lambda_0))),$$  \hfill (3.4)

$$\mu = \dim(\ker(F(\lambda))), \quad 0 < |\lambda - \lambda_0| \ll 1,$$  \hfill (3.5)
and, since \( \det(F) \in O(\lambda_0) \),

\[
\mu = 0 \iff \det(F) \neq 0.
\]  

(3.6)

But an interpretation of the numbers \( \{m_{\mu+j}\}_{j=1}^g \) is not as straightforward. Thus we now give an important example showing how to interpret this theorem in a special case to the structure of the Jordan normal form of a matrix corresponding to an eigenvalue. Moreover, the following example will help to motivate the discussion in the next section.

**Example 1.** Let \( A \in M_n(\mathbb{C}) \) and define \( F(\lambda) := A - \lambda I_n \) for \( \lambda \in \mathbb{C} \). Let \( \lambda_0 \) be an eigenvalue of \( A \). Then the hypotheses of Theorem 9 are satisfied and so \( F \) is equivalent at \( \lambda_0 \) to a diagonal matrix

\[
G(\lambda) = \text{diag}\{(\lambda - \lambda_0)^{m_1}, \ldots, (\lambda - \lambda_0)^{m_g}, 1, \ldots, 1\}.
\]  

(3.7)

where \( m_1 \geq \cdots \geq m_g \geq 1 \) and \( \{m_j\}_{j=1}^g \subseteq \mathbb{N} \). The numbers \( g, m_1, \ldots, m_g, \) and \( m := m_1 + \cdots + m_g \) are called the geometric multiplicity, partial multiplicities, and algebraic multiplicity, respectively, of the eigenvalue \( \lambda_0 \) of the matrix \( A \). It follows from [18, §2.2] and [18, p. 657, §A.3, Theorem A.3.4] that the Jordan normal form of the matrix \( A \) corresponding to the eigenvalue \( \lambda_0 \) consists of \( g \) Jordan blocks of order \( m_1, \ldots, m_g \), respectively, with eigenvalue \( \lambda_0 \), that is, there exists matrices \( S \in M_n(\mathbb{C}) \) and \( W_0 \in M_{n-m}(\mathbb{C}) \) such that \( S \) is an invertible matrix, \( \lambda_0 \) is not an eigenvalue of the matrix \( W_0 \), and \( S^{-1}AS \) is the block diagonal matrix

\[
S^{-1}AS = \text{diag}\{J_{m_1}(\lambda_0), \ldots, J_{m_g}(\lambda_0), W_0\},
\]  

(3.8)
where $J_{m_j}(\lambda_0)$ is the $m_j \times m_j$ Jordan block given by

$$J_{m_j}(\lambda_0) = \begin{bmatrix} \lambda_0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_0 \end{bmatrix}, \text{ for } j = 1, \ldots, g, \quad (3.9)$$

where all unmarked elements are zeros.

In the next section we use Theorem 9 and this example to analogously define for an eigenvalue of a holomorphic matrix function, its geometric multiplicity, partial multiplicities, and algebraic multiplicity. We also introduce the notion of a Jordan chain for a holomorphic matrix function and again use this example to compare it with the standard definition of a Jordan chain of a matrix (see [35, §6.3] for the standard definitions in the matrix case).

### 3.1.1 Local Spectral Theory of Holomorphic Matrix Functions

For holomorphic matrix functions, a local spectral theory can be given based on the local Smith form. We now give a brief exposition of this theory.

**Definition 5** Let $F \in O_{n \times n}(\lambda_0)$. Then $\ker(F(\lambda_0))$ is called the **eigenspace** of $F$ at $\lambda_0$. If $\varphi \in \ker(F(\lambda_0))$ and $\varphi \neq 0$ then $\varphi$ is called an **eigenvector** of $F$ corresponding to the eigenvalue $\lambda_0$.

**Definition 6** If $F \in O_{n \times n}(\lambda_0)$ and $\lambda_0$ is an eigenvalue of $F$ then we say $\lambda_0$ is an eigenvalue of $F$ of **finite (infinite) algebraic multiplicity** if $\det(F) \neq 0$ ($\det(F) = 0$).

**Definition 7** Let $F$ be any holomorphic matrix function such that $F \in O_{n \times n}(\lambda_0)$ and $\lambda_0$ an
eigenvalue of \( F \) of finite algebraic multiplicity. By Theorem 9 and (3.6) there exists unique numbers \( g, m_1, \ldots, m_g \in \mathbb{N} \) satisfying \( m_1 \geq \cdots \geq m_g \) such that the local Smith form of \( F \) corresponding to the eigenvalue \( \lambda_0 \) is

\[
G(\lambda) = \text{diag}\{(\lambda - \lambda_0)^{m_1}, \ldots, (\lambda - \lambda_0)^{m_g}, 1, \ldots, 1\}.
\] (3.10)

The positive integers \( g, m_1, \ldots, m_g, \) and \( m := m_1 + \cdots + m_g \) are called the geometric multiplicity, partial multiplicities, and algebraic multiplicity, respectively, of the eigenvalue \( \lambda_0 \) of \( F \). We say \( \lambda_0 \) is a semisimple eigenvalue of \( F \) if \( m_1 = \cdots = m_g = 1 \).

The following proposition gives a simple but important characterization of the geometric and algebraic multiplicity.

**Proposition 10** Suppose \( F \in \mathcal{O}_{n \times n}(\lambda_0) \) and \( \lambda_0 \) is an eigenvalue of \( F \) of finite algebraic multiplicity. Let \( g \) and \( m \) be the geometric multiplicity and algebraic multiplicity of the eigenvalue \( \lambda_0 \) of \( F \). Then \( g \) is the dimension of \( \ker(F(\lambda_0)) \) and \( m \) is the order of the zero of the function \( \det(F(\lambda)) \) at \( \lambda = \lambda_0 \).

**Proof.** Let \( m_1 \geq \cdots \geq m_g \) be the partial multiplicities of the eigenvalue \( \lambda_0 \) of \( F \) so that \( m = m_1 + \cdots + m_g \). By Theorem 9 we know that \( F \) is equivalent at \( \lambda_0 \) to its local Smith form. Thus there exists \( N, M \in \mathcal{O}_{n \times n}(\lambda_0) \) and an \( r > 0 \) such that \( N(\lambda_0), M(\lambda_0) \) are invertible matrices and

\[
F(\lambda) = N(\lambda) \text{diag}\{(\lambda - \lambda_0)^{m_1}, \ldots, (\lambda - \lambda_0)^{m_g}, 1, \ldots, 1\}M(\lambda),
\]

for every \( \lambda \in B(\lambda_0, r) \). It follows from this representation that the first \( g \) columns of the matrix \( M(\lambda_0)^{-1} \) are a basis for \( \ker(F(\lambda_0)) \) so that \( g = \dim(\ker(F(\lambda_0))) \). Moreover, since
\[ \det(N), \det(M) \in \mathcal{O}(\lambda_0) \text{ with } \det(N(\lambda_0)), \det(M(\lambda_0)) \neq 0, \text{ we have} \]

\[
\det(F(\lambda)) = \det(N(\lambda)) \text{ diag}\{(\lambda - \lambda_0)^{m_1}, \ldots, (\lambda - \lambda_0)^{m_g}, 1, \ldots, 1\} M(\lambda)
\]

\[
= (\lambda - \lambda_0)^m \det(N(\lambda)) \det(M(\lambda))
\]

\[
= (\lambda - \lambda_0)^m \det(N(\lambda_0)) \det(M(\lambda_0)) + O((\lambda - \lambda_0)^{m+1}), \text{ as } \lambda \to \lambda_0.
\]

Hence \( m \) is the order of the zero of the function \( \det(F(\lambda)) \) at \( \lambda = \lambda_0 \). This completes the proof. \( \blacksquare \)

**Definition 8** Let \( F \) be any holomorphic matrix function such that \( F \in \mathcal{O}_{n \times n}(\lambda_0) \) and \( \lambda_0 \) an eigenvalue of \( F \). A sequence \( \{\varphi_j\}_{j=0}^{l-1} \subseteq \mathbb{C}^n \), where \( l \in \mathbb{N} \), is called a **Jordan chain** of \( F \) of **length** \( l \) corresponding to the eigenvalue \( \lambda_0 \) (associated with the eigenvector \( \varphi_0 \)) if

\[
\varphi_0 \neq 0 \text{ and } \sum_{h=0}^{j} \frac{1}{h!} F^{(h)}(\lambda_0) \varphi_{j-h} = 0, \quad j = 0, \ldots, l - 1.
\]

(3.11)

The maximal length of a Jordan chain of \( F \) corresponding to the eigenvalue \( \lambda_0 \) and associated with \( \varphi_0 \) is called the **multiplicity of the eigenvector** \( \varphi_0 \) and is denoted by \( m(\varphi_0) \), i.e.,

\[
m(\varphi_0) := \sup\{l \in \mathbb{N} : \{\varphi_j\}_{j=0}^{l-1} \text{ satisfies (3.11)}\}.
\]

(3.12)

**Example 2.** Let \( A \in M_n(\mathbb{C}) \) and define \( F(\lambda) := A - \lambda I_n \) for \( \lambda \in \mathbb{C} \). Then by Definition 5, \( \lambda_0 \) is an eigenvalue of \( A \) if and only if \( \lambda_0 \) is an eigenvalue of \( F \). We will now show that a sequence of vectors \( \{\varphi_j\}_{j=0}^{l-1} \subseteq \mathbb{C}^n \) is a Jordan chain of \( A \) of length \( l \) corresponding to the eigenvalue \( \lambda_0 \) (see [35, p. 230, §6.3] for definition) if and only if \( \{\varphi_j\}_{j=0}^{l-1} \) is a Jordan chain of \( F \) of length \( l \) corresponding to the eigenvalue \( \lambda_0 \).
Suppose \( \lambda_0 \) is an eigenvalue of \( A \) and \( \{ \varphi_j \}_{j=0}^{l-1} \) is a Jordan chain of \( A \) of length \( l \) corresponding to the eigenvalue \( \lambda_0 \), i.e.,

\[
\varphi_0 \neq 0, \quad (A - \lambda_0 I_n)\varphi_j = \varphi_{j-1}, \quad j = 0, \ldots, l - 1
\]

where \( \varphi_{-1} := 0 \). Then \( \varphi_0 \neq 0 \) and

\[
\sum_{h=0}^{j} \frac{1}{h!} F^{(h)}(\lambda_0)\varphi_{j-h} = F^{(0)}(\lambda_0)\varphi_j + F^{(1)}(\lambda_0)\varphi_{j-1} = (A - \lambda_0 I_n)\varphi_j - \varphi_{j-1} = 0
\]

for \( j = 0, \ldots, l - 1 \). This implies by Definition 8 that the sequence \( \{ \varphi_j \}_{j=0}^{l-1} \) is a Jordan chain of \( F \) of length \( l \) corresponding to the eigenvalue \( \lambda_0 \).

Conversely, if \( \{ \varphi_j \}_{j=0}^{l-1} \) is a Jordan chain of \( F \) of length \( l \) corresponding to the eigenvalue \( \lambda_0 \), then by Definition 8 this sequence satisfies

\[
\varphi_0 \neq 0 \quad \text{and} \quad 0 = \sum_{h=0}^{j} \frac{1}{h!} F^{(h)}(\lambda_0)\varphi_{j-h} = (A - \lambda_0 I_n)\varphi_j - \varphi_{j-1}, \quad j = 0, \ldots, l - 1,
\]

where \( \varphi_{-1} := 0 \). This implies \( \{ \varphi_j \}_{j=0}^{l-1} \subseteq \mathbb{C}^n \) is a Jordan chain of \( A \) of length \( l \) corresponding to the eigenvalue \( \lambda_0 \).

**Definition 9** Let \( F \in \mathcal{O}_{n \times n}(\lambda_0) \). A vector-valued function \( \varphi \in \mathcal{O}_{n \times 1}(\lambda_0) \) is called a generating function of order \( l \) for \( F \) at the eigenvalue \( \lambda_0 \), where \( l \in \mathbb{N} \), if \( \varphi(\lambda_0) \neq 0 \) and the function \( F \varphi \in \mathcal{O}_{n \times 1}(\lambda_0) \) has a zero of order \( \geq l \) at \( \lambda_0 \), i.e.,

\[
\varphi(\lambda_0) \neq 0 \quad \text{and} \quad F(\lambda)\varphi(\lambda) = O((\lambda - \lambda_0)^l), \quad \text{as} \ \lambda \to \lambda_0 \quad (3.13)
\]

**Proposition 11** Let \( F \in \mathcal{O}_{n \times n}(\lambda_0) \). Then \( \varphi \in \mathcal{O}_{n \times 1}(\lambda_0) \) is a generating function of order \( l \)
for $F$ at the eigenvalue $\lambda_0$ if and only if $\lambda_0$ is an eigenvalue $F$ and the sequence

$$\varphi_j := \frac{1}{j!} \varphi^{(j)}(\lambda_0), \quad j = 0, \ldots, l - 1$$

(3.14)

is a Jordan chain of $F$ of length $l$ corresponding to the eigenvalue $\lambda_0$.

**Proof.** This statement and its proof can be found in [44, p. 57, §II.11, Lemma 11.3]. The result follows from the fact if $F \in \mathcal{O}_{n \times n}(\lambda_0)$ and $\varphi \in \mathcal{O}_{n \times 1}(\lambda_0)$ then $F\varphi \in \mathcal{O}_{n \times 1}(\lambda_0)$ and hence

$$F(\lambda)\varphi(\lambda) = \sum_{j=0}^{\infty} \frac{1}{j!} F^{(j)}(\lambda_0) (\lambda - \lambda_0)^j \sum_{j=0}^{\infty} \frac{1}{j!} \varphi^{(j)}(\lambda_0) (\lambda - \lambda_0)^j$$

$$= \sum_{j=0}^{\infty} \sum_{h=0}^{j} \frac{1}{h!} F^{(h)}(\lambda_0) \frac{1}{(j-h)!} \varphi^{(j-h)}(\lambda_0)$$

$$= \sum_{j=0}^{l-1} \sum_{h=0}^{j} \frac{1}{h!} F^{(h)}(\lambda_0) \frac{1}{(j-h)!} \varphi^{(j-h)}(\lambda_0) + O((\lambda - \lambda_0)^l), \quad \text{as } \lambda \to \lambda_0.$$

**Definition 10** Let $F \in \mathcal{O}_{n \times n}(\lambda_0)$ such that $\lambda_0$ an eigenvalue of $F$ of finite algebraic multiplicity. We say $\{\varphi_{i,j}\}_{i=0}^{m_j-1}$, $j = 1, \ldots, g$ is a **canonical system** of Jordan chains of $F$ corresponding to the eigenvalue $\lambda_0$ if the following three conditions are satisfied:

1. The numbers $g$, $m_1, \ldots, m_g$, and $m := m_1 + \cdots + m_g$ are the geometric multiplicity, partial multiplicities, and algebraic multiplicity, respectively, of the eigenvalue $\lambda_0$ of $F$.

2. The sequence $\{\varphi_{i,j}\}_{i=0}^{m_j-1}$ is a Jordan chain of $F$ of length $m_j$, for $j = 1, \ldots, g$ and $m_1 \geq \cdots \geq m_g \geq 1$.

3. The vectors $\varphi_{0,j}$, $j = 1, \ldots, g$ form a basis for the eigenspace $\ker(F(\lambda_0))$.  

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Example 3. Let $A \in M_n(\mathbb{C})$ and define $F(\lambda) := A - \lambda I_n$ for $\lambda \in \mathbb{C}$. Then it can be shown that $\{\varphi_{i,j}\}_{i=0}^{m_j-1}$, $j = 1, \ldots , g$ is a canonical system of Jordan chains of $F$ corresponding to the eigenvalue $\lambda_0$ if and only if the set of vectors $\bigcup_{j=1}^g \{\varphi_{i,j}\}_{i=0}^{m_j-1}$ form a Jordan basis for the generalized eigenspace of the matrix $A$ corresponding to the eigenvalue $\lambda_0$ (see [35, p. 232, §6.4] for the definition) where $\{\varphi_{i,j}\}_{i=0}^{m_j-1}$ is a Jordan chain of $A$ of length $m_j$ corresponding to the eigenvalue $\lambda_0$ and $m_1 \geq \cdots \geq m_g \geq 1$.

Lemma 12 The relation $\sim$ defined by $F \sim G$ if $F,G \in \mathcal{O}_{n \times n}(\lambda_0)$ are equivalent at $\lambda_0$, is an equivalence relation on $\mathcal{O}_{n \times n}(\lambda_0)$.

Proof. First, its obvious the relation $\sim$ is reflexive since the identity matrix $I_n$ as a constant function belongs in $\mathcal{O}_{n \times n}(\lambda_0)$. Now lets show the relation is symmetric. Suppose $F,G \in \mathcal{O}_{n \times n}(\lambda_0)$ and $F \sim G$. This implies there exists $N,M \in \mathcal{O}_{n \times n}(\lambda_0)$ and an $r > 0$ such that $F(\lambda) = N(\lambda)G(\lambda)M(\lambda)$ for $|\lambda - \lambda_0| < r$ and $N(\lambda_0), M(\lambda_0)$ are invertible matrices. It follows from [45, p. 7, §1.2, Proposition 1.2.5] that there exists a possibly smaller $r > 0$ such that $N^{-1}(\lambda) := N(\lambda)^{-1}$ and $M^{-1}(\lambda) := M(\lambda)^{-1}$ exist for $|\lambda - \lambda_0| < r$ and $N^{-1}, M^{-1} \in \mathcal{O}(B(\lambda_0, r), M_n(\mathbb{C}))$. But then $N^{-1}, M^{-1} \in \mathcal{O}_{n \times n}(\lambda_0)$, $N^{-1}(\lambda), M^{-1}(\lambda_0)$ are invertible, and $G(\lambda) = N^{-1}(\lambda)F(\lambda)M^{-1}(\lambda)$ for $|\lambda - \lambda_0| < r$. This proves $G \sim F$ and so $\sim$ is symmetric. We now prove $\sim$ is transitive. Indeed, let $F \sim G$ and $G \sim H$. Then there exists $N_1, N_2, M_1, M_2 \in \mathcal{O}_{n \times n}(\lambda_0)$ and an $r > 0$ such that $F(\lambda) = N_1(\lambda)G(\lambda)M_1(\lambda) = N_1(\lambda)N_2(\lambda)H(\lambda)M_1(\lambda)M_2(\lambda)$ for $|\lambda - \lambda_0| < r$ but the products $N_1N_2$ and $M_1M_2$ are in $\mathcal{O}_{n \times n}(\lambda_0)$ and are invertible at $\lambda_0$. Thus $F \sim H$ and so $\sim$ is transitive. This proves the relation $\sim$ is an equivalence relation on $\mathcal{O}_{n \times n}(\lambda_0)$.

Proposition 13 Let $F,G$ be any holomorphic matrix function such that $F,G \in \mathcal{O}_{n \times n}(\lambda_0)$ are equivalent at $\lambda_0$. Then

1. $\lambda_0$ is an eigenvalue of $F$ of finite algebraic multiplicity if and only if $\lambda_0$ is an eigenvalue
of $G$ of finite algebraic multiplicity.

2. Let $\lambda_0$ be an eigenvalue of $F$ of finite algebraic multiplicity. Then the local Smith forms of $F$ and $G$ corresponding to the eigenvalue $\lambda_0$ are equal and, in particular, the geometric, partial, and algebraic multiplicities of the eigenvalue $\lambda_0$ of $F$ and $G$ are the same.

3. Let $\lambda_0$ be an eigenvalue of $F$ of finite algebraic multiplicity. Let $N, M \in \mathcal{O}_{n \times n}(\lambda_0)$ be such that $N(\lambda_0), M(\lambda_0)$ are invertible matrices and

$$F(\lambda) = N(\lambda)G(\lambda)M(\lambda)$$

(3.15)

for every $|\lambda - \lambda_0| < r$. If $\{\varphi_j\}_{j=0}^l$ is a Jordan chain of $F$ of length $l$ corresponding to the eigenvalue $\lambda_0$ then $\{\tilde{\varphi}_j\}_{j=0}^l$ is a Jordan chain of $G$ of length $l$ corresponding to the eigenvalue $\lambda_0$ where

$$\tilde{\varphi}_j := \frac{1}{j!} \sum_{h=0}^{j} \frac{1}{h!} M^{(h)}(\lambda_0) \varphi_{j-h}, \quad j = 0, \ldots, l - 1.$$  

(3.16)

**Proof.** Let $F, G$ be any holomorphic matrix function such that $F, G \in \mathcal{O}_{n \times n}(\lambda_0)$ are equivalent at $\lambda_0$. It follows from Lemma 12 and Theorem 9 that the local Smith forms of $F$ and $G$ are equal. This implies that $\lambda_0$ is an eigenvalue of $F$ finite algebraic multiplicity if and only if $\lambda_0$ is an eigenvalue of $G$ finite algebraic multiplicity. It also implies that if $\lambda_0$ is an eigenvalue of $F$ finite algebraic multiplicity then the geometric, partial, and algebraic multiplicities of the eigenvalue $\lambda_0$ of $F$ and $G$ are the same.

Suppose that $\lambda_0$ is an eigenvalue of $F$ finite algebraic multiplicity. Let $N, M \in \mathcal{O}_{n \times n}(\lambda_0)$ be such that $N(\lambda_0), M(\lambda_0)$ are invertible matrices and

$$F(\lambda) = N(\lambda)G(\lambda)M(\lambda)$$

(3.17)
for every $|\lambda - \lambda_0| < r$. Suppose $\{\varphi_j\}_{j=0}^l$ is a Jordan chain of $F$ of length $l$ corresponding to the eigenvalue $\lambda_0$. Define the vector-valued function $\varphi \in \mathcal{O}_{n\times 1}(\lambda_0)$ by

$$\varphi(\lambda) := \sum_{j=0}^{l-1} \varphi_j(\lambda - \lambda_0)^j, \quad \lambda \in \mathbb{C}.$$ 

It follows from Proposition 11 that $\varphi$ is a generating function of order $l$ for $F$ at the eigenvalue $\lambda_0$. This implies

$$\varphi(\lambda_0) \neq 0, \quad F(\lambda)\varphi(\lambda) = O((\lambda - \lambda_0)^l), \quad \text{as } \lambda \to \lambda_0.$$ 

But since $N \in \mathcal{O}_{n\times n}(\lambda_0)$ and $N(\lambda_0)$ is invertible this implies there exists a possibly smaller $r > 0$ such that $N^{-1}(\lambda) := N(\lambda)^{-1}$ exists for $|\lambda - \lambda_0| < r$ and $N^{-1} \in \mathcal{O}(B(\lambda_0, r), M_n(\mathbb{C}))$. This implies

$$G(\lambda)M(\lambda)\varphi(\lambda) = N^{-1}(\lambda)F(\lambda)\varphi(\lambda) = O((\lambda - \lambda_0)^l), \quad \text{as } \lambda \to \lambda_0.$$ 

But since $M \in \mathcal{O}_{n\times n}(\lambda_0)$ and $M(\lambda_0)$ is invertible then $M\varphi \in \mathcal{O}_{n\times 1}(\lambda_0)$ and

$$M(\lambda_0)\varphi(\lambda_0) \neq 0, \quad G(\lambda)M(\lambda)\varphi(\lambda) = O((\lambda - \lambda_0)^l), \quad \text{as } \lambda \to \lambda_0.$$ 

This implies by Definition 9 that $M\varphi$ is a generating function of order $l$ for $F$ at the eigenvalue $\lambda_0$ and so by Proposition 11, $\{\tilde{\varphi}_j\}_{j=0}^l$ is a Jordan chain of $G$ of length $l$ corresponding to the eigenvalue $\lambda_0$ where

$$\tilde{\varphi}_j := \frac{1}{j!} \frac{d^j}{d\lambda^j}(M(\lambda)\varphi(\lambda))|_{\lambda=\lambda_0} = \frac{1}{j!} \sum_{h=0}^{j} \frac{1}{h!} M^{(h)}(\lambda_0)\varphi_{j-h}, \quad j = 0, \ldots, l - 1.$$ 

This completes the proof.  

**Theorem 14** Let $F \in \mathcal{O}_{n\times n}(\lambda_0)$ such that $\lambda_0$ an eigenvalue of $F$ of finite algebraic mul-
tlicity. Then there exists a canonical system of Jordan chains of $F$ corresponding to the eigenvalue $\lambda_0$.

**Proof.** Let $F \in \mathcal{O}_{n \times n}(\lambda_0)$ such that $\lambda_0$ an eigenvalue of $F$ of finite algebraic multiplicity. Denote by $g, m_1, \ldots, m_g, m$ the geometric, partial, and algebraic multiplicities, respectively, of the eigenvalue $\lambda_0$ of $F$ where $m_1 \geq \cdots \geq m_g \geq 1$ and $m = m_1 + \cdots + m_g$. By Theorem 9 and Lemma 12 there exists $N, M \in \mathcal{O}_{n \times n}(\lambda_0)$ and an $r > 0$ such that $N(\lambda_0), M(\lambda_0)$ are invertible matrices and

$$G(\lambda) = N(\lambda)F(\lambda)M(\lambda),$$

for every $\lambda \in B(\lambda_0, r)$ where

$$G(\lambda) = \text{diag}\{(\lambda - \lambda_0)^{m_1}, \ldots, (\lambda - \lambda_0)^{m_g}, 1, \ldots, 1\}.$$

Let $e_1, \ldots, e_n$ denote the standard orthonormal basis vectors in $\mathbb{C}^n$. Define vector-valued functions $\varphi_j \in \mathcal{O}_{n \times 1}(\lambda_0), j = 1, \ldots, g$ by

$$\varphi_j(\lambda) = e_j, \quad \lambda \in \mathbb{C}, \quad j = 1, \ldots, g. \quad (3.18)$$

Then since

$$\varphi_j(\lambda_0) = e_j \neq 0, \quad G(\lambda)\varphi_i(\lambda) = (\lambda - \lambda_0)^{m_j}e_j = O((\lambda - \lambda_0)^{m_i}), \quad \text{as } \lambda \to \lambda_0, \quad (3.19)$$

each $\varphi_j$ is a generating function of order $m_j$ for $G$ at the eigenvalue $\lambda_0$ and so by Proposition 11 the sequence $\{\varphi_{i,j}\}_{i=0}^{m_j-1}$ defined by

$$\varphi_{i,j} := \frac{1}{i!}\varphi_j^{(i)}(\lambda_0) = \delta_{0,i}e_j, \quad i = 0, \ldots, m_j - 1, \quad (3.20)$$

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where $\delta_{0,i}$ is the Kronecker delta symbol, is a Jordan chain of $G$ of length $m_j$ corresponding to the eigenvalue $\lambda_0$, for $j = 1, \ldots, g$. It follows that $\{\varphi_{i,j}\}_{i=0}^{m_j-1}$ is canonical system of Jordan chains for $G$ corresponding to the eigenvalue $\lambda_0$, that is, we have constructed a canonical system of Jordan chains for the local Smith form of $F$ corresponding to the eigenvalue $\lambda_0$. By Proposition 13 the sequence $\{\tilde{\varphi}_{i,j}\}_{i=0}^{m_j-1}$ defined by

$$\tilde{\varphi}_{i,j} := \frac{1}{i!} \sum_{h=0}^{i} \frac{1}{h!} M^{(h)}(\lambda_0) \varphi_{i-h,j}, \quad i = 0, \ldots, m_j - 1$$

(3.21)

is a Jordan chain of $F$ of length $m_j$ corresponding to the eigenvalue $\lambda_0$, for $j = 1, \ldots, g$. Moreover, since $e_1, \ldots, e_g$ is a basis for $\ker G(\lambda_0)$, $N(\lambda_0)$, $M(\lambda_0)$ are invertible, and $N(\lambda_0)G(\lambda_0) = F(\lambda_0)M(\lambda_0)$ then $\tilde{\varphi}_{0,j} = M(\lambda_0)e_j$, $j = 1, \ldots, g$ is a basis for $\ker F(\lambda_0)$. Thus we have shown the numbers $g$, $m_1, \ldots, m_g$, and $m = m_1 + \cdots + m_g$ are the geometric multiplicity, partial multiplicities, and algebraic multiplicity, respectively, of the eigenvalue $\lambda_0$ of $F$, the sequence $\{\varphi_{i,j}\}_{i=0}^{m_j-1}$ is a Jordan chain of $F$ of length $m_j$, for $j = 1, \ldots, g$ and $m_1 \geq \cdots \geq m_g \geq 1$, and the vectors $\varphi_{0,j}$, $j = 1, \ldots, g$ form a basis for the eigenspace $\ker(F(\lambda_0))$. Therefore, by Definition 10, $\{\varphi_{i,j}\}_{i=0}^{m_j-1}$, $j = 1, \ldots, g$ is a canonical system of Jordan chains of $F$ corresponding to the eigenvalue $\lambda_0$. This completes the proof.

It will be important in this thesis to know when an eigenvalue of a holomorphic matrix function is both of finite algebraic multiplicity and semisimple. The next proposition gives us a simple sufficient condition for this to be true.

**Proposition 15** If $\lambda_0$ is an eigenvalue of $F \in \mathcal{O}_{n \times n}(\lambda_0)$ and the equation

$$F^{(1)}(\lambda_0)\varphi_0 = -F(\lambda_0)\varphi_1$$

(3.22)

has no solution for $\varphi_0 \in \ker(F(\lambda_0))/\{0\}$ and $\varphi_1 \in \mathbb{C}^n$ then $\lambda_0$ is an eigenvalue of $F$ of finite algebraic multiplicity and a semisimple eigenvalue of $F$. 

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Proof. Let the hypotheses of this proposition be true for some $F \in \mathcal{O}_{n\times n}(\lambda_0)$. Then the hypothesis that \eqref{eq:3.22} has no solution for $\varphi_0 \in \ker(F(\lambda_0))/\{0\}$ and $\varphi_1 \in \mathbb{C}^n$ is equivalent by Definition 8 to there are no Jordan chains of $F$ of length $l \geq 2$ corresponding to the eigenvalue $\lambda_0$. Thus if $\lambda_0$ was an eigenvalue of $F$ but not of finite algebraic multiplicity then by Theorem 9 we could find a vector-valued function $\varphi \in \mathcal{O}_{n\times 1}(\lambda_0)$ and an $r > 0$ such that $F(\lambda)\varphi(\lambda) = 0$ for $|\lambda - \lambda_0| < r$. By Proposition 11 we could find a Jordan chain of order $l$ for any $l \in \mathbb{N}$ and in particular, for $l \geq 2$, a contradiction. Hence $\lambda_0$ must be an eigenvalue of $F$ of finite algebraic multiplicity. If $\lambda_0$ was an eigenvalue of $F$ but was not a semisimple eigenvalue of $F$ then by Theorem 14 there would exist a Jordan chain of $F$ of length $\geq 2$, a contradiction. Therefore, $\lambda_0$ is an eigenvalue of $F$ of finite algebraic multiplicity and a semisimple eigenvalue of $F$. This completes the proof.

Another important result needed in this thesis is given by the next proposition.

**Proposition 16** Let $F \in \mathcal{O}_{n\times n}(\lambda_0)$ and let $f \in \mathcal{O}_{n\times n}(z_0)$ be such that $f(z_0) = \lambda_0$. Define $G(z) := F(f(z))$ for all $z$ such that the composition is well-defined. Then $G \in \mathcal{O}_{n\times n}(z_0)$ and $\lambda_0$ is an eigenvalue of $F$ if and only if $z_0$ is an eigenvalue of $G$. Moreover, if $\lambda_0$ is an eigenvalue of $F$ and $f'(z_0) \neq 0$ then

1. $\lambda_0$ is an eigenvalue of $F$ of finite algebraic multiplicity if and only if $z_0$ is an eigenvalue of $G$ of finite algebraic multiplicity.

2. Let $\lambda_0$ be an eigenvalue of $F$ of finite algebraic multiplicity. Then the geometric, partial, and algebraic multiplicities of the eigenvalue $\lambda_0$ of $F$ and of the eigenvalue $z_0$ of $G$ are in one-to-one correspondence.

Proof. Let $F \in \mathcal{O}_{n\times n}(\lambda_0)$ and let $f \in \mathcal{O}_{n\times n}(z_0)$ such that $f(z_0) = \lambda_0$. It follows there exists an $\epsilon > 0$ such that $F \in \mathcal{O}(B(\lambda_0,\epsilon),M_n(\mathbb{C}))$. Thus there exists a $\delta > 0$ such that $f \in \mathcal{O}(B(\lambda_0,\delta),\mathbb{C})$ and if $z \in B(\lambda_0,\delta)$ then $f(z) \in B(\lambda_0,\epsilon)$. It then follows $G(z) = F(f(z))$
is well-defined on $B(\lambda_0, \delta)$ and $G'(z) = f'(z)F'(f(z))$ for every $z \in B(\lambda_0, \delta)$. Thus $G \in O(B(z_0, \delta), M_n(\mathbb{C}))$ and so $G \in O_{n \times n}(z_0)$. It follows from the fact $\det F(\lambda_0) = \det G(z_0)$ that $\lambda_0$ is an eigenvalue of $F$ if and only if $z_0$ is an eigenvalue of $G$.

Suppose $\lambda_0$ is an eigenvalue of $F$. Then by the local Smith form, Theorem 9, there exists $N, M \in O_{n \times n}(\lambda_0)$ and an $r > 0$ with $r \leq \epsilon$ such that $N(\lambda_0), M(\lambda_0)$ are invertible matrices and

$$F(\lambda) = N(\lambda) \operatorname{diag}\{0_\mu, (\lambda - \lambda_0)^{m_{\mu+1}}, \ldots, (\lambda - \lambda_0)^{m_g}, I_{n-g}\}M(\lambda),$$

for every $\lambda \in B(\lambda_0, r)$, where $0 \leq \mu \leq g \leq n$, $0_\mu$ is the $\mu \times \mu$ zero matrix, $I_{n-g}$ is the $(n-g) \times (n-g)$ identity matrix, the number of zeros down the diagonal is $\mu$, when $\mu = g$ there are no $(\lambda - \lambda_0)$ terms appearing in the diagonal, when $\mu < g$ we have $m_{\mu+1} \geq \ldots \geq m_g \geq 1$ with $\{m_{\mu+j}\}_{j=1}^g \subseteq \mathbb{N}$, and when $g = n$ there are no ones appearing in the diagonal.

Again we can find a $\delta_1 \leq \delta$ such that $f(z) \in B(\lambda_0, r)$ for $z \in B(z_0, \delta_1)$. But from what we just proved this implies for $\tilde{N}(z) := N(f(z))$ and $\tilde{M}(z) := M(f(z))$, for all $z$ such that the composition is well-defined, that $\tilde{N}, \tilde{M} \in O_{n \times n}(z_0)$ with $\tilde{N}(z_0) = N(\lambda_0), \tilde{M}(z_0) = M(\lambda_0)$ invertible matrices such that

$$G(z) = F(f(z)) = \tilde{N}(z) \operatorname{diag}\{0_\mu, (f(z) - \lambda_0)^{m_{\mu+1}}, \ldots, (f(z) - \lambda_0)^{m_g}, I_{n-g}\}\tilde{M}(z)$$

for every $z \in B(z_0, \delta_1)$.

Now since $f'(z_0) \neq 0$ then for the function $f_0(z) := \frac{f(z)}{z-z_0}$ for $z \in B(z_0, \delta_1)$ we have $O(B(z_0, \delta_1), \mathbb{C})$ with $f_0(z_0) = f'(z_0) \neq 0$. Thus there exists a $\delta_2 > 0$ with $\delta_2 \leq \delta_1$ such that $f_0^{-1}(z) := \frac{z-z_0}{f(z)}$ with $z \in B(z_0, \delta_2)$ is well-defined and $f_0^{-1} \in O(B(z_0, \delta_2), \mathbb{C})$. 
Define the matrix $\hat{M}(z)$ for $z \in B(z_0, \delta_2)$ by

$$\hat{M}(z) = \text{diag}\{I_\mu, f_0(z)^{m_{\mu + 1}}, \ldots, f_0(z)^{m_g}, I_{n-g}\}\tilde{M}(z).$$

It follows from this that $\hat{M} \in O(B(z_0, \delta_2), M_n(\mathbb{C}))$ with $\hat{M}(z_0)$ an invertible matrix and $\hat{M}(z)$ is invertible for every $z \in B(z_0, \delta_2)$. Moreover, since we have $(f(z) - \lambda_0)^j = (z - z_0)^j f_0(z)^j$, it follows that

$$G(z) = F(f(z))$$

$$= \tilde{N}(z) \text{diag}\{0, (f(z) - \lambda_0)^{m_{\mu + 1}}, \ldots, (f(z) - \lambda_0)^{m_g}, I_{n-g}\}\hat{M}(z)$$

$$= \tilde{N}(z) \text{diag}\{0, (z - z_0)^{m_{\mu + 1}}, \ldots, (z - z_0)^{m_g}, I_{n-g}\}\hat{M}(z)$$

for every $z \in B(z_0, \delta_2)$.

Therefore by the uniqueness portion of Theorem 9 it follows that the local Smith form of $F$ corresponding to the eigenvalue $\lambda_0$ is

$$\text{diag}\{0, (\lambda - \lambda_0)^{m_{\mu + 1}}, \ldots, (\lambda - \lambda_0)^{m_g}, I_{n-g}\}$$

and the local Smith form of $G$ corresponding to the eigenvalue $z_0$ is

$$\text{diag}\{0, (z - z_0)^{m_{\mu + 1}}, \ldots, (z - z_0)^{m_g}, I_{n-g}\}.$$

The proof of the proposition now follows from the uniqueness of those local Smith forms.

The following example arises in our thesis in Floquet theory when studying the Jordan normal form of the monodromy matrix for canonical systems of periodic differential equations with period $d$. 

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Example 4. Let \( A \in M_n(\mathbb{C}) \) and define \( F(\lambda) := A - \lambda I_n \) for \( \lambda \in M_n(\mathbb{C}) \). Define \( G(k) := F(e^{ikd}) = A - e^{ikd}I_n \), for \( k \in \mathbb{C} \). Let \( \lambda_0 \) be an eigenvalue of the matrix \( A \). Then we know from example 1 that if we let \( g, m_1 \geq \cdots \geq m_g, m := m_1 + \cdots + m_g \) denote the geometric, partial, and algebraic multiplicities, respectively, of the eigenvalue \( \lambda_0 \) of \( F \) then the Jordan normal form of the matrix \( A \) corresponding to the eigenvalue \( \lambda_0 \) has exactly \( g \) Jordan blocks of order \( m_1, \ldots, m_g \), respectively, with eigenvalue \( \lambda_0 \). Now let \( k_0 \in \mathbb{C} \) such that \( e^{ikd} = \lambda_0 \). Then, since \( \frac{d}{dk} e^{ikd} = ide^{ikd} \neq 0 \) for any \( k \), it follows from Proposition 16 that \( k_0 \) is an eigenvalue of \( G \) with finite algebraic multiplicity such that its geometric, partial, and algebraic multiplicities are \( g, m_1, \ldots, m_g \) and \( m \), respectively. By the same proposition the converse is also true. We conclude by noting that it can be helpful sometimes to study the spectrum of the matrix \( A \) by studying the spectrum of the holomorphic matrix function \( G(k) := A - e^{ikd}I_n \). This perspective is found useful when considering perturbation problems such as when \( A := A(\omega) \) depends on a perturbation parameter \( \omega \).

3.2 On Holomorphic Matrix-Valued Functions of Two Variables

We introduce the following notation and convention for this section:

(i) \( \mathcal{O}(U, E) := \{ f : U \to E \mid f \text{ is holomorphic on } U \} \) where \( U \) is an open connected set in \( \mathbb{C}^2 \), \( (E, \| \cdot \|) \) is a Banach space, and a function \( f : U \to E \) is said to be holomorphic on \( U \) provided the partial derivatives of \( f \) exist at each point in \( U \), i.e., the limits

\[
\left. \frac{\partial f}{\partial \lambda} \right|_{(\lambda, z) = (\lambda_0, z_0)} := f_\lambda(\lambda_0, z_0) := \lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^{-1} (f(\lambda, z_0) - f(\lambda_0, z_0))
\]

\[
\left. \frac{\partial f}{\partial z} \right|_{(\lambda, z) = (\lambda_0, z_0)} := f_z(\lambda_0, z_0) := \lim_{z \to z_0} (z - z_0)^{-1} (f(\lambda_0, z) - f(\lambda_0, z_0))
\]

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exist for every \((\lambda_0, z_0) \in U\).

(ii) \(\mathcal{O}(\lambda_0, z_0) := \bigcup_{r > 0} \mathcal{O}(B((\lambda_0, z_0), r), \mathbb{C})\) – the set of scalar functions holomorphic at \((\lambda_0, z_0) \in \mathbb{C}^2\).

(iii) \(\mathcal{O}_{m \times n}(\lambda_0, z_0) := \bigcup_{r > 0} \mathcal{O}(B((\lambda_0, z_0), r), M_{m,n}(\mathbb{C}))\) – the set of \(m \times n\) matrix-valued functions holomorphic at \((\lambda_0, z_0) \in \mathbb{C}^2\).

(iv) As convention we identify \(\mathcal{O}_{1 \times 1}(\lambda_0, z_0)\) with \(\mathcal{O}(\lambda_0, z_0)\).

(v) If \(f_1, f_2 \in \mathcal{O}_{m \times n}(\lambda_0, z_0)\) then we use the notation \(f_1 = f_2\) to mean there exists an \(r > 0\) such that \(f_1, f_2 \in \mathcal{O}(B((\lambda_0, z_0), r), M_{m,n}(\mathbb{C}))\) and \(f_1(\lambda, z) = f_2(\lambda, z)\) for

\[||((\lambda, z) - (\lambda_0, z_0)||c < r.\]

The following is an important characterization of holomorphic functions as analytic functions:

**Lemma 17** If \(f \in \mathcal{O}_{m \times n}(\lambda_0, z_0)\) then \(f(\lambda, z)\) is analytic at \((\lambda, z) = (\lambda_0, z_0)\), that is, the partial and mixed partial derivatives \(f^{(j_1, j_2)}(\lambda_0, z_0) := \frac{\partial^{j_1+j_2} f}{\partial^{j_1} \lambda \partial^{j_2} z}|_{(\lambda, z) = (\lambda_0, z_0)}\) exist for every \(j_1, j_2 \in \mathbb{N} \cup \{0\}\) and there exists an \(r > 0\) such that

\[f(\lambda, z) = \sum_{j_1, j_2 = 0}^{\infty} \frac{1}{j_1! j_2!} f^{(j_1, j_2)}(\lambda_0, z_0)(\lambda - \lambda_0)^{j_1}(z - z_0)^{j_2}, \quad |\lambda - \lambda_0| < r, |z - z_0| < r\]

where the power series on the right converges absolutely in the \(M_{m,n}(\mathbb{C})\) norm to \(f(\lambda, z)\) for \((\lambda, z) \in B(\lambda_0, r) \times B(z_0, r)\).

**Proof.** The space \(M_{m,n}(\mathbb{C})\) is finite dimensional and since all norms on a finite dimensional space are equivalent, we need only show that the statement is true for the norm given by

\[||[a_{i,j}]_{i=1,j=1}^{m,n}||_{\infty} := \max_{i,j} |a_{i,j}|, \quad [a_{i,j}]_{i=1,j=1}^{m,n} \in M_{m,n}(\mathbb{C}).\]

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Thus it follows from this that \( O_{m\times n}((\lambda_0, z_0)) = M_{m,n}(O((\lambda_0, z_0))) \) and the statement will be true if we can show that its true for every \( f \in O((\lambda_0, z_0)) \). But this statement for functions in \( O((\lambda_0, z_0)) \) is a well known fact from the theory of functions of several complex variables [17, p. 39, Theorem 3.2] and follows from a deep theorem of Hartogs (see [17, p. 21, Theorem 1.6]).

In the spectral perturbation theory of holomorphic matrix functions, the following theorem plays a central role:

**Theorem 18 (Weierstrass Preparation Theorem)** If \( f \in O((\lambda_0, z_0)), f(\lambda_0, z_0) = 0, \) and \( f \neq 0, \) then there exists an \( r > 0 \) such that

\[
f(\lambda, z) = (z - z_0)^\mu f_0(\lambda, z)f_1(\lambda, z), \quad f_1(\lambda, z) \neq 0, \quad \text{and} \quad \mu, m \in \mathbb{N} \cup \{0\}, \quad a_j \in O(B(z_0, r), \mathbb{C}) \quad \text{with} \quad a_j(z_0) = 0 \quad \text{for} \quad j = 1, \ldots, m - 1, \quad \text{and} \quad f_1 \in O(B(\lambda_0, r) \times B(z_0, r), \mathbb{C}).
\]

**Proof.** The proof of this statement follows from [17, p. 55, §I.4, Theorem 4.1].

It follows from this that to study the zero set of the function \( f \in O((\lambda_0, z_0)) \) near \((\lambda_0, z_0)\) we need to study the zero set of the monic polynomial \( f_0 \) in \( \lambda \) (if \( m \geq 1 \)) given by (3.24), whose coefficients \( a_1, \ldots, a_m \) are in \( O(z_0) \) and satisfy \( a_1(z_0) = \cdots = a_m(z_0) = 0 \).

Hence we define the set

\[
O_0((\lambda_0, z_0)) := \{ f \in O((\lambda_0, z_0)) \mid f \text{ has the form in (3.24)} \}.
\]
Let \( f \in \mathcal{O}_0((\lambda_0, z_0)) \). Then \( f(\lambda, z_0) = (\lambda - \lambda_0)^m \) for some \( m \in \mathbb{N} \cup \{0\} \). We call \( \deg f := m \) the degree of \( f \).

A function \( f \in \mathcal{O}_0((\lambda_0, z_0)) \) with \( f \neq 1 \) is said to be irreducible if whenever \( f = p_1 p_2 \) with \( p_1, p_2 \in \mathcal{O}_0((\lambda_0, z_0)) \) then \( p_1 = 1 \) or \( p_2 = 1 \).

**Theorem 19** Every \( f \in \mathcal{O}_0((\lambda_0, z_0)) \) with \( f \neq 1 \), factors uniquely into a finite product of irreducibles, i.e., there exists unique irreducibles \( p_1, \ldots, p_l \in \mathcal{O}_0((\lambda_0, z_0)) \) (not necessarily distinct) such that

\[
    f = \prod_{j=1}^l p_j. \tag{3.26}
\]

Furthermore, \( \deg f = \deg p_1 + \cdots + \deg p_l \). Moreover, there exists an \( r > 0 \) such that for each \( j = 1, \ldots, l \) and for each fixed \( z \) satisfying \( 0 < |z - z_0| < r \), the polynomial \( p_j(\lambda, z) \) has only simple roots.

**Proof.** The uniqueness of a factorization into irreducibles follows from [4, p. 397, §A.2.6, Theorem 1] and our choice of notation in A.2.(v). From the definition of the degree of an element from \( \mathcal{O}_0((\lambda_0, z_0)) \) we have

\[
    (\lambda - \lambda_0)^{\deg f} = f(\lambda, z_0) = \prod_{j=1}^l p_j(\lambda, z_0) = \prod_{j=1}^l (\lambda - \lambda_0)^{\deg p_j} = (\lambda - \lambda_0)^{\deg p_1 + \cdots + \deg p_l}
\]

implying \( \deg f = \deg p_1 + \cdots + \deg p_l \). The latter part of this theorem follows from [4, p. 399, §A.3.2, Theorem 2].

**Definition 11** Let \( q \in \mathbb{N} \) and \( r > 0 \). If \( v \in \mathcal{O}(B(0, \sqrt[q]{r}), M_{m,n}(\mathbb{C})) \), where \( \sqrt[q]{r} \) denotes the real \( q \)th root of \( r \), and is given by the convergent power series

\[
    v(\varepsilon) = \sum_{j=0}^{\infty} c_j \varepsilon^j, \quad |\varepsilon| < \sqrt[q]{r} \tag{3.27}
\]
then the multi-valued function
\[
u(z) := v((z - z_0)^{\frac{1}{q}}) = \sum_{j=0}^{\infty} c_j(z - z_0)^{\frac{j}{q}}, \quad |z - z_0| < r, \tag{3.28}
\]
is called a convergent \textbf{Puiseux series} expanded about \(z_0\) with \textbf{limit point} \(c_0\), domain \(B(z_0, r)\), and \textbf{period} \(q\).

Fix any branch of the \(q\)th root function and denote its evaluation at \((z - z_0)\) by \((z - z_0)^{1/q}\) (e.g., using the principal branch of the \(q\)th root \((z - z_0)^{1/q} = |z - z_0|^{1/q}e^{i\frac{1}{q}\arg(z - z_0)}\) where \(-\pi < \arg(z - z_0) \leq \pi\)). Then the single-valued functions \(u_0, \ldots, u_{q-1}\) given by the convergent series
\[
u_h(z) := v(\zeta^h(z - z_0)^{1/q}) = \sum_{j=0}^{\infty} c_j(\zeta^h(z - z_0)^{1/q})^j, \quad |z - z_0| < r, \quad h = 0, \ldots, q - 1, \tag{3.29}
\]
where \(\zeta\) is any primitive \(q\)th root of unity (e.g., \(\zeta = e^{i2\pi/q}\)), are called the \textbf{branches} of the Puiseux series \(u\). The \textbf{values} of the Puiseux series \(u\) at a point \(z \in B(z_0, r)\) are the values of its branches at this point, namely, \(u_0(z), \ldots, u_{q-1}(z)\). The \textbf{graph} of the Puiseux series \(u\) is the union of the graphs of its branches, namely, \(\bigcup_{h=0}^{q-1}\{ (z, u_h(z)) : |z - z_0| < r \}\).

**Theorem 20** Let \(f \in \mathcal{O}_0((\lambda_0, z_0))\) with \(f \neq 1\). Let \(f = \prod_{j=1}^{l} p_j\) be a factorization of \(f\) into unique irreducibles \(p_1, \ldots, p_l \in \mathcal{O}_0((\lambda_0, z_0))\) (not necessarily distinct) with \(\deg f = \deg p_1 + \cdots + \deg p_l\) and \(r > 0\) be such that for each \(j = 1, \ldots, l\) and for each fixed \(z\) satisfying \(0 < |z - z_0| < r\), the polynomial \(p_j(\lambda, z)\) has only simple roots. Then there exists \(l\) convergent Puiseux series \(\lambda_1(z), \ldots, \lambda_l(z)\) expanded about \(z_0\) with limit point \(\lambda_0\), domain \(B(z_0, r)\), and periods \(\deg p_1, \ldots, \deg p_l\), respectively, such that for every \((\lambda, z) \in B(z_0, r) \times \mathbb{C}\),
\[
p_j(\lambda, z) = \prod_{h=0}^{\deg p_j - 1} (\lambda - \lambda_j(z), h), \quad j = 1, \ldots, l \tag{3.30}
\]
and

\[ f(\lambda, z) = \prod_{j=1}^{l} \prod_{h=0}^{\deg p_j - 1} (\lambda - \lambda_{j,h}(z)), \quad (3.31) \]

where \( \lambda_{j,h}(z) \), \( h = 0, \ldots, \deg p_j - 1 \) denotes all the branches of the Puiseux series \( \lambda_j(z) \), for \( j = 1, \ldots, l \).

**Proof.** This statement follows from [4, p. 406, §A.5.4, Theorem 3].

3.3 On the Perturbation Theory for Holomorphic Matrix Functions

In this section, we consider a holomorphic matrix-valued function of two variables \( L \in \mathcal{O}_{n \times n}((\lambda_0, z_0)) \) which satisfies \( \det(L(\lambda_0, z_0)) = 0 \). For such a function, we can consider \( L(\lambda, z) \) as a family, indexed by a perturbation parameter \( z \), of holomorphic matrix functions in the variable \( \lambda \). We will give basic facts regarding the dependency on \( z \) of the eigenvalues \( \lambda(z) \) and corresponding eigenvectors \( \varphi(z) \) of the holomorphic matrix function \( L(\lambda, z) \) when \( (\lambda, z) \) is near \( (\lambda_0, z_0) \). In order to avoid confusion we denote by \( L(\cdot, z) \), the holomorphic matrix function \( L(\lambda, z) \) in the variable \( \lambda \) for a fix \( z \).

3.3.1 Eigenvalue Perturbations

We begin by defining the spectrum for a holomorphic matrix-valued function of two variables in a manner similar to the one variable case.

**Definition 12** If \( L \in \mathcal{O}(U, M_n(\mathbb{C})) \) then the spectrum of \( L \), denoted by \( \sigma(L) \), is the subset
of $U$ such that the matrix $L(\lambda, z)$ is not invertible, i.e.,

$$\sigma(L) := \{(\lambda, z) \in U \mid \det L(\lambda, z) = 0\}. \quad (3.32)$$

Notice that $(\lambda_0, z_0) \in \sigma(L)$ if and only if $\lambda_0 \in \sigma(L(\cdot, z_0))$, that is, $\lambda_0$ is an eigenvalue of the holomorphic matrix function $L(\cdot, z_0) \in \mathcal{O}(\lambda_0)$. Thus the spectral theory developed in the previous sections for holomorphic matrix functions applies.

Conceptually, we can consider the eigenvalues $\lambda(z)$ of $L(\cdot, z)$ as a multi-valued function of $z$ implicitly defined by the equation $\det L(\lambda, z) = 0$. Thus it is useful to introduce the notation $\sigma(L)^{-1}$ for its graph, i.e.,

$$\sigma(L)^{-1} := \{(z, \lambda) \mid (\lambda, z) \in \sigma(L)\}. \quad (3.33)$$

With this in mind, the next theorem can be interpreted as a description of the local analytic properties of this multi-valued function.

**Theorem 21** Suppose $L \in \mathcal{O}(U, M_n(\mathbb{C}))$ such that $(\lambda_0, z_0) \in \sigma(L)$. If $\lambda_0$ an eigenvalue of $L(\cdot, z_0)$ with algebraic multiplicity $m$ then for any $\varepsilon > 0$ there exists an $\delta > 0$ such that $\sigma(L)^{-1} \cap B(z_0, \delta) \times B(\lambda_0, \varepsilon)$ is the union of the graphs of $l$ convergent Puiseux series $\lambda_1(z), \ldots, \lambda_l(z)$ expanded about $z_0$ with limit point $\lambda_0$, domain $B(z_0, \delta)$, and periods $q_1, \ldots, q_l$, respectively, which satisfy $m = q_1 + \cdots + q_l$.

**Proof.** First, $L \in \mathcal{O}(U, M_n(\mathbb{C}))$ implies $f := \det(L) \in \mathcal{O}(U, \mathbb{C})$ since the determinant of a matrix is just a polynomial in its entries. Hence $\sigma(L)$ is the zero set of the holomorphic function $f : U \rightarrow \mathbb{C}$ and since $(\lambda_0, z_0) \in \sigma(L)$ then $f(\lambda_0, z_0) = 0$. Suppose $\lambda_0$ an eigenvalue of $L(\cdot, z_0)$ with algebraic multiplicity $m$. Then by Proposition 10 the order of the zero of $f(\lambda, z_0) = \det(L(\lambda, z_0))$ at $\lambda = \lambda_0$ is $m$ and so $f(\cdot, z_0) \neq 0$. By the Weierstrass
Preparation Theorem, Theorem 18 above, there exists $f_0 \in \mathcal{O}_0((\lambda_0, z_0))$ with degree $m$ and $f_1 \in \mathcal{O}((\lambda_0, z_0))$ with $f_1(\lambda_0, z_0) \neq 0$ such that $f = f_0 f_1$ in $\mathcal{O}((\lambda_0, z_0))$. By Theorem 19, there exists unique irreducibles $p_1, \ldots, p_l \in \mathcal{O}_0((\lambda_0, z_0))$ (not necessarily distinct) such that in $\mathcal{O}((\lambda_0, z_0))$ we have

$$f = f_1 f_0 = f_1 \prod_{j=1}^l p_j.$$ 

Moreover, by the same theorem, $\deg f_0 = \deg p_1 + \cdots + \deg p_l$ and there exists an $r > 0$ such that for each $j = 1, \ldots, l$ and for each fixed $z$ satisfying $0 < |z - z_0| < r$, the polynomial $p_j(\lambda, z)$ has only simple roots. By taking a smaller $r$, we may also assume without loss of generality that $f, f_1, p_1, \ldots, p_l \in \mathcal{O}(B(\lambda_0, r) \times B(z_0, r), \mathbb{C})$ and

$$f(\lambda, z) = f_1(\lambda, z) \prod_{j=1}^l p_j(\lambda, z), \quad f_1(\lambda, z) \neq 0,$$

for every $(\lambda, z) \in B(\lambda_0, r) \times B(z_0, r)$. We now define the numbers $q_j := \deg p_j$, $j = 1, \ldots, l$. Thus $m = \deg f_0 = \deg p_1 + \cdots + \deg p_l = q_1 + \cdots + q_l$. By Theorem 20 there exists $l$ convergent Puiseux series $\lambda_1(z), \ldots, \lambda_l(z)$ expanded about $z_0$ with limit point $\lambda_0$, domain $B(z_0, r)$, and periods $q_1, \ldots, q_l$, respectively, such that

$$p_j(\lambda, z) = \prod_{h=0}^{q_j-1} (\lambda - \lambda_{j,h}(z)), \quad j = 1, \ldots, l$$

where $\lambda_{j,h}(z)$, $h = 0, \ldots, q_j - 1$ denotes all the branches of the Puiseux series $\lambda_j(z)$, for $j = 1, \ldots, l$. This implies

$$\det(L(\lambda, z)) = f_1(\lambda, z) \prod_{j=1}^l \prod_{h=0}^{q_j-1} (\lambda - \lambda_{j,h}(z)), \quad f_1(\lambda, z) \neq 0,$$

for every $(\lambda, z) \in B(\lambda_0, r) \times B(z_0, r)$. 

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Let \( \varepsilon > 0 \) be given. Then since

\[
\lim_{z \to z_0} \lambda_{j,h}(z) = \lambda_{j,h}(z_0) = \lambda_0, \quad h = 0, \ldots, q_j - 1, \quad j = 1, \ldots, l,
\]

there exists \( \delta > 0 \) with \( \delta \leq r \) such that

\[
|\lambda_{j,h}(z) - \lambda_0| < \min\{\varepsilon, r\}, \quad h = 0, \ldots, q_j - 1, \quad j = 1, \ldots, l,
\]

if \( |z - z_0| < \delta \). Then, since \( B(\lambda_0, \varepsilon) \times B(z_0, \delta) \subseteq B(\lambda_0, r) \times B(z_0, r) \), we have

\[
\sigma(L) \cap B(\lambda_0, \varepsilon) \times B(z_0, \delta) = \sigma(L) \cap B(\lambda_0, r) \times B(z_0, r) \cap B(\lambda_0, \varepsilon) \times B(z_0, \delta)
\]

\[
= \bigcup_{j=1}^{l} \bigcup_{h=0}^{q_j-1} \{ (\lambda_{j,h}(z), z) : (\lambda_{j,h}(z), z) \in B(\lambda_0, r) \times B(z_0, r) \} \cap B(\lambda_0, \varepsilon) \times B(z_0, \delta)
\]

\[
= \bigcup_{j=1}^{l} \bigcup_{h=0}^{q_j-1} \{ (\lambda_{j,h}(z), z) \mid z \in B(z_0, \delta) \}
\]

It follows from this that if we just take the convergent Puiseux series \( \lambda_1(z), \ldots, \lambda_l(z) \) to have the domain \( B(z_0, \delta) \) instead of \( B(z_0, r) \) then we have \( l \) convergent Puiseux series expanded about \( z_0 \) with limit point \( \lambda_0 \), domain \( B(z_0, \delta) \), and periods \( q_1, \ldots, q_l \), respectively, which satisfy \( m = \deg q_1 + \cdots + \deg q_l \) and such that the union of their graphs is

\[
\bigcup_{j=1}^{l} \bigcup_{h=0}^{q_j-1} \{ (z, \lambda_{j,h}(z)) \mid z \in B(z_0, \delta) \} = \{ (z, \lambda) \mid (\lambda, z) \in \sigma(L) \cap B(\lambda_0, \varepsilon) \times B(z_0, \delta) \}
\]

\[
= \sigma(L)^{-1} \cap B(z_0, \delta) \times B(\lambda_0, \varepsilon).
\]

This completes the proof. \( \blacksquare \)

Let \( L \in \mathcal{O}(U, M_n(\mathbb{C})) \) and assume that \( (\lambda_0, z_0) \in \sigma(L) \) and \( \lambda_0 \) an eigenvalue of \( L(\cdot, z_0) \) with finite algebraic multiplicity. Denote by \( g, m_1 \geq \cdots \geq m_g \), and \( m = m_1 + \cdots + m_g \) the geometric multiplicity, partial multiplicities, and algebraic multiplicity, respectively of the
eigenvalue $\lambda_0$ of $L(\cdot, z_0)$. Then it follows Theorem 21 that given an $\epsilon > 0$ there exists a $\delta > 0$ such that, for $|z - z_0| < \delta$, the spectrum of $L(\lambda, z)$ in $|\lambda - \lambda_0| < \epsilon$, that is, $\sigma(L(\cdot, z)) \cap B(\lambda_0, \epsilon)$, consists of $m$ eigenvalues and they are the values of $l$ convergent Puiseux series of the form

$$
\lambda_j(z) = \lambda_0 + \sum_{s=1}^{\infty} c_{j,s}(z - z_0)^{\frac{s}{q_j}}, \quad |z - z_0| < \delta, \quad j = 1, \ldots, l
$$

where there periods satisfy $m = q_1 + \ldots + q_l$. Specifically, letting $\lambda_{j,h}(z)$, $h = 0, \ldots, q_j - 1$ denote all the branches of the Puiseux series $\lambda_j(z)$, for $j = 1, \ldots, l$ then

$$
\sigma(L(\cdot, z)) \cap B(\lambda_0, \epsilon) = \bigcup_{j=1}^{l} \{\lambda_{j,h}(z) : h = 0, \ldots, q_j - 1\}
$$

for each $z \in B(z_0, \delta)$.

In general the only connection between the numbers $l$ and $q_1, \ldots, q_l$ with the geometric multiplicity $g$ and partial multiplicities $m_1, \ldots, m_g$ is the equality $m = \sum_{j=1}^{l} q_j = \sum_{j=1}^{g} m_j$. Because of this two important problems arise in the spectral perturbation theory of holomorphic matrix functions:

1. Find general conditions involving just the partial derivatives of $L$ at $(\lambda_0, z_0)$ up to first order, namely, $L(\lambda_0, z_0)$, $L_\lambda(\lambda_0, z_0)$, and $L_z(\lambda_0, z_0)$ such that we may determine the values $l, q_1, \ldots, q_l$ and, in particular, those conditions which imply $l = g$ and $q_j = m_j$, for $j = 1, \ldots, g$.

2. Find formulas for the first order coefficients of those Puiseux series, i.e., formulas for coefficient $c_{j,1}, j = 1, \ldots, l$ in (3.34).

Several papers have addressed these problems (see [23, 34, 37–39]) and significant progress was made towards their resolution.
3.3.2 Eigenvector Perturbations

We will now proceed to give a useful description for the dependency on $z$ of the perturbed eigenvectors $\varphi(z)$ of the holomorphic matrix function $L(\lambda, z)$ corresponding to perturbed eigenvalue $\lambda(z)$ when $(\lambda, z)$ is near a point $(\lambda_0, z_0) \in \sigma(L)$.

**Definition 13** Let $L \in \mathcal{O}(U, M_n(\mathbb{C}))$. We say $\lambda(z)$ is a **eigenvalue Puiseux series** of $L(\cdot, z)$ provided it is a convergent Puiseux series whose values at each point $z$ in its domain are eigenvalues of $L(\cdot, z)$. If

$$
\lambda(z) = \lambda_0 + \sum_{s=1}^{\infty} c_s(z - z_0)^{\frac{s}{q}}, \quad |z - z_0| < \delta
$$

is a eigenvalue Puiseux series of $L(\cdot, z)$ then we say $\varphi(z)$ is an **eigenvector Puiseux series** of $L(\cdot, z)$ corresponding to the eigenvalue Puiseux series $\lambda(z)$ provided it is a convergent Puiseux series with

$$
\varphi(z) = \beta_0 + \sum_{s=1}^{\infty} \beta_s(z - z_0)^{\frac{s}{q}}, \quad |z - z_0| < r
$$

where $0 < r \leq \delta$, $\{\beta_s\}_{s=0}^{\infty} \subseteq \mathbb{C}^n$, $\beta_0 \neq 0$, and such that if we fix any branch of the $q$th root function and denote its evaluation at $(z - z_0)$ by $(z - z_0)^{1/q}$ then their branches

$$
\lambda_h(z) = \lambda_0 + \sum_{s=1}^{\infty} c_s(\zeta^h(z - z_0)^{1/q})^s, \quad |z - z_0| < r, \quad h = 0, \ldots, q,
$$

$$
\varphi_h(z) = \beta_0 + \sum_{s=1}^{\infty} \beta_s(\zeta^h(z - z_0)^{1/q})^s, \quad |z - z_0| < r, \quad h = 0, \ldots, q,
$$

where $\zeta$ is any primitive $q$th root of unity, satisfy

$$
L(\lambda_h(z), z)\varphi_h(z) = 0, \quad \varphi_h(z) \neq 0, \quad h = 0, \ldots, q - 1, \quad |z - z_0| < r.
$$
We call the pair \((\lambda(z), \varphi(z))\) an **eigenpair Puiseux series** of \(L(\cdot, z)\).

The following lemma tells us that to every eigenvalue Puiseux series there exists a corresponding eigenvector Puiseux series.

**Lemma 22** Suppose \(L \in \mathcal{O}(U, M_n(\mathbb{C}))\). If \(\lambda(z)\) is a eigenvalue Puiseux series of \(L(\cdot, z)\) then there exists an eigenvector Puiseux series \(\varphi(z)\) of \(L(\cdot, z)\) corresponding to the eigenvector Puiseux series \(\lambda(z)\).

**Proof.** Assume \(\lambda(z)\) be an eigenvalue Puiseux series of \(L(\cdot, z)\). This implies for some \(q \in \mathbb{N}, \delta > 0, v \in \mathcal{O}(B(0, \sqrt[\delta]{\cdot}), \mathbb{C})\), and some \((\lambda_0, z_0) \in U, \lambda(z)\) is a convergent Puiseux series with

\[
\lambda(z) = v((z-z_0)^\frac{1}{q}) = \lambda_0 + \sum_{s=1}^{\infty} c_s(z-z_0)^\frac{s}{q}, \quad |z-z_0| < \delta
\]  

(3.41)

and taking any fixed branch of the \(q\)th root and denoting its evaluation at \(z-z_0\) by \((z-z_0)^{1/q}\), its branches

\[
\lambda_h(z) = v(\zeta^h(z-z_0)^{1/q}) = \lambda_0 + \sum_{s=1}^{\infty} c_s(\zeta^h(z-z_0)^{1/q})^s, \quad |z-z_0| < r, \quad h = 0, \ldots, q, \quad (3.42)
\]

where \(\zeta\) is any primitive \(q\)th root of unity, satisfy \((\lambda_h(z), z) \in U\) for \(|z-z_0| < \delta\) and

\[
\det L(\lambda_h(z), z) = 0, \quad |z-z_0| < \delta, \quad h = 0, \ldots, q-1,
\]  

(3.43)

since the values \(\lambda_h(z), h = 0, \ldots, q-1\) for \(|z-z_0| < \delta\) are eigenvalues of \(L(\cdot, z)\) by assumption.

We introduce a new variable \(\varepsilon \in B(0, \sqrt[\delta]{\cdot})\). Then \(z(\varepsilon) := \varepsilon^q + z_0 \in B(z_0, \delta)\) is an analytic function. Moreover, for each \(\varepsilon \in B(0, \sqrt[\delta]{\cdot})\) there exists an \(h \in \{0, \ldots, q-1\}\) such that \(\varepsilon = \zeta^h(z(\varepsilon)-z_0)^{1/q}\). This implies \((v(\varepsilon), z(\varepsilon)) = (\lambda_h(z(\varepsilon)), z(\varepsilon)) \in U\) and hence \(\det L(v(\varepsilon), z(\varepsilon)) = \)
\[ \text{det } L(\lambda_h(z), z) = 0. \] Thus we have shown for every \( \varepsilon \in B(0, \sqrt{r}) \), the analytic functions \( v(\varepsilon) \) and \( z(\varepsilon) \) satisfy \((v(\varepsilon), z(\varepsilon)) \in U \) and so have ranges in the domain of the analytic function \( L \) and satisfy

\[ \text{det } L(v(\varepsilon), z(\varepsilon)) = 0. \tag{3.44} \]

But this implies the function \( F : B(0, \sqrt{\delta}) \to M_n(\mathbb{C}) \) defined by \( F(\varepsilon) := L(v(\varepsilon), z(\varepsilon)) \) is an analytic function as well, i.e., \( F \in \mathcal{O}(B(0, \sqrt{\delta}), M_n(\mathbb{C})) \), and hence \( \text{det } F \in \mathcal{O}(B(0, \sqrt{\delta}), \mathbb{C}) \) with

\[ \text{det } F = 0. \tag{3.45} \]

But this implies that 0 is an eigenvalue of the holomorphic matrix function \( F \) of infinite algebraic multiplicity. Consider the local Smith form \( G \) of \( F \) corresponding to the eigenvalue 0 of infinite algebraic multiplicity. By Theorem 9 it must be of the form

\[ G(\varepsilon) = \text{diag}\{0, \ldots, 0, \varepsilon^{m'_{\mu+1}}, \ldots, \varepsilon^{m'_{g}}, 1, \ldots, 1\} \tag{3.46} \]

where \( 1 \leq \mu \leq g = \dim \ker F(0) = \dim \ker L(\lambda_0, z_0) \leq n \), the number of zeros down the diagonal is \( 1 \leq \mu = \dim \ker F(\varepsilon) = \dim \ker L(v(\varepsilon), z(\varepsilon)) \) for \( 0 < \varepsilon \ll 1 \), when \( \mu = g \) there are no \( \varepsilon \) terms down the diagonal, when \( \mu < g \) we have \( m'_{\mu+1} \geq \cdots \geq m'_{g} \geq 1 \) with \( \{m'_{\mu+j}\}_{j=1}^{g} \subseteq \mathbb{N} \), and when \( g = n \) there are no ones appearing in the diagonal. Hence by Theorem 9 and Lemma 12 there exists an \( r > 0 \) with \( 0 < \sqrt{r} \leq \sqrt{\delta} \) and matrix functions \( M, N \in \mathcal{O}(B(0, \sqrt{r}), M_n(\mathbb{C})) \) such that \( M(\varepsilon), N(\varepsilon) \) are invertible matrices for every \( \varepsilon \in B(0, \sqrt{r}) \) and satisfy

\[ \text{diag}\{0, \ldots, 0, \varepsilon^{m'_{\mu+1}}, \ldots, \varepsilon^{m'_{g}}, 1, \ldots, 1\} = N(\varepsilon) F(\varepsilon) M(\varepsilon) \]

\[ = N(\varepsilon) L(v(\varepsilon), z(\varepsilon)) M(\varepsilon), \quad |\varepsilon| < \sqrt{r}. \tag{3.47} \]
Denote the standard orthonormal basis vectors in $\mathbb{C}^n$ by $e_1, \ldots, e_n$. Then

$$
\phi_j(\varepsilon) := M(\varepsilon)e_j, \quad j = 1, \ldots, \mu
$$

(3.48)

are the first $\mu$ columns of $M(\varepsilon)$ and so are a basis for $\ker L(v(\varepsilon), z(\varepsilon))$ for every $\varepsilon \in B(0, \sqrt{r})$. Hence for $j = 1, \ldots, \mu$ we have $\phi_j \in \mathcal{O}(B(0, \sqrt{r}), \mathbb{C}^n)$ and $\phi_j(\varepsilon) \neq 0$ for every $\varepsilon \in B(0, \sqrt{r})$.

In particular, $\phi_1 \in \mathcal{O}(B(0, \sqrt{r}), \mathbb{C}^n)$ and satisfies

$$
L(v(\varepsilon), z(\varepsilon))\varphi_1(\varepsilon) = 0, \quad \varphi_1(\varepsilon) \neq 0, \quad |\varepsilon| < \sqrt{r}.
$$

(3.49)

Now since $\phi_1 \in \mathcal{O}(B(0, \sqrt{r}), \mathbb{C}^n)$ it follows that it has a power series expansion about the point $z_0$ which converges for $|\varepsilon| < \sqrt{r}$. Thus we have

$$
\phi_1(z) = \beta_0 + \sum_{s=1}^{\infty} \beta_s \varepsilon^s, \quad |\varepsilon| < \sqrt{r}.
$$

(3.50)

where $\{\beta_s\}_{s=0}^{\infty} \subseteq \mathbb{C}^n$, $\beta_0 \neq 0$, and the series converges absolutely in the $\mathbb{C}^n$ norm for every $\varepsilon \in B(0, \sqrt{r})$. It follows that a convergent Puiseux series $\varphi(z)$ is given by

$$
\varphi(z) = \phi_1((z - z_0)^{1/q}) = \beta_0 + \sum_{s=1}^{\infty} \beta_s (z - z_0)^{s/q}, \quad |z - z_0| < r.
$$

(3.51)

We will now show that $\varphi(z)$ is an eigenvector Puiseux series of $L(\cdot, z)$ corresponding to the eigenvalue Puiseux series $\lambda(z)$. Fix any branch of the $q$th root function denoting its evaluation at $(z - z_0)$ by $(z - z_0)^{1/q}$ and let $\zeta$ be any primitive $q$th root of unity. Now if $|z - z_0| < r$ then $\varepsilon_h(z) := \zeta^h(z - z_0)^{1/q} \in B(0, \sqrt{r}) \subseteq B(0, \sqrt{r})$ for $h = 0, \ldots, q$ and so it follows that the branches of $\lambda(z)$ and $\varphi(z)$ are given by the convergent series

$$
\lambda_h(z) = \lambda_0 + \sum_{s=1}^{\infty} c_s (\zeta^h(z - z_0)^{1/q})^s, \quad |z - z_0| < r, \quad h = 0, \ldots, q.
$$

(3.52)
Theorem 23

Suppose \( z = |z - z_0| < r \), \( h = 0, \ldots, q \).

Moreover, if \( |z - z_0| < r \) then we have \( \varphi_h(z) = \zeta^h(z - z_0)^{1/q} \) \( \in B(0, \sqrt[r]{r}) \subseteq B(0, \sqrt[q]{\delta}) \) implies \( z = z(\varphi_h(z)) \), \( \lambda_h(z) = v(\varphi_h(z)) \), \( (v(\varphi_h(z)), z(\varphi_h(z))) = (\lambda_h(z), z) \in U \), \( \varphi_h(z) = \phi_1(\varphi_h(z)) \neq 0 \), and

\[
L(\lambda_h(z), z)\varphi_h(z) = L(v(\varphi_h(z)), z(\varphi_h(z)))\phi_1(\varphi_h(z)) = 0 \tag{3.54}
\]

for \( h = 0, \ldots, q - 1 \). Therefore \( \varphi(z) \) is an eigenvector Puiseux series of \( L(\cdot, z) \) corresponding to the eigenvalue Puiseux series \( \lambda(z) \). This completes the proof. \( \blacksquare \)

Theorem 23 Suppose \( L \in \mathcal{O}(U, M_n(\mathbb{C})) \) such that \( (\lambda_0, z_0) \in \sigma(L) \). If \( \lambda_0 \) an eigenvalue of \( L(\cdot, z_0) \) with algebraic multiplicity \( m \) then for any \( \varepsilon > 0 \) there exists a \( r > 0 \) and \( l \) eigenvalue Puiseux series \( (\lambda_1(z), \varphi_1(z)), \ldots, (\lambda_l(z), \varphi_l(z)) \) expanded about \( z_0 \) with domain \( B(z_0, r) \) and periods \( q_1, \ldots, q_l \), respectively, satisfying \( m = q_1 + \cdots + q_l \) such that if we fix any branch of the \( q \)th root function, for \( q \in \{q_1, \ldots, q_l\} \), denoting its evaluation at \( (z - z_0) \) by \( (z - z_0)^{1/q} \) and let \( \zeta_q \) be any primitive \( q \)th root of unity then their branches given by the convergent series

\[
\lambda_{j,h}(z) = \lambda_0 + \sum_{s=1}^{\infty} \beta_{j,s}(\zeta_q^h(z - z_0)^{1/q})^s, \quad |z - z_0| < r, \quad h = 0, \ldots, q_j - 1, \tag{3.55}
\]

\[
\varphi_{j,h}(z) = \beta_{j,0} + \sum_{s=1}^{\infty} \beta_{j,h}(\zeta_q^h(z - z_0)^{1/q})^s, \quad |z - z_0| < r, \quad h = 0, \ldots, q_j - 1. \tag{3.56}
\]

satisfy

\[
\det L(\lambda_{j,h}(z), z) = 0, \quad h = 0, \ldots, q_j - 1, \quad j = 1, \ldots, l, \quad |z - z_0| < r \tag{3.57}
\]

\[
\sigma(L) \cap B(\lambda_0, \varepsilon) \times B(z_0, r) = \bigcup_{j=1}^{l} \bigcup_{h=1}^{q_j-1} \{ (\lambda_{j,h}(z), z) : |z - z_0| < r \} \tag{3.58}
\]

\[
\sigma(L(\cdot, z)) \cap B(\lambda_0, \varepsilon) = \bigcup_{j=1}^{l} \bigcup_{h=1}^{q_j-1} \{ \lambda_{j,h}(z) \}, \quad |z - z_0| < r \tag{3.59}
\]
Proof. This theorem follows directly from Theorem 21 and Lemma 22.

Now in general the zero-order eigenvectors \( \{ \beta_{j,0} \}_{j=1}^g \subseteq \ker L(\lambda_0, z_0) \) need not be a basis for \( \ker L(\lambda_0, z_0) \), the eigenspace of \( L(\cdot, z_0) \) corresponding to the eigenvalue \( \lambda_0 \). Because of this two important problems arise in the spectral perturbation theory of holomorphic matrix functions:

1. Find general conditions involving just the partial derivatives of \( L \) at \( (\lambda_0, z_0) \) up to first order, namely, \( L(\lambda_0, z_0) \), \( L(\lambda_0, z_0) \), and \( L(\lambda_0, z_0) \) such that we may determine whether or not we can choose eigenpair Puiseux series \( (\lambda_1(z), \varphi_1(z)), \ldots, (\lambda_l(z), \varphi_l(z)) \) of \( L(\cdot, z) \) as given in the above theorem so that their zero-order eigenvectors \( \{ \varphi_j(z_0) \}_{j=1}^g \) span the whole eigenspace \( \ker L(\lambda_0, z_0) \).

2. Find formulas to determine the zero-order eigenvectors \( \{ \beta_{j,0} \}_{j=1}^g \).

The literature which addresses these problems is scarce and this author is only aware of [23, 34] which addresses these problems for general perturbations of holomorphic matrix functions. Even in the special case \( L(\lambda, z) = A(z) - \lambda I \), few results exist with the exception of [4, p. 270, Theorem 4; pp. 310-311, Theorem 8, Corollary 2, & Corollary 3], [41, Theorem 2], [58, p. 64, Chap. 2, §7.10, Theorem 6], and [47, Theorem 2.2]. In particular, the second problem remains open even in this case, when total splitting at the first-order approximation (see [4, p. 311] for definition) does not occur, and seems a difficult problem to resolve.
3.3.3 Analytic Eigenvalues and Eigenvectors

In this section we give a survey of the main results from [23, 34] that we will need in our thesis.

Preliminaries. We first begin with some preliminaries.

Definition 14 Let \( F \in \mathcal{O}(U, M_{m,n}(\mathbb{C})) \) where \( U \subseteq \mathbb{C} \) is an open connected set. Define \( U^* := \{ \lambda \mid \bar{\lambda} \in U \} \). The matrix function \( F^* : U^* \to M_{m,n}(\mathbb{C}) \) defined by \( F^*(\lambda) = F(\bar{\lambda})^* \) is called the adjoint of the holomorphic matrix function \( F \).

Lemma 24 If \( F \in \mathcal{O}(U, M_{m,n}(\mathbb{C})) \) where \( U \subseteq \mathbb{C} \) is an open connected set then \( U^* \) is an open connected set and \( F^* \in \mathcal{O}(U^*, M_{m,n}(\mathbb{C})) \).

Proof. If \( F \in \mathcal{O}(U, M_{m,n}(\mathbb{C})) \) then \( F = [a_{ij}]_{i=1,j=1}^{m,n} \) where \( a_{ij} \in \mathcal{O}(U, \mathbb{C}) \). Now obviously \( U^* \) is an open connected set and from complex analysis we know \( a_{ij}^*(\lambda) := \overline{a_{ij}(\bar{\lambda})} \) belongs to \( \mathcal{O}(U^*, \mathbb{C}) \). But this implies \( F^* = [a_{ji}^*]_{i=1,j=1}^{n,m} \in \mathcal{O}(U^*, M_{m,n}(\mathbb{C})) \). This completes the proof.

Proposition 25 Let \( F \in \mathcal{O}(U, M_n(\mathbb{C})) \). Then

1. \( \lambda_0 \in \mathbb{C} \) is an eigenvalue of \( F \) if and only if \( \bar{\lambda}_0 \) is an eigenvalue of \( F^* \).

2. \( \lambda_0 \) is an eigenvalue of \( F \) of finite algebraic multiplicity if and only if \( \bar{\lambda}_0 \) is an eigenvalue of \( F^* \) of finite algebraic multiplicity.

3. If \( \lambda_0 \) is an eigenvalue of \( F \) of finite algebraic multiplicity then the geometric, partial, and algebraic multiplicities of \( \lambda_0 \) coincide with the geometric, partial, and algebraic multiplicities, respectively, of the eigenvalue \( \bar{\lambda}_0 \) of \( F^* \).

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Proof. The first and second properties follows from the equality \( \det F(\lambda_0) = \det F(\lambda_0)^* = \det F^*(\overline{\lambda_0}) \) for any \( \lambda_0 \in U \). To prove the third part let \( \lambda_0 \) be an eigenvalue of \( F \) of finite algebraic multiplicity and let \( g, m_1 \geq \cdots \geq m_g, m := m_1 + \cdots + m_g \) be its geometric, partial, and algebraic multiplicities, respectively. Then by Theorem 9 there exists \( N, M \in \mathcal{O}(\lambda_0) \) and an \( r > 0 \) such that \( M(\lambda_0), N(\lambda_0) \) are invertible and for every \( \lambda \in B(\lambda_0, r) \) we have

\[
F(\lambda) = N(\lambda) \text{diag}\{(\lambda - \lambda_0)^{m_1}, \ldots, (\lambda - \lambda_0)^{m_g}, 1, \ldots, 1\} M(\lambda).
\]

where if \( g = n \) there are no ones appearing in the diagonal. But this implies for any \( \lambda \in B(\lambda_0, r) \) we have

\[
F^*(\lambda) = M^*(\lambda) \text{diag}\{(\lambda - \overline{\lambda_0})^{m_1}, \ldots, (\lambda - \overline{\lambda_0})^{m_g}, 1, \ldots, 1\} N^*(\lambda).
\]

where if \( g = n \) there are no ones appearing in the diagonal. And since \( M^*, N^* \in \mathcal{O}(\overline{\lambda_0}) \) with \( M^*(\overline{\lambda_0}) = M(\lambda_0)^*, N^*(\overline{\lambda_0}) = N(\lambda_0)^* \) are invertible, this implies that \( G^*(\lambda) = \text{diag}\{(\lambda - \overline{\lambda_0})^{m_1}, \ldots, (\lambda - \overline{\lambda_0})^{m_g}, 1, \ldots, 1\} \) is the local Smith form of \( F^* \) corresponding to the eigenvalue \( \overline{\lambda_0} \). Therefore, \( g, m_1 \geq \cdots \geq m_g, m \) are the geometric, partial, and algebraic multiplicities, respectively, of the eigenvalue \( \overline{\lambda_0} \) of \( F^* \). This completes the proof. ■

Definition 15 Let \( U \subseteq \mathbb{C}^2 \). Define \( U^* := \{ (\xi, \eta) \mid (\overline{\xi}, \overline{\eta}) \in U \} \). If \( L \in \mathcal{O}(U, M_n(\mathbb{C})) \) where \( U \) is an open connected set then the matrix function \( L^* \) defined by \( L^*(\xi, \eta) = L(\overline{\xi}, \overline{\eta})^* \) is called the adjoint of the holomorphic matrix function \( L \).

Lemma 26 If \( L \in \mathcal{O}(U, M_n(\mathbb{C})) \) where \( U \subseteq \mathbb{C}^2 \) is an open connected set then \( U^* \) is an open connected set and \( L^* \in \mathcal{O}(U^*, M_n(\mathbb{C})) \).

Proof. Let \( L \in \mathcal{O}(U, M_n(\mathbb{C})) \) where \( U \subseteq \mathbb{C}^2 \) is an open connected set. Obviously \( U^* \) is an open connected set. The adjoint of \( L \) is \( L^*(\xi, \eta) := \overline{L(\overline{\xi}, \overline{\eta})} \). Now for any \( (\xi_0, \eta_0) \in U^* \)
there exist \((\lambda_0, z_0) \in U\) such that \((\xi_0, \eta_0) = (\lambda_0, z_0)\). We define \(F_1(\xi) := L(\xi, z_0) = L^*(\xi, \eta_0)\) and \(F_2(\eta) := \overline{L(\lambda_0, \eta)} = L^*(\xi_0, \eta)\). It follows from the one variable case that \(F_1 \in \mathcal{O}(\xi_0)\) and \(F_2 \in \mathcal{O}(\eta_0)\). This implies the partial derivatives of \(L^*(\xi, \eta)\) exist at \((\xi_0, \eta_0)\). But since this is true for every \((\xi_0, \eta_0) \in U^*\) then \(L^* \in \mathcal{O}(U^*, \mathbb{C})\). This completes the proof.

\[\lambda(z) \in \sigma(L(\cdot, z)) \text{ if and only if } \overline{\lambda(\eta)} \in \sigma(L^*(\cdot, \eta)), \]

where \(\eta = z\). Moreover,

\[\sigma(L^*) = \sigma(L)^* . \]  

\[\text{(3.61)}\]

**Proof.** If \(\lambda(z) \in \sigma(L(\cdot, z))\) then \(0 = \det L(\lambda(z), z) = \det L^*(\overline{\lambda(z)}, \overline{z}) = \det L^*(\overline{\lambda(\eta)}, \eta)\) where \(\eta = \overline{z}\). Hence \(\overline{\lambda(\eta)} \in \sigma(L^*(\cdot, \eta))\). Conversely, if \(\overline{\lambda(\eta)} \in \sigma(L^*(\cdot, \eta))\) then letting \(\eta = \overline{z}\) we have \(0 = \det L^*(\overline{\lambda(\eta)}, \eta) = \det L^*(\overline{\lambda(z)}, \overline{z}) = \det L(\lambda(z), z)^* = \det L(\lambda(z), z)\). Hence \(\lambda(z) \in \sigma(L(\cdot, z))\). The latter part of this statement follows from the identity \(\det L^*(\xi, \eta) = \det L(\overline{\xi}, \overline{\eta})^* = \det L(\overline{\xi}, \overline{\eta})\) for any \((\xi, \eta) \in U^*\). This completes the proof.

**Definition 16** Suppose that \(\lambda(z)\) is a convergent Puiseux series given by

\[\lambda(z) = \sum_{s=0}^{\infty} c_s(z - z_0)^{\frac{s}{q}}, \quad |z - z_0| < r\]  

where \(\{c_s\}_{s=0}^{\infty} \subseteq \mathbb{C}\). Then a convergent Puiseux series is given by

\[\lambda^*(\eta) := \sum_{s=0}^{\infty} \overline{c_s}(\eta - \overline{z_0})^{\frac{s}{q}}, \quad |\eta - \overline{z_0}| < r.\]  

We call the Puiseux series \(\lambda^*(\eta)\) the **adjoint** of the Puiseux series \(\lambda(z)\).

**Lemma 28** Let \(L \in \mathcal{O}(U, M_n(\mathbb{C}))\). Then \(\lambda(z)\) is an eigenvalue Puiseux series of \(L(\cdot, z)\) if
and only if $\lambda^*(\eta)$ is an eigenvalue Puiseux series of $L^*(\cdot, \eta)$, where $\eta = \overline{\eta}$. Moreover, if $\lambda_h(z)$, $h = 0, \ldots, q - 1$ are all the branches of $\lambda(z)$ and $\lambda_h^*(\eta)$, $h = 0, \ldots, q - 1$ are all the branches of $\lambda^*(\eta)$ then for $\eta = \overline{\eta}$ we have

$\bigcup_{\eta = 0}^{q-1} \{ \lambda_h^*(\eta) \} = \bigcup_{\eta = 0}^{q-1} \{ \lambda_h(\overline{\eta}) \}$, \hspace{1cm} (3.64)

$\bigcup_{\eta = 0}^{q-1} \{ \lambda_h(z) \} = \bigcup_{\eta = 0}^{q-1} \{ \lambda_h^*(\overline{\eta}) \}$. \hspace{1cm} (3.65)

**Proof.** It’s obvious that

$$\lambda(z) = \lambda_0 + \sum_{s=1}^{\infty} c_s (z - z_0)^{\frac{s}{q}}, \quad |z - z_0| < r$$ \hspace{1cm} (3.66)

is a convergent Puiseux series if and only if its adjoint

$$\lambda^*(\eta) := \overline{\lambda}_0 + \sum_{s=1}^{\infty} \overline{c}_s (\eta - \overline{z}_0)^{\frac{s}{q}}, \quad |\eta - \overline{z}_0| < r$$ \hspace{1cm} (3.67)

is a convergent Puiseux series since $(\lambda^*)^*(z) = \lambda(z)$. Denote the principal branch of the $q$th root by $\sqrt[q]{z - z_0} := \sqrt[q]{|z - z_0|} e^{\frac{i}{q} \arg(z - z_0)}$ for $0 \leq \arg(z - z_0) \leq 2\pi$ and let $\zeta := e^{i2\pi/q}$. Denote another branch of the $q$th root by $(z - z_0)^{1/q} := \sqrt[q]{|z - z_0|} e^{\frac{1}{q} \arg(z - z_0)}$ for $-\pi \leq \arg(z - z_0) < \pi$ and notice $\overline{\zeta} = e^{-i2\pi/q}$ is a primitive $q$th root of unity. A quick calculation shows

$$\sqrt[q]{z - z_0} = (z - z_0)^{1/q},$$ \hspace{1cm} (3.68)

$$\overline{(z - z_0)^{1/q}} = \sqrt[q]{z - z_0},$$ \hspace{1cm} (3.69)

for every $z \in \mathbb{C}$. 

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Now all the branches of $\lambda(z)$ and $\lambda^*(\eta)$ are given by the convergent series

\[
\lambda_h(z) = \lambda_0 + \sum_{s=1}^{\infty} c_s (\zeta^h \sqrt[q]{z - z_0})^s, \quad h = 0, \ldots, q - 1, \quad |z - z_0| < r,
\]

\[
\lambda^*_h(\eta) = \sum_{s=1}^{\infty} c_s (\zeta^h (\eta - z_0)^{1/q})^s, \quad h = 0, \ldots, q - 1, \quad |\eta - z_0| < r.
\]

Making the substituions $\eta = \overline{z}$ and $z = \overline{\eta}$ as input to those branches and then using (3.68) and (3.69), we prove the identities

\[
\lambda_h(z) = \lambda^*_h(\overline{z}), \quad h = 0, \ldots, q - 1, \quad |z - z_0| < r, \quad (3.70)
\]

\[
\lambda^*_h(\eta) = \lambda_h(\overline{\eta}), \quad h = 0, \ldots, q - 1, \quad |\eta - z_0| < r. \quad (3.71)
\]

This proves

\[
\bigcup_{h=0}^{q-1} \{\lambda^*_h(\eta)\} = \bigcup_{h=0}^{q-1} \{\lambda_h(\overline{\eta})\}, \quad \bigcup_{h=0}^{q-1} \{\lambda_h(z)\} = \bigcup_{h=0}^{q-1} \{\lambda^*_h(\overline{z})\}
\]

which is just the set of all values of the Puiseux series $\lambda^*(\eta)$ and $\lambda(z)$ at $\eta$ and $z$, respectively, where $\eta = \overline{z}$. And hence, since for any convergent Puiseux series with say period $q$, the set of all values at a point in its domain is independent of our choice of the branch of the $q$th root function and primitive $q$th root of unity that we use, this proves the latter part of the statement of this lemma. It now follows this and Lemma 27 that the values of the branches of $\lambda(z)$ are eigenvalues of $L(\cdot, z)$ if and only if the branches of its adjoint $\lambda^*(\eta)$ are eigenvalues of $L^*(\cdot, \eta)$, where $\eta = \overline{z}$. And hence $\lambda(z)$ is an eigenvalue Puiseux series of $L(\cdot, z)$ if and only if $\lambda^*(\eta)$ is an eigenvalue Puiseux series of $L^*(\cdot, \eta)$, where $\eta = \overline{z}$. This completes the proof.

**On Results from the Spectral Perturbation Theory.** We are now ready to begin giving some of the major results from [23, 34] that we will need in our thesis. The following definition comes from [34].

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Definition 17 Let $L \in \mathcal{O}(U, M_n(\mathbb{C}))$ and $(\lambda_0, z_0) \in U$. Let $(\lambda(z), \varphi(z))$ be an eigenpair Puiseux series of $L(\cdot, z)$ such that $\lambda(z)$ and $\varphi(z)$ are expanded about $z_0$ with limit point $\lambda_0$ and $\beta_0$, respectively. The vector $\beta_0$ is called a generating eigenvector of $L(\cdot, z)$ (at the point $(\lambda_0, z_0)$ and associated with $\lambda(z)$).

Remark. It should be noted that in the case $L \in \mathcal{O}(U, M_n(\mathbb{C}))$ with $(\lambda_0, z_0) \in U$ and $\lambda_0$ is an eigenvalue of $L(\cdot, z_0)$ of finite algebraic multiplicity, that by Theorem 23 there exists an eigenpair Puiseux series $(\lambda(z), \varphi(z))$ of $L(\cdot, z)$ such that $\lambda(z)$ and $\varphi(z)$ are expanded about $z_0$ with limit point $\lambda_0$ and $\beta_0$, respectively, for some $\beta_0$. And hence a generating eigenvector $\beta_0$ of $L(\cdot, z)$ at the point $(\lambda_0, z_0)$ and associated with $\lambda(z)$ exists. And, in particular, we have $\beta_0 \neq 0$ and $\beta_0 \in \ker(L(\lambda_0, z_0))$. Moreover, by Lemma 28 and Lemma 22 it follows there exists an eigenpair Puiseux series $(\lambda^*(\eta), \gamma(\eta))$ of $L^*(\cdot, \eta)$ such that $\lambda^*(\eta)$ and $\gamma(\eta)$ are expanded about $\overline{z_0}$ with limit point $\overline{\lambda_0}$ and $\gamma_0$, respectively, for some $\gamma_0$. Thus $\gamma_0$ is a generating eigenvector of $L^*(\cdot, \eta)$ at the point $(\overline{\lambda_0}, \overline{z_0})$ and associated with $\lambda^*(\eta)$, where $\lambda^*(\eta)$ is the adjoint of the Puiseux series $\lambda(z)$ and $L^*$ is the adjoint of $L$. And, in particular, $\gamma_0 \neq 0$ and $\gamma_0 \in \ker(L^*(\overline{\lambda_0}, \overline{z_0})) = \ker(L(\lambda_0, z_0))^*$. 

The next proposition is equivalent to Lemma 7 from [34] and its proof comes from [23]. As we will shall soon see this proposition is actually one of the key results in the spectral perturbation theory of holomorphic matrix functions.

Proposition 29 Let $L \in \mathcal{O}(U, M_n(\mathbb{C}))$ and $(\lambda_0, z_0) \in U$. Let $\lambda_1(z), \lambda_2(z)$ be any two eigenvalue Puiseux series of $L$ expanded about $z_0$ with common limit point $\lambda_0$. Suppose that for some branch $\lambda_{1, h_1}(z)$ of $\lambda_1(z)$ and some branch $\lambda_{2, h_2}(z)$ of $\lambda_2(z)$ there existed an $r > 0$ such that $\lambda_{1, h_1}(z) \neq \lambda_{2, h_2}(z)$ for $|z - z_0| < r$. Then for any generating eigenvector $\beta_0$ of $L(\cdot, z)$ at the point $(\lambda_0, z_0)$ and associated with $\lambda_1(z)$ and for any generating eigenvector $\gamma_0$ of $L^*(\cdot, \eta)$
at the point \((\overline{\lambda_0}, \overline{z_0})\) and associated with \(\lambda_2^* (\eta)\) we have

\[
\langle L_\lambda (\lambda_0, z_0) \beta_0, \gamma_0 \rangle \subset 0.
\]  

(3.72)

**Proof.** By assumption there exists eigenvalue Puiseux series

\[
\lambda_1 (z) = \sum_{s=0}^{\infty} c_{1,s} (z - z_0)^{\frac{s}{q_1}}, \quad |z - z_0| < r
\]

\[
\lambda_2 (z) = \sum_{s=0}^{\infty} c_{2,s} (z - z_0)^{\frac{s}{q_2}}, \quad |z - z_0| < r
\]

of \(L(\cdot, z)\) such that for some fixed branch of the \(q_1\)th and \(q_2\)th root functions say \(f_1\) and \(f_2\) with its evaluation at \((z - z_0)\) given by \(f_1(z - z_0)\) and \(f_2(z - z_0)\), respectively, and some fixed primitive \(q_1\)th and \(q_2\)th root of unity say \(\zeta_1\) and \(\zeta_2\), respectively, we have all their branches given by the convergent series

\[
\lambda_{1,h} (z) = \lambda_0 + \sum_{s=1}^{\infty} c_{1,s} (\zeta_1^h f_1(z - z_0))^s, \quad h = 0, \ldots, q_1 - 1, \quad |z - z_0| < r
\]

\[
\lambda_{2,h} (z) = \lambda_0 + \sum_{s=1}^{\infty} c_{2,s} (\zeta_2^h f_2(z - z_0))^s, \quad h = 0, \ldots, q_2 - 1, \quad |z - z_0| < r
\]

and for some \(h_1 \in \{0, \ldots, q_1 - 1\}\) and \(h_2 \in \{0, \ldots, q_2 - 1\}\),

\[
\lambda_{1,h_1} (z) \neq \lambda_{2,h_2} (z), \quad |z - z_0| < r.
\]

Now let \(\beta_0\) be a generating eigenvector of \(L(\cdot, z)\) at the point \((\lambda_0, z_0)\) and associated with \(\lambda_1 (z)\) and \(\gamma_0\) be a generating eigenvector of \(L^* (\cdot, \eta)\) at the point \((\overline{\lambda_0}, \overline{z_0})\) and associated with \(\lambda_2^* (\eta)\). By definition of generating eigenvector, there exists an eigenpair Puiseux series \((\lambda (z), \varphi (z))\) of \(L (\cdot, z)\) such that \(\lambda (z)\) and \(\varphi (z)\) are expanded about \(z_0\) with limit point \(\lambda_0\) and \(\beta_0\), respectively, and there exists an eigenpair Puiseux series \((\lambda_2^* (\eta), \gamma (\eta))\) of \(L^* (\cdot, \eta)\) such

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that \( \lambda(z) \) and \( \varphi(z) \) are expanded about \( z_0 \) with limit point \( \overline{\lambda_0} \) and \( \gamma_0 \), respectively.

Now because of the definition of eigenvector Puiseux series and the definition of adjoint of a Puiseux series and because none of the properties just mentioned change by going to a smaller domain for \( z \), i.e., taking \( r \) smaller, we can assume without loss of generality that all the branches of \( \varphi(z) \), \( \lambda^*_2(\eta) \), \( \gamma(\eta) \) are given by the convergent series

\[
\varphi_h(z) = \beta_0 + \sum_{s=1}^{\infty} \beta_{1,s}(\zeta_1 h f_1(z - z_0))^s, \quad h = 0, \ldots, q_1 - 1, \quad |z - z_0| < r,
\]

\[
\lambda^*_{2,h}(\eta) = \lambda_0 + \sum_{s=1}^{\infty} c_{2,s}(\zeta_2 h f_2(\eta - \overline{z_0}))^s, \quad h = 0, \ldots, q_2 - 1, \quad |\eta - \overline{z_0}| < r,
\]

\[
\gamma_h(\eta) = \lambda_0 + \sum_{s=1}^{\infty} \chi_s(\zeta_2 h f_2(\eta - \overline{z_0}))^s, \quad h = 0, \ldots, q_2 - 1, \quad |\eta - \overline{z_0}| < r,
\]

respectively, and satisfy

\[
L(\lambda_{1,h}(z), z)\varphi_h(z) = 0, \quad h = 0, \ldots, q_1 - 1, \quad |z - z_0| < r, \quad |\eta - \overline{z_0}| < r,
\]

\[
L^*(\lambda^*_{2,h}(\eta), \eta)\gamma_h(\eta) = 0, \quad h = 0, \ldots, q_2 - 1, \quad |\eta - \overline{z_0}| < r.
\]

Now it follows from Lemma 28 that

\[
\bigcup_{h=0}^{q_2-1} \{\lambda^*_2(\eta)\} = \bigcup_{h=0}^{q_2-1} \{\overline{\lambda_{2,h}(\eta)}\}, \quad |\eta - \overline{z_0}| < r.
\]

Thus for every \( \eta \in B(\overline{z_0}, r) \) there exists \( h(\eta) \in \{0, \ldots, q_2 - 1\} \) such that

\[
\lambda^*_{2,h(\eta)}(\eta) = \overline{\lambda_{2,h(\eta)}(\eta)}.
\]

Hence if \( |z - z_0| < r \) then for \( \eta = z \) we have

\[
0 = L^*(\lambda^*_{2,h(\eta)}(\eta), \eta)\gamma_{h(\eta)}(\eta) = L(\lambda_{2,h(\eta)}(\eta), \overline{\eta})^*\gamma_{h(\eta)}(\eta) = L(\lambda_{2,h(\eta)}(z), z)^*\gamma_{h(z)}(z)
\]
and thus

$$\langle \gamma_{h(z)}(z), L(\lambda_2, h_2(z), z) \varphi_{h_1}(z) \rangle = \langle L(\lambda_2, h_2(z), z)^* \gamma_{h(z)}(z), \varphi_{h_1}(z) \rangle_C = 0.$$  

Now it follows from Lemma 17, since $L \in \mathcal{O}(U, M_n(\mathbb{C}))$ and $(\lambda_0, z_0) \in U$, that $L$ is analytic at $(\lambda, z) = (\lambda_0, z_0)$, i.e., the partial and mixed partial derivatives $L^{(j_1,j_2)}(\lambda_0, z_0) := \frac{\partial^{j_1+j_2}L}{\partial^{j_1} \lambda \partial^{j_2} z}|_{(\lambda, z) = (\lambda_0, z_0)}$ exist for every $j_1, j_2 \in \mathbb{N} \cup \{0\}$ and there exists an $r_0 > 0$ such that

$$L(\lambda, z) = \sum_{j_1,j_2=0}^{\infty} \frac{1}{j_1! j_2!} L^{(j_1,j_2)}(\lambda_0, z_0) (\lambda - \lambda_0)^{j_1} (z - z_0)^{j_2}, \ |\lambda - \lambda_0| < r_0, |z - z_0| < r_0$$

where the power series on the right converges absolutely in the $M_n(\mathbb{C})$ norm to $L(\lambda, z)$ for $(\lambda, z) \in B(\lambda_0, r_0) \times B(z_0, r_0)$. We may assume without loss of generality that $r_0 \leq r$.

Thus from the facts above and the facts

$$x^j - y^j = (x - y) \sum_{l=0}^{j-1} x^l y^{j-1-l}, \ x, y \in \mathbb{C}, \ j \in \mathbb{N}$$

$$\lim_{z \to z_0} \lambda_{1,h_1}(z) = \lim_{z \to z_0} \lambda_{1,h_2}(z) = \lambda_0$$

$$\lim_{z \to z_0} \gamma_{h(z)}(z) = \lim_{z \to z_0} \gamma_{h(z)}(z) = \gamma_0, \ h = 0, \ldots, q_2 - 1$$

it follows that

$$\frac{(\lambda_{1,h_1}(z) - \lambda_0)^j - (\lambda_{2,h_2}(z) - \lambda_0)^j}{\lambda_{1,h_1}(z) - \lambda_{2,h_2}(z)} = o(1), \ \text{as} \ z \to z_0, \ \text{if} \ j \geq 2$$

and so

$$\frac{L(\lambda_{1,h_1}(z), z) - L(\lambda_{2,h_2}(z), z)}{\lambda_{1,h_1}(z) - \lambda_{2,h_2}(z)} = L^{(1,0)}(\lambda_0, z_0) + o(1), \ \text{as} \ z \to z_0$$
which implies

\[
0 = \frac{\langle \gamma_h(z), L(\lambda_{1, h_1}(z), z) \varphi_{h_1}(z) \rangle_C - \langle \gamma_h(z), L(\lambda_{2, h_2}(z), z) \varphi_{h_1}(z) \rangle_C}{\lambda_{1, h_1}(z) - \lambda_{2, h_2}(z)} \\
= \frac{\langle \gamma_h(z), \frac{L(\lambda_{1, h_1}(z), z) - L(\lambda_{2, h_2}(z), z)}{\lambda_{1, h_1}(z) - \lambda_{2, h_2}(z)} \varphi_{h_1}(z) \rangle_C}{\lambda_{1, h_1}(z) - \lambda_{2, h_2}(z)} \\
= \langle \gamma_0, L^{(1, 0)}(\lambda_0, z_0) \beta_0 \rangle_C + o(1), \quad \text{as } z \to z_0.
\]

Hence we must have

\[
0 = \langle \gamma_0, L^{(1, 0)}(\lambda_0, z_0) \beta_0 \rangle_C = \langle \gamma_0, L(\lambda_0, z_0) \beta_0 \rangle_C = \langle \overline{L(\lambda_0, z_0) \beta_0}, \gamma_0 \rangle_C,
\]

where by our notation \( L(\lambda_0, z_0) := L^{(1, 0)}(\lambda_0, z_0) \). Therefore \( \langle L(\lambda_0, z_0) \beta_0, \gamma_0 \rangle_C = 0 \). This completes the proof.

\textbf{Lemma 30} Let \( \lambda(z) \) be a convergent Puiseux series expanded about \( z_0 \) with domain \( B(z_0, r_0) \) and period \( q \). Suppose \( \lambda(z) \) is not a single-valued analytic function of \( z \) in \( B(z_0, r_0) \). Then if we let \( \lambda_1(z), \ldots, \lambda_{q-1}(z) \) be all the branches of \( \lambda(z) \) with respect to a fix branch of the \( q \)th root function and a fix primitive \( q \)th root of unity, there exists \( h_1, h_2 \in \{0, \ldots, q-1\} \) and an \( r > 0 \) with \( r \leq r_0 \) such that

\[
\lambda_{h_1}(z) \neq \lambda_{h_2}(z), \quad 0 < |z - z_0| < r.
\]

\textbf{Proof.} Suppose \( \lambda(z) \) is a convergent Puiseux series expanded about \( z_0 \) with domain \( B(z_0, r_0) \) and period \( q \), where

\[
\lambda(z) = \sum_{s=0}^{\infty} c_s (z - z_0)^{\frac{s}{q}}, \quad |z - z_0| < r_0,
\]

and suppose that \( \lambda(z) \) is not a single-valued analytic function of \( z \) in \( B(z_0, r_0) \), i.e., there exists \( s \in \mathbb{N} \) such that \( \frac{s}{q} \notin \mathbb{N} \) and \( c_s \neq 0 \). We will denote by \( s_0 \) the smallest such \( s \) with that
property. Note that for the Puiseux series $\lambda(z)$ with the series representation given above we must have the period $q \geq 2$.

Now fix a branch of the $q$th root and denote its evaluation at $z - z_0$ by $(z - z_0)^{1/q}$ and fix some primitive $q$th root of unity denoting it by $\zeta$. Then the branches of $\lambda(z)$ are given by the convergent series

$$\lambda_h(z) = \sum_{s=0}^{\infty} c_s (\zeta^h (z - z_0)^{1/q})^s, \quad |z - z_0| < r_0, \quad h = 0, \ldots, q - 1.$$  

By the definition of $s_0$ there exists an $l \in \mathbb{N} \cup \{0\}$ such that $\frac{s_0 - 1}{q} = l$, $c_{s_0} \neq 0$, and

$$\lambda_h(z) = \sum_{j=0}^{l} c_{jq}(z - z_0)^j + c_{s_0} (\zeta^h (z - z_0)^{1/q})^{s_0} + \sum_{s=s_0+1}^{\infty} c_s (\zeta^h (z - z_0)^{1/q})^s,$$

for $|z - z_0| < r_0$ and $h = 0, \ldots, q - 1$. It follows from this that for any $h_1, h_2 \in \{0, \ldots, q - 1\}$ we have

$$|\lambda_{h_1}(z) - \lambda_{h_2}(z)| = |c_{s_0}||\zeta^{h_1 s_0} - \zeta^{h_2 s_0}| |z - z_0|^{\frac{s_0}{q}} + O(|z - z_0|^{\frac{s_0 + 1}{q}}), \quad \text{as } z \to z_0.$$

The leading order term here can only be zero if $|\zeta^{h_1 s_0} - \zeta^{h_2 s_0}| = 0$ which would imply $\zeta^{(h_1 - h_2)s_0} = 1$ and since $\zeta$ is a primitive $q$th root of unity this implies $\frac{(h_1 - h_2)s_0}{q} \in \mathbb{Z}$. Thus letting $h_1 := 1, h_2 := 0 \in \{0, \ldots, q - 1\}$ (since $q \geq 2$) then, because $\frac{(h_1 - h_2)s_0}{q} = \frac{s_0}{q} \not\in \mathbb{Z}$, we have $|\zeta^{h_1 s_0} - \zeta^{h_2 s_0}| \neq 0$ and so

$$\frac{|\lambda_{h_1}(z) - \lambda_{h_2}(z)|}{|c_{s_0}||\zeta^{h_1 s_0} - \zeta^{h_2 s_0}| |z - z_0|^{\frac{s_0}{q}}} = 1 + o(1), \quad z \neq z_0, \quad \text{as } z \to z_0.$$

Therefore, since $\lim_{z \to z_0} \lambda_h(z) = c_0$ for $h = 0, \ldots, q - 1$, it follows there exists $r > 0$ with $r \leq r_0$ such that

$$\lambda_{h_1}(z) \neq \lambda_{h_2}(z), \quad 0 < |z - z_0| < r.$$
This completes the proof.

The following theorem is Theorem 9 from [34].

**Theorem 31** Let $L \in \mathcal{O}(U, M_n(\mathbb{C}))$ and suppose $(\lambda_0, z_0) \in U$. Let $\lambda(z)$ be an eigenvalue Puiseux series of $L(\cdot, z)$ expanded about $z_0$ with limit point $\lambda_0$. Assume that for every generating eigenvector $\beta$ of $L(\cdot, z)$ at the point $(\lambda_0, z_0)$ and associated with $\lambda(z)$ there exists a generating eigenvector $\gamma$ of $L^*(\cdot, \eta)$ at the point $(\overline{\lambda_0}, \overline{z_0})$ and associated with $\lambda^*(\eta)$ such that

$$
\langle L_\lambda(\lambda_0, z_0)\beta, \gamma \rangle_{\mathbb{C}} \neq 0, \quad (3.74)
$$

where $\lambda^*(\eta)$ denotes the adjoint of $\lambda(z)$ and $L^*$ denotes the adjoint of $L$. Then $\lambda(z)$ is a single-valued analytic function of $z$ and there exists an eigenvector Puiseux series $\varphi(z)$ of $L(\cdot, z)$ corresponding to $\lambda(z)$ which is also a single-valued analytic function of $z$.

**Proof.** Let $L \in \mathcal{O}(U, M_n(\mathbb{C}))$ and suppose $(\lambda_0, z_0) \in U$. Let $\lambda(z)$ be an eigenvalue Puiseux series of $L(\cdot, z)$ expanded about $z_0$, domain $B(z_0, r_0)$, with limit point $\lambda_0$, and period $q$. Suppose the hypotheses of this theorem are true for $\lambda(z)$. But suppose that $\lambda(z)$ was not a single-valued analytic function of $z \in B(z_0, r_0)$. Then by Lemma 30, if we let $\lambda_1(z), \ldots, \lambda_{q-1}(z)$ be all the branches of $\lambda(z)$ with respect to a fix branch of the $q$th root function and a fix primitive $q$th root of unity, there exists $h_1, h_2 \in \{0, \ldots, q - 1\}$ and an $r > 0$ with $r \leq r_0$ such that

$$
\lambda_{h_1}(z) \neq \lambda_{h_2}(z), \quad 0 < |z - z_0| < r. \quad (3.75)
$$

Now by Lemma 22 there exists an eigenvector Puiseux series $\varphi(z)$ of $L(\cdot, z)$ corresponding to the eigenvalue Puiseux series $\lambda(z)$. Let $\beta_0$ denote the limit point of $\varphi(z)$ when expanded about $z_0$. By definition $\beta_0$ is a generating eigenvector of $L(\cdot, z)$ at the point $(\lambda_0, z_0)$ and
associated with $\lambda(z)$ and so by our hypotheses their exists a eigenvalue Puiseux series $\gamma(\eta)$, expanded about $z_0$ with limit point $\gamma_0$, of $L^*(\cdot, \eta)$ corresponding to the eigenvalue Puiseux series $\lambda^*(\eta)$ such that

$$\langle L_\lambda(\lambda_0, z_0) \beta_0, \gamma_0 \rangle \mathbb{C} \neq 0. \quad (3.76)$$

But according to Proposition 29 with $\lambda_1(z) := \lambda_2(z) = \lambda(z)$ we must have

$$\langle L_\lambda(\lambda_0, z_0) \beta_0, \gamma_0 \rangle \mathbb{C} = 0. \quad (3.77)$$

This is a contradiction. Therefore, $\lambda(z)$ is single-valued analytic function of $z \in B(z_0, r_0)$ and so by Lemma 22 there exists an eigenvector Puiseux series $\varphi(z)$ of $L(\cdot, z)$ corresponding to $\lambda(z)$ which is also a single-valued analytic function of $z$. This completes the proof.  \(\blacksquare\)

The following proposition comes from [23, Lemma 4.1].

**Proposition 32** Let $L \in \mathcal{O}(U, M_n(\mathbb{C}))$ and suppose $(\lambda_0, z_0) \in \sigma(L)$. Assume $(\lambda(z), \varphi(z))$ is an eigenpair Puiseux series of $L(\cdot, z)$ such that both $\lambda(z)$ and $\varphi(z)$ are analytic functions at $z_0$ with $\lambda(z_0) = \lambda_0$. Denote the order of the zero of the function $\lambda(z) - \lambda_0$ at $z_0$ by $q$. Then $\varphi(z)$ is a generating function of order $q$ for $L(\lambda_0, \cdot)$ at the eigenvalue $z_0$.

**Proof.** By hypotheses there exists an $r > 0$ such that $B(\lambda_0, r) \times B(z_0, r) \subseteq U$, $\lambda(\cdot) \in \mathcal{O}(B(z_0, r), \mathbb{C})$, and $\varphi(\cdot) \in \mathcal{O}(B(z_0, r), \mathbb{C}^n)$ such that $\varphi(z_0) \neq 0$ and $L(\lambda(z), z)\varphi(z) = 0$ for every $z \in B(z_0, r)$. Thus it follows from this by using the power series expansion of $L$ at $(\lambda_0, z_0)$ from Lemma 17, that

$$\varphi(z_0) \neq 0, \quad L(\lambda_0, z)\varphi(z) = L(\lambda_0, z)\varphi(z) - L(\lambda(z), z)\varphi(z)$$

$$= (L(\lambda_0, z) - L(\lambda(z), z))\varphi(z)$$

$$= -(\lambda(z) - \lambda_0)L_\omega(\lambda_0, z_0)\varphi(z) + o(\lambda(z) - \lambda_0)$$
\[ -\lambda^{(q)}(z_0)(z - z_0)^q L_\omega(\lambda_0, z_0)\varphi(z_0) + o((z - z_0)^q) \]
\[ = O((z - z_0)^q), \quad \text{as } z \to z_0. \]

By definition of generating function this implies \( \varphi(z) \) is a generating function of order \( q \) for \( L(\lambda_0, \cdot) \) at the eigenvalue \( z_0 \). \( \blacksquare \)
Chapter 4

Canonical Equations: A Model for Studying Slow Light

4.1 Introduction

In this chapter we consider a general nondissipative but dispersive model for wave propagation using an important class of periodic differential and differential-algebraic equations (DAEs) called canonical equations [31]. This model is general enough to include electromagnetic waves governed by the time-harmonic Maxwell’s equations for lossless one-dimensional photonic crystals whose constituent layers can be any combination of isotropic, anisotropic, or bianisotropic materials with or without material dispersion (i.e., frequency-dependent response of materials). This makes our work particularly significant in the study of slow light since metamaterials are widening the range of potential photonic crystals that can be fabricated and so a model like ours that has the ability to analysis slow light phenomena for a broad range of photonic crystals is in need.
4.1.1 Model Formulation

In this chapter\(^1\), we will denote a wave by \(y\) and its evaluation at a position \(x\) by \(y(x)\), where \(x \in \mathbb{R}\), \(y(x) \in \mathbb{C}^N\), and \(y \in (L^2_{\text{loc}}(\mathbb{R}))^N\). The frequency domain \(\Omega\) is an open connected set in \(\mathbb{C}\) and the real frequency domain \(\Omega_\mathbb{R} := \Omega \cap \mathbb{R}\) is nonempty.

Wave propagation will be governed by canonical systems of differential equations with periodic coefficients that depend holomorphically on the frequency parameter \(\omega\), i.e., differential equations of the form

\[
\mathcal{G} y'(x) = V(x, \omega)y(x),
\]

where the leading matrix coefficient \(\mathcal{G} \in M_N(\mathbb{C})\) and the matrix-valued function \(V : \mathbb{R} \times \Omega \to M_N(\mathbb{C})\), the Hamiltonian, have the properties:

(i) \(\mathcal{G}^* = -\mathcal{G}\)

(ii) \(V(x, \omega)^* = V(x, \omega)\), for each \(\omega \in \Omega_\mathbb{R}\) and almost every \(x \in \mathbb{R}\)

(iii) \(V(x + d, \omega) = V(x, \omega)\), for each \(\omega \in \Omega\) and almost every \(x \in \mathbb{R}\)

(iv) \(V \in \mathcal{O}(\Omega, M_N(L^p(\mathbb{T})))\) as a function of frequency where

\[
p = \begin{cases} 
1 & \text{if } \det(\mathcal{G}) \neq 0, \\
2 & \text{if } \det(\mathcal{G}) = 0. 
\end{cases}
\]

In the case \(\det(\mathcal{G}) = 0\), the domain of equation (4.1) and definition of its leading term are

\[
\mathcal{D} := \{y \in (L^2_{\text{loc}}(\mathbb{R}))^N : P_\perp y \in (W^{1,1}_{\text{loc}}(\mathbb{R}))^N\}, \quad \mathcal{G} y' := \mathcal{G}(P_\perp y)',
\]

\(^1\)See section 4.5 for notation and convention.
where $P, P_\perp \in M_N(\mathbb{C})$ denote the projections onto the kernel and range of $\mathcal{G}$, respectively. In the case $\det(\mathcal{G}) \neq 0$, we take the standard domain $\mathcal{D} := (W^{1,1}_{\text{loc}}(\mathbb{R}))^N$ and definition of $y'$ as the weak derivative of the entries of $y$.

The equations of the form (4.1) with $\det(\mathcal{G}) = 0$ are differential-algebraic equations (DAEs) and are not just ODEs. Their solutions must satisfy an algebraic equation as well, namely, $PV(x, \omega)y(x) = 0$ for a.e. $x \in \mathbb{R}$. Thus, in order to guarantee the existence of a nontrivial solution for each frequency and a solution space which depends holomorphically on frequency, we impose an additional hypothesis for equations of the form (4.1) whenever $\det(\mathcal{G}) = 0$, namely, $(\mathcal{G} + PV(x, \omega)P)^{-1}$ exists for each $\omega \in \Omega$ and a.e. $x \in \mathbb{R}$ and as a function of frequency

$$(\mathcal{G} + PV)^{-1} \in \mathcal{O}(\Omega, M_N(L^\infty(0,d))). \quad (4.3)$$

We will refer to this as the index-1 hypothesis.

Differential-algebraic equations of the form given by (4.1), for real frequencies, are known as canonical equations \cite{31} and, in the case $\det(\mathcal{G}) \neq 0$, they are known as canonical differential equations, linear canonical systems, linear Hamiltonian systems, or Hamiltonian equations. We distinguish between the two classes of canonical equations of the form (4.1) by referring to those with $\det(\mathcal{G}) = 0$ as canonical DAEs and those with $\det(\mathcal{G}) \neq 0$ as canonical ODEs.

We introduce the following definitions into our model:

**Definition 18** We say $y$ is a **solution** of equation (4.1) provided $y \in \mathcal{D}$ and equation (4.1) is satisfied for some $\omega \in \Omega$ and for a.e. $x$ in $\mathbb{R}$. In which case we call $\omega$ its **frequency**.

**Definition 19** We say $y$ is a **Floquet solution** provided for some $\omega \in \Omega$, it is a solution
of equation (4.1) with $\omega$ as its frequency and has the Floquet representation
\begin{equation}
    y(x) = e^{ikx} \sum_{j=0}^{m} x^j u_j(x),
\end{equation}
where $m \in \mathbb{N} \cup \{0\}$ and $u_j \in \mathcal{D} \cap (L^p(\mathbb{T}))^N$, for $j = 1, \ldots, m$. We call $k$, $(k, \omega)$, and $\lambda := e^{ikd}$ its wavenumber, wavenumber-frequency pair, and Floquet multiplier, respectively.

If $u_m \neq 0$ then we call $m$ its order. We call $y$ a Bloch solution if it is a Floquet solution with $m = 0$. If, in addition, $y$ is a Bloch solution with real frequency then we call it a propagating wave if it has a real wavenumber and an evanescent wave otherwise.

**Definition 20** We define the Bloch variety to be the set
\begin{equation}
    \mathcal{B} := \{(k, \omega) \in \mathbb{C} \times \Omega \mid (k, \omega) \text{ is the wavenumber-frequency pair of some nontrivial Bloch solution of equation (4.1)}\}
\end{equation}
and the real Bloch variety to be the set
\begin{equation}
    \mathcal{B}_\mathbb{R} := \mathcal{B} \cap \mathbb{R}^2.
\end{equation}

**Definition 21** We define the dispersion relation denoted by $\omega = \omega(k)$ as the multi-valued function whose graph is the Bloch variety. We say a point $(k_0, \omega_0) \in \mathcal{B}_\mathbb{R}$ is amicable if for every $\epsilon > 0$ there exists an $\delta > 0$ such that $\mathcal{B} \cap B(k_0, \delta) \times B(\omega_0, \epsilon)$ is the union of the graphs of a finite number of real analytic functions, say $\omega_1(k), \ldots, \omega_g(k)$, with the property $\omega_1(k_0) = \cdots = \omega_g(k_0) = \omega_0$. We call these functions the band functions and their derivatives $\frac{d\omega_1}{dk}, \ldots, \frac{d\omega_g}{dk}$ the group velocities. The intersection of the graph of a band function with $\mathbb{R}^2$ is called a band.

**Definition 22** For any solution $y$ of equation (4.1) with real frequency $\omega_0$ we define its
energy flux and energy density to be the functions of position given by

\[ \langle iG_y, y \rangle_{\mathbb{C}} \]  

(4.7)

\[ \langle V_\omega(\cdot, \omega_0)y, y \rangle_{\mathbb{C}} \]  

(4.8)

respectively, where \( V_\omega \) denotes the derivative of the Hamiltonian \( V \) with respect to frequency in the \( M_N(L^p(\mathbb{T})) \) norm. We define its average energy flux and average energy density to be the averages

\[ \frac{1}{d} \int_0^d \langle iG_y(x), y(x) \rangle_{\mathbb{C}} dx \]  

(4.9)

\[ \frac{1}{d} \int_0^d \langle V_\omega(x, \omega_0)y(x), y(x) \rangle_{\mathbb{C}} dx \]  

(4.10)

respectively. We also define its energy velocity as the ratio of its average energy flux to its average energy density provided the latter is nonzero, i.e.,

\[ \frac{\frac{1}{d} \int_0^d \langle iG_y(x), y(x) \rangle_{\mathbb{C}} dx}{\frac{1}{d} \int_0^d \langle V_\omega(x, \omega_0)y(x), y(x) \rangle_{\mathbb{C}} dx}. \]  

(4.11)

Remark: The integrals (4.9) and (4.10) are well-defined because Corollary 40, Corollary 43, and Theorem 50 show the functions (4.7) and (4.8) are in \( L_{\text{loc}}^1(\mathbb{R}) \).

Definition 23 We say \((k_0, \omega_0) \in B_{\mathbb{R}}\) is a point of definite type for the canonical equations in (4.1) and say these equations are of definite type at this point provided the average energy density of any nontrivial Bloch solution \( y \) with \((k_0, \omega_0)\) as its wavenumber-frequency pair is nonzero, i.e.,

\[ \frac{1}{d} \int_0^d \langle V_\omega(x, \omega_0)y(x), y(x) \rangle_{\mathbb{C}} dx \neq 0. \]  

(4.12)
The canonical equations in (4.1) along with the definitions just given constitute the model of wave propagation that we will analyze in this thesis. Moreover, based on our studies on electromagnetic wave propagation in one-dimensional photonic crystals, it is found by physical considerations that a point of definite type is the right assumption in which to begin a perturbation analysis from stationary points of the dispersion relation.

We are particularly interested in the perturbation analysis of canonical equations in the slow wave regime in which points in the Bloch variety \((k, \omega)\) belong to a small neighborhood of a point \((k_0, \omega_0)\) in the real Bloch variety where \((k_0, \omega_0)\) is both a point of definite type for the canonical equations (such a point is amicable (see Theorem 48 and Theorem 50)) and a stationary point of the dispersion relation, i.e., at \((k_0, \omega_0)\) one of the bands has a stationary point. For this model and with this regime in mind, we give the following definition for a slow wave:

**Definition 24** Let \((k_0, \omega_0)\) be any point which is amicable and \(0 < r \ll 1\). Then any propagating Bloch wave \(y\) with wavenumber-frequency pair \((k, \omega)\) belonging to a band with a stationary point at \((k_0, \omega_0)\) and satisfying \(0 < \| (k, \omega) - (k_0, \omega_0) \|_C < r\) is called a slow wave.

The main purpose of this chapter is to begin developing a mathematical framework for this model, including the Floquet, spectral, and perturbation theory, and to use this framework to analyze in detail the slow wave regime.

### 4.2 Canonical ODEs

In this section we consider the canonical ODEs

\[
\mathcal{J} \psi'(x) = A(x, \omega) \psi(x), \quad \psi \in (W^{1,1}_{\text{loc}}(\mathbb{R}))^n
\]
where, as discussed in section 4.1.1, the leading matrix coefficient $\mathcal{J} \in M_n(\mathbb{C})$ and the Hamiltonian $A : \mathbb{R} \times \Omega \to M_n(\mathbb{C})$ have the properties:

(i) $\det(\mathcal{J}) \neq 0$, $\mathcal{J}^* = -\mathcal{J}$,

(ii) $A(x, \omega)^* = A(x, \omega)$, for each $\omega \in \Omega$ and a.e. $x \in \mathbb{R}$,

(iii) $A(x + d, \omega) = A(x, \omega)$, for each $\omega \in \Omega$ and a.e. $x \in \mathbb{R}$,

(iv) $A \in \mathcal{O}(\Omega, M_N(L^1(\mathbb{T})))$ as a function of frequency.

Denote by $A_\omega$ the derivative of the Hamiltonian $A$ with respect to frequency in the $M_n(L^1(\mathbb{T}))$ norm.

### 4.2.1 Preliminaries

We begin with some preliminary results on the solutions of the canonical ODEs in (4.13) and the Floquet theory relating to the matricant and monodromy matrix of these canonical ODEs. The dependency of the matricant and monodromy matrix on frequency is also discussed. We will hold off on proving the results of this subsection until section 4.4. The statements in this subsection are known, but we state them here in order to setup the main results of this chapter. We also will give proofs of these statements for completeness in section 4.4.

**Proposition 33** For each $\omega \in \Omega$, let $\Psi(\cdot, \omega)$ denote the fundamental matrix solution of the canonical ODEs in (4.13) satisfying $\Psi(0, \omega) = I_n$, i.e., the unique function $\Psi(\cdot, \omega) \in M_n(W^{1,1}_{loc}(\mathbb{R}))$ satisfying a.e. the matrix differential equation with initial condition

$$\mathcal{J} \Psi'(x) = A(x, \omega) \Psi(x), \quad \Psi(0) = I_n.$$  

(4.14)
Then the matrix-valued function \( \Psi : \mathbb{R} \times \Omega \to M_n(\mathbb{C}) \) has the following properties:

(i) \( \psi \) is a solution of the canonical ODEs in (4.13) at the frequency \( \omega \in \Omega \) if and only if \( \psi = \Psi(\cdot, \omega)\gamma \) for some \( \gamma \in \mathbb{C}^n \). Moreover, \( \gamma \) is unique.

(ii) For every \((x, \omega) \in \mathbb{R} \times \Omega\),

\[
\Psi(x, \omega) = I_n + \int_0^x \mathcal{J}^{-1}A(t, \omega)\Psi(t, \omega)dt. \tag{4.15}
\]

(iii) For every \( x \in \mathbb{R} \), \( \Psi(x, \cdot) \in \mathcal{O}(\Omega, M_n(\mathbb{C})) \). We will denote the partial derivative of the function \( \Psi \) with respect to frequency in the \( M_n(\mathbb{C}) \) norm by \( \Psi_\omega \).

(iv) For every \((x, \omega) \in \mathbb{R} \times \Omega\), \( \Psi_\omega(\cdot, \omega) \in M_n(W^{1,1}_{\text{loc}}(\mathbb{R})) \) and

\[
\Psi_\omega(x, \omega) = \int_0^x \mathcal{J}^{-1}A_\omega(t, \omega)\Psi(t, \omega) + \mathcal{J}^{-1}A(t, \omega)\Psi_\omega(t, \omega)dt \quad \tag{4.16}
\]

(v) For every \((x, \omega) \in \mathbb{R} \times \Omega\), \( \Psi^{-1}(x, \omega) := \Psi(x, \omega)^{-1} \) exists and \( \Psi^{-1}(\cdot, \omega) \in M_n(W^{1,1}_{\text{loc}}(\mathbb{R})) \).

(vi) For every \((x, \omega) \in \mathbb{R} \times \Omega\),

\[
\Psi(x + d, \omega) = \Psi(x, \omega)\Psi(d, \omega). \tag{4.17}
\]

Now for each fixed \( \omega \in \Omega \), \( \Psi(\cdot, \omega) \) and \( \Psi(d, \omega) \) are the matricant and monodromy matrix, respectively, of the canonical ODEs in (4.13). Thus the previous proposition implies the following:

**Theorem 34 (Floquet-Lyapunov)** Let \( \omega \in \Omega \). Then there exists a matrix \( K(\omega) \in M_n(\mathbb{C}) \) and a function \( F(\cdot, \omega) \in M_n(W^{1,1}_{\text{loc}}(\mathbb{R})) \) such that for every \( x \in \mathbb{R} \),

\[
\Psi(x, \omega) = F(x, \omega)e^{ixK(\omega)},
\]

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\[ F(x + d, \omega) = F(x, \omega), \ F(0, \omega) = I_n, \ \ F^{-1}(x, \omega) := F(x, \omega)^{-1} \text{ exists, and } F^{-1}(\cdot, \omega) \in M_n(W_{\text{loc}}^{1,1}(\mathbb{R})). \]

**Theorem 35** If \( \psi \) is a Floquet solution of the canonical ODEs in (4.13) with wavenumber-frequency pair \((k, \omega)\), Floquet multiplier \( \lambda = e^{ikd} \), and order \( m \) then \( \psi = \Psi(\cdot, \omega)\gamma \) where \( \gamma \) is a generalized eigenvector of \( \Psi(d, \omega) \) of order \( m + 1 \) corresponding to the eigenvalue \( \lambda \). Conversely, if \( \gamma \) is a generalized eigenvector of order \( m + 1 \) of the monodromy matrix \( \Psi(d, \omega) \) corresponding to the eigenvalue \( \lambda \) then for any \( k \in \mathbb{C} \) such that \( \lambda = e^{ikd} \), \( \psi = \Psi(\cdot, \omega)\gamma \) is a Floquet solution of the canonical ODEs in (4.13) with wavenumber-frequency pair \((k, \omega)\), Floquet multiplier \( \lambda = e^{ikd} \), and order \( m \).

**Corollary 36** Let \( \omega \in \Omega \). Let \( \{\gamma_j\}_{j=1}^n \) be a Jordan basis for the matrix \( \Psi(d, \omega) \). For \( j = 1, \ldots, n \), let \( l_j, \lambda_j, \) and \( k_j \) be such that \( \gamma_j \) is a generalized eigenvalue of \( \Psi(d, \omega) \) of order \( l_j \) corresponding to the eigenvalue \( \lambda_j = e^{ikjd} \). Define \( \{\psi_j\}_{j=1}^n \) by \( \psi_j := \Psi(\cdot, \omega)\gamma_j \) for \( j = 1, \ldots, n \). Then the following statements are true:

(i) The set of solutions of the canonical ODEs in (4.13) at the frequency \( \omega \) is a vector space over \( \mathbb{C} \) and \( \{\psi_j\}_{j=1}^n \) is a basis for this space.

(ii) For \( j = 1, \ldots, n \), \( \psi_j \) is a Floquet solution of the canonical ODEs in (4.13) with wavenumber-frequency pair \((k_j, \omega)\), Floquet multiplier \( \lambda_j = e^{ikjd} \), and order \( l_j - 1 \).

**Corollary 37** Let \( \mathcal{B} \) denote the Bloch variety of the canonical ODEs in (4.13). Then, for any \( l \in \mathbb{Z} \),

\[ \mathcal{B} = (2\pi l/d, 0) + \mathcal{B}. \]  

(4.18)

**Corollary 38** Define the function \( D : \mathbb{C} \times \Omega \to \mathbb{C} \) by

\[ D(k, \omega) := \det \left( e^{ikd}I_n - \Psi(d, \omega) \right). \]  

(4.19)
Then $D$ is a nonconstant holomorphic function and its zero set is the Bloch variety of the canonical ODEs in (4.13), i.e.,

\[ \mathcal{B} = \{(k, \omega) \in \mathbb{C} \times \Omega \mid D(k, \omega) = 0\}. \quad (4.20) \]

### 4.2.2 Energy Flux, Energy Density, and their Averages

The following results describe how the energy flux, energy density, their averages, and points of definite type of the canonical ODEs in (4.13) are related to the leading matrix coefficient, the matricant, and the monodromy matrix.

We start by considering the energy flux and its average. The following proposition is the key result, as we will see in this chapter, in the study of energy flux for canonical equations.

**Proposition 39** For each $\omega \in \Omega_\mathbb{R}$ we have $\Psi(\cdot, \omega)^* i \mathcal{J} \Psi(\cdot, \omega) \in M_n(W^{1,1}_{\text{loc}}(\mathbb{R}))$ and

\[ \Psi(\cdot, \omega)^* i \mathcal{J} \Psi(\cdot, \omega) = i \mathcal{J}. \quad (4.21) \]

This proposition is important because it tells us the energy flux of any solution of canonical ODEs with real frequency is a conserved quantity. A more precise statement of this is found in the next corollary.

**Corollary 40** For each $\omega \in \Omega_\mathbb{R}$ and every $\gamma_1, \gamma_2 \in \mathbb{C}^n$, if $\psi_1 = \Psi(\cdot, \omega) \gamma_1$ and $\psi_2 = \Psi(\cdot, \omega) \gamma_2$ then $\langle i \mathcal{J} \psi_1, \psi_2 \rangle_{\mathbb{C}} \in W^{1,1}_{\text{loc}}(\mathbb{R})$ and

\[ \langle i \mathcal{J} \psi_1, \psi_2 \rangle_{\mathbb{C}} = \langle i \mathcal{J} \gamma_1, \gamma_2 \rangle_{\mathbb{C}}, \quad (4.22) \]

\[ \frac{1}{d} \int_0^d \langle i \mathcal{J} \psi_1(x), \psi_2(x) \rangle_{\mathbb{C}} dx = \langle i \mathcal{J} \gamma_1, \gamma_2 \rangle_{\mathbb{C}}. \quad (4.23) \]

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We will now consider the energy density and its average.

**Lemma 41** For each \( \omega \in \Omega_\mathbb{R} \) we have \( A_\omega(\cdot, \omega) \in M_n(L^1(\mathbb{T})) \) and

\[
A_\omega(\cdot, \omega)^* = A_\omega(\cdot, \omega). \tag{4.24}
\]

**Theorem 42** For each \( \omega \in \Omega_\mathbb{R} \) we have

\[
\Psi(\cdot, \omega)^* \mathcal{J} \Psi(\cdot, \omega) \in M_n(W^{1,1}_\text{loc}(\mathbb{R})),
\]

\[
\Psi(\cdot, \omega)^* A_\omega(\cdot, \omega) \Psi(\cdot, \omega) \in M_n(L^1_{\text{loc}}(\mathbb{R})),
\]

\[
\Psi(\cdot, \omega)^* A_\omega(\cdot, \omega) \Psi(\cdot, \omega) = (\Psi(\cdot, \omega)^* \mathcal{J} \Psi(\cdot, \omega))', \tag{4.25}
\]

\[
\frac{1}{d} \int_0^d \Psi(t, \omega)^* A_\omega(t, \omega) \Psi(t, \omega) dt = \frac{1}{d} \Psi(d, \omega)^* \mathcal{J} \Psi(d, \omega). \tag{4.26}
\]

**Corollary 43** For each \( \omega \in \Omega_\mathbb{R} \) and every \( \gamma_1, \gamma_2 \in \mathbb{C}^n \), if \( \psi_1 = \Psi(\cdot, \omega) \gamma_1 \) and \( \psi_2 = \Psi(\cdot, \omega) \gamma_2 \) then \( \langle \Psi(\cdot, \omega)^* \mathcal{J} \Psi(\cdot, \omega) \gamma_1, \gamma_2 \rangle_\mathbb{C} \in W^{1,1}_\text{loc}(\mathbb{R}), \langle A_\omega(\cdot, \omega) \psi_1, \psi_2 \rangle_\mathbb{C} \in L^1_{\text{loc}}(\mathbb{R}) \) and

\[
\langle A_\omega(\cdot, \omega) \psi_1, \psi_2 \rangle_\mathbb{C} = \langle \Psi(\cdot, \omega)^* \mathcal{J} \Psi(\cdot, \omega) \gamma_1, \gamma_2 \rangle_\mathbb{C}', \tag{4.27}
\]

\[
\frac{1}{d} \int_0^d \langle A_\omega(x, \omega) \psi_1(x), \psi_2(x) \rangle_\mathbb{C} dx = \frac{1}{d} \langle \Psi(d, \omega)^* \mathcal{J} \Psi(d, \omega) \gamma_1, \gamma_2 \rangle_\mathbb{C}. \tag{4.28}
\]

**4.2.3 On Points of Definite Type for Canonical ODEs**

Let \( (k, \omega) \in \mathcal{B}_\mathbb{R} \) and on the eigenspace of the monodromy matrix \( \Psi(d, \omega) \) corresponding to the eigenvalue \( e^{ikd} \) we define the sesquilinear form \( q_{(k, \omega)} \) by

\[
q_{(k, \omega)}(\gamma_1, \gamma_2) := \frac{1}{d} \int_0^d \langle A_\omega(x, \omega) \psi_1(x), \psi_2(x) \rangle_\mathbb{C} dx = \frac{1}{d} \langle \Psi(d, \omega)^* \mathcal{J} \Psi(d, \omega) \gamma_1, \gamma_2 \rangle_\mathbb{C}, \tag{4.31}
\]

\[
\gamma_1, \gamma_2 \in \ker(e^{ikd}I_n - \Psi(d, \omega)), \quad \psi_1 = \Psi(\cdot, \omega) \gamma_1, \quad \psi_2 = \Psi(\cdot, \omega) \gamma_2.
\]

It follows from Corollary 43 that \( q_{(k, \omega)} \) is well-defined.
Lemma 44 For each \((k, \omega) \in \mathcal{B}_\mathbb{R}\), \(q_{(k, \omega)}\) is a Hermitian form, that is, \(q_{(k, \omega)}\) satisfies

\[
q_{(k, \omega)}(\gamma_1, \gamma_2) = \overline{q_{(k, \omega)}(\gamma_2, \gamma_1)}
\]

(4.32)

for every \(\gamma_1, \gamma_2 \in \ker(e^{ikd}I_n - \Psi(d, \omega))\). In particular, \(q_{(k, \omega)}(\gamma, \gamma) \in \mathbb{R}\) for every \(\gamma \in \ker(e^{ikd}I_n - \Psi(d, \omega))\).

Denote the sign function by \(\text{sgn}(x) := \frac{x}{|x|} \in \{-1, 1\}\), for \(x \in \mathbb{R}\)\/{0}\).

Proposition 45 The canonical ODEs in (4.13) are of definite type at a point \((k_0, \omega_0) \in \mathcal{B}_\mathbb{R}\) if and only if \(q_{(k_0, \omega_0)}\) is a definite sesquilinear form which is bounded, i.e., the following properties hold:

(i) For every nonzero \(\gamma \in \ker(e^{ikd}I_n - \Psi(d, \omega))\) we have

\[
q_{(k_0, \omega_0)}(\gamma, \gamma) \in \mathbb{R}/\{0\}
\]

(4.33)

and its sign \(\text{sgn}(q_{(k_0, \omega_0)}(\gamma, \gamma)) =: \text{sgn}(q_{(k_0, \omega_0)})\) is independent of the choice of \(\gamma\).

(ii) The sesquilinear form

\[
\langle \gamma_1, \gamma_2 \rangle_{(k_0, \omega_0)} := \text{sgn}(q_{(k_0, \omega_0)})q_{(k_0, \omega_0)}(\gamma_1, \gamma_2),
\]

(4.34)

\(\gamma_1, \gamma_2 \in \ker(e^{ikd}I_n - \Psi(d, \omega_0))\)

is inner product on \(\ker(e^{ikd}I_n - \Psi(d, \omega_0))\).

(iii) There exists constants \(C_1, C_2 > 0\) such that

\[
C_1||\gamma||^2_C \leq |q_{(k_0, \omega_0)}(\gamma, \gamma)| \leq C_2||\gamma||^2_C, \text{ for all } \gamma \in \ker(e^{ikd}I_n - \Psi(d, \omega_0)).
\]

(4.35)
Lemma 46  The family of sesquilinear forms \( \{ q_{(k, \omega)} \}_{(k, \omega) \in \mathcal{B}_R} \) depends continuously on the indexing parameter \((k, \omega)\) in the following sense:

If \( \{ (k_j, \omega_j) \}_{j \in \mathbb{N}} \subseteq \mathcal{B}_R, \{ \gamma_j \}_{j \in \mathbb{N}} \subseteq \mathbb{C}^n \) such that \( \gamma_j \in \ker(e^{ik_jd}I_n - \Psi(d, \omega_j)) \) with \( ||\gamma_j||_C = 1 \) for all \( j \in \mathbb{N} \) and

\[
(k_j, \omega_j) \xrightarrow{||||_C} (k_0, \omega_0) \quad \text{and} \quad \gamma_j \xrightarrow{||||_C} \gamma_0, \quad \text{as } j \to \infty
\]

then \( (k_0, \omega_0) \in \mathcal{B}_R, \gamma_0 \in \ker(e^{ik_0d}I_n - \Psi(d, \omega_0)) \) with \( ||\gamma_0||_C = 1 \) and

\[
q_{(k_j, \omega_j)}(\gamma_j, \gamma_j) \xrightarrow{||} q_{(k_0, \omega_0)}(\gamma_0, \gamma_0) \quad \text{as } j \to \infty.
\]  

(4.36)

Theorem 47  If \((k_0, \omega_0) \in \mathcal{B}_R\) is a point of definite type for the canonical ODEs in (4.13) then there exists an \( r > 0 \) such that every \((k, \omega) \in B((k_0, \omega_0), r) \cap \mathcal{B}_R\) is a point of definite type and

\[
\sgn(q_{(k, \omega)}) = \sgn(q_{(k_0, \omega_0)}).
\]  

(4.37)

4.2.4  Perturbation Theory for Canonical ODEs

This section contains the main results of this chapter on the perturbation theory for canonical ODEs.

Theorem 48  Suppose the canonical ODEs in (4.13) are of definite type at \((k_0, \omega_0) \in \mathcal{B}_R\). Let \( g \) be the number of Jordan blocks (geometric multiplicity) in the Jordan form of the monodromy matrix \( \Psi(d, \omega_0) \) corresponding to the eigenvalue \( \lambda_0 = e^{ik_0d} \) and \( m_1 \geq \cdots \geq m_g \geq 1 \) the dimensions of each of those Jordan blocks (partial multiplicities). Define \( m := m_1 + \cdots + m_g \) (algebraic multiplicity). Let \( \epsilon > 0 \) be given. Then there exists an \( \delta > 0 \) such
(i) The order of the zero of $D(k_0, \omega)$ at $\omega = \omega_0$ is $g$ and the order of the zero of $D(k, \omega_0)$ at $k = k_0$ is $m$.

(ii) The set $\mathcal{B} \cap B(k_0, \delta) \times B(\omega_0, \epsilon)$ is the union of the graphs of $g$ nonconstant real analytic functions $\omega_1(k), \ldots, \omega_g(k)$ given by the convergent power series

$$\omega_j(k) = \omega_0 + \nu_{j,m_j}(k - k_0)^{m_j} + \sum_{l = m_j + 1}^{\infty} \nu_{j,l}(k - k_0)^l, \quad |k - k_0| < \delta$$

where

$$\nu_{j,m_j} \neq 0,$$  \hspace{1cm}  (4.39)

for $j = 1, \ldots, g$. Moreover, there exists analytic functions $\varphi_1(k), \ldots, \varphi_g(k)$ belonging to $O(B(k_0, \delta), \mathbb{C}^n)$ such that

$$\varphi_j(k) \neq 0, \quad \Psi(d, \omega_j(k))\varphi_j(k) = e^{ikd}\varphi_j(k), \quad |k - k_0| < \delta, \quad j = 1, \ldots, g,$$  \hspace{1cm}  (4.40)

the vectors $\varphi_1(k), \ldots, \varphi_g(k)$ are linearly independent for $|k - k_0| < \delta$, and, in particular, the vectors $\varphi_1(k_0), \ldots, \varphi_g(k_0)$ form a basis for $\ker(e^{ikd}I_n - \Psi(d, \omega_0))$.

**Corollary 49** The conditions

$$D(k_0, \omega_0) = 0, \quad \frac{\partial D}{\partial \omega}(k_0, \omega_0) \neq 0, \quad (k_0, \omega_0) \in \mathbb{R} \times \Omega_\mathbb{R}$$  \hspace{1cm}  (4.41)

are satisfied if and only if $(k_0, \omega_0) \in \mathcal{B}_\mathbb{R}$ is a point of definite type for the canonical ODEs in (4.13) and the Jordan normal form of the monodromy matrix $\Psi(d, \omega_0)$ corresponding to the eigenvalue $\lambda_0 = e^{ikd}$ consists of a single Jordan block, i.e., $\dim \ker(\lambda_0 I_n - \Psi(d, \omega_0)) = 1$.  

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Remark. In this case, by denoting \( \lambda_0 := e^{ik_0 d} \), the conditions of the previous corollary are
\[
\det(\lambda_0 I_n - \Psi(d,\omega)) = D(k_0,\omega) = 0 \quad \text{and} \quad \frac{\partial}{\partial \omega} \det(\lambda I_n - \Psi(d,\omega))|_{(\omega_0,\lambda_0)} = \frac{\partial D}{\partial \omega}(k_0,\omega_0) \neq 0,
\]
but this is just the generic condition (2.1) in chapter 2 (see also [61, See (1.1)]). This condition was thoroughly investigated in chapter 2 (see also [61, See (1.1)]) for general matrix perturbations. And, in particular, in Theorem 2 of chapter 2 (see also [61, Theorem 3.1]) explicit recursive formulas are given for the unique eigenvalue Puiseux series \( \lambda(\omega) \) which satisfies \( \lambda(\omega_0) = \lambda_0 \) and for its associated eigenvector Puiseux series of the monodromy matrix \( \Psi(d,\omega) \) near the frequency \( \omega = \omega_0 \). Moreover, their Puiseux series coefficients up to the second order are conveniently listed in Corollary 4 of chapter 2 (see also [61, Corollary 3.3]).

4.3 Canonical DAEs

In this section we consider the canonical DAEs

\[
\mathcal{G} \dot{y}(x) = V(x,\omega)y(x), \quad y \in \mathcal{D}
\]

(4.42)

where, as discussed in §1, the leading matrix coefficient \( \mathcal{G} \in M_n(\mathbb{C}) \) and the Hamiltonian \( V : \mathbb{R} \times \Omega \to M_N(\mathbb{C}) \) have the properties:

(i) \( \det(\mathcal{G}) = 0, \mathcal{G}^* = -\mathcal{G} \),

(ii) \( V(x,\omega)^* = V(x,\omega) \), for each \( \omega \in \Omega_\mathbb{R} \) and a.e. \( x \in \mathbb{R} \),

(iii) \( V(x + d,\omega) = V(x,\omega) \), for each \( \omega \in \Omega \) and a.e. \( x \in \mathbb{R} \),

(iv) \( V \in \mathcal{O}(\Omega, M_N(L^2(\mathbb{T}))) \) as a function of frequency.
Denote by $V_\omega$ the derivative of the Hamiltonian $V$ with respect to frequency in the $M_n(L^2(\mathbb{T}))$ norm.

The domain $\mathcal{D}$ of the canonical DAEs in (4.42) and definition of its leading term are

$$\mathcal{D} := \{ y \in (L^2_{\text{loc}}(\mathbb{R}))^N : P_\perp y \in (W^{1,1}_{\text{loc}}(\mathbb{R}))^N \}, \quad \mathcal{G}y' := \mathcal{G}(P_\perp y)' \quad \text{(4.43)}$$

where $P, P_\perp \in M_N(\mathbb{C})$ denote the projections onto the kernel and range of $\mathcal{G}$, respectively.

We shall assume that the canonical DAEs in (4.42) satisfy the index-1 hypothesis, namely,

$$(\mathcal{G} + PVP)^{-1} \in \mathcal{O}(\Omega, M_N(L^\infty(0,d))). \quad \text{(4.44)}$$

The goal of this section is to describe the theory of canonical DAEs including the Floquet and spectral perturbation theory in terms of the theory developed in the previous section for canonical ODEs. We will only present the statement of our main result, the proof and consequences will appear in [60].

### 4.3.1 The Correspondence between Canonical DAEs and Canonical ODEs

The following represents our main result in the study of canonical DAEs:

**Theorem 50** Let $n := \dim \text{ran}(\mathcal{G})$. Then there exists an $n \times n$ system of canonical ODEs

$$\mathcal{J} \psi'(x) = A(x,\omega)\psi(x), \quad \psi \in (W^{1,1}_{\text{loc}}(\mathbb{R}))^n \quad \text{(4.45)}$$

with $\mathcal{J} \in M_n(\mathbb{C})$, $A \in \mathcal{O}(\Omega, M_n(L^1(\mathbb{T})))$, and a function $Q \in \mathcal{O}(\Omega, M_{N \times n}(L^2(\mathbb{T})))$ such that the multiplication map $Q(\cdot, \omega) : (W^{1,1}_{\text{loc}}(\mathbb{R}))^n \to \mathcal{D}$ is injective and the following properties
hold:

(i) The solutions of the canonical DAEs in (4.42) with frequency \( \omega \) are given by \( y = Q(\cdot, \omega) \psi \) where \( \psi \) is a solution of the canonical ODEs in (4.45) with frequency \( \omega \).

(ii) If \( y = Q(\cdot, \omega) \psi \) then \( y \) is Floquet solution of the canonical DAEs with wavenumber-frequency pair \((k, \omega)\), Floquet multiplier \( \lambda = e^{ikd} \), and order \( m \) if and only if \( \psi \) is a Floquet solution of the canonical ODEs with wavenumber-frequency pair \((k, \omega)\), Floquet multiplier \( \lambda = e^{ikd} \), and order \( m \).

(iii) If \( y = Q(\cdot, \omega_0) \psi \) is a solution of the canonical DAEs with frequency \( \omega_0 \in \Omega_{\mathbb{R}} \) then its energy flux and energy density are in \( L^{_1}_{\text{loc}}(\mathbb{R}) \) and

\[
\langle i \mathcal{G} y, y \rangle_{\mathbb{C}} = \langle i J \psi, \psi \rangle_{\mathbb{C}}, \quad \langle V_{\omega}(\cdot, \omega_0) y, y \rangle_{\mathbb{C}} = \langle A_{\omega}(\cdot, \omega_0) \psi, \psi \rangle_{\mathbb{C}}. \tag{4.46}
\]

To summarize this theorem, the study of canonical DAEs (with index-1 hypothesis), including the Floquet, spectral, and perturbation theory, is reduced to the study of canonical ODEs and the theory already developed in this chapter. In \([60]\) we will elaborate on this correspondence in more detail and discuss the spectral perturbation theory as well.

### 4.4 Proofs

#### Proofs for Section 4.2.1

We begin by introducing a differential operator associated to the canonical ODEs in (4.13).

We consider the frequency dependent operator

\[
T(\omega) \psi := \psi' - \mathcal{J}^{-1} A(\cdot, \omega) \psi, \quad \psi \in (W^{1,1}_{\text{loc}}(\mathbb{R}))^n, \quad \omega \in \Omega. \tag{4.47}
\]
Lemma 51  For each \( \omega \in \Omega \), \( T(\omega) \) is a linear operator with \( T(\omega) : (W^{1,1}_{\text{loc}}(\mathbb{R}))^n \to (L^1_{\text{loc}}(\mathbb{R}))^n \).

Proof.  We begin by proving the operator \( T(\omega) : (W^{1,1}_{\text{loc}}(\mathbb{R}))^n \to (L^1_{\text{loc}}(\mathbb{R}))^n \) given by (4.47) is well-defined. Let \( \psi \in (W^{1,1}_{\text{loc}}(\mathbb{R}))^n \). Then by definition of the space \( (W^{1,1}_{\text{loc}}(\mathbb{R}))^n \) we have \( \psi' \in (L^1_{\text{loc}}(\mathbb{R}))^n \). We now show \( J^{-1}A(\cdot, \omega)\psi \in (L^1_{\text{loc}}(\mathbb{R}))^n \). To do this let \((a, b) \subseteq \mathbb{R}\) be any bounded interval. It follows from Lemma 88 that \( A(\cdot, \omega)|_{(a, b)} \in M_n(L^1(a, b)) \) and from Lemma 69 that \( \psi|_{(a, b)} \in (W^{1,1}(a, b))^n \). It then follows from Lemma 83 that \( A(\cdot, \omega)|_{(a, b)}\psi|_{(a, b)} \in (L^1(a, b))^n \). From this it follows that \( (J^{-1}A(\cdot, \omega)\psi)|_{(a, b)} = J^{-1}A(\cdot, \omega)|_{(a, b)}\psi|_{(a, b)} \in (L^1(a, b))^n \) for every bounded interval \((a, b) \subseteq \mathbb{R}\), this implies \( J^{-1}A(\cdot, \omega)\psi \in (L^1_{\text{loc}}(\mathbb{R}))^n \). And hence \( \psi' - J^{-1}A(\cdot, \omega)\psi \in (L^1_{\text{loc}}(\mathbb{R}))^n \). This proves that the operator \( T(\omega) : (W^{1,1}_{\text{loc}}(\mathbb{R}))^n \to (L^1_{\text{loc}}(\mathbb{R}))^n \) given by (4.47) is well-defined. Therefore, since it's obvious the operator \( T(\omega) \) is linear, the proof is complete.

Lemma 52  \( \psi \) is a solution of the canonical ODEs in (4.13) with frequency \( \omega \in \Omega \) if and only if \( \psi \in \ker(T(\omega)) \).

Proof.  Let \( \psi \) is a solution of the canonical ODEs in (4.13) with frequency \( \omega \in \Omega \). This means by Definition 18 that \( \psi \in (W^{1,1}_{\text{loc}}(\mathbb{R}))^n \) and for a.e. \( x \in \mathbb{R} \) \( \psi \) satisfies the equation

\[
J\psi'(x) = A(x, \omega)\psi(x).
\]

Hence for a.e. \( x \in \mathbb{R} \) satisfies the equation

\[
\psi'(x) = J^{-1}A(x, \omega)\psi(x).
\]
But this implies, since as we showed in the proof of Lemma 4.47 that \( J^{-1}A(\cdot, \omega)\psi \in (L^1_{\text{loc}}(\mathbb{R}))^n \), that \( T(\omega)\psi = \psi' - J^{-1}A(\cdot, \omega)\psi = 0 \) in \((L^1_{\text{loc}}(\mathbb{R}))^n\). This proves \( \psi \in \ker(T(\omega)) \).

Conversely, if \( \psi \in \ker(T(\omega)) \) then \( \psi \in (W^{1,1}_{\text{loc}}(\mathbb{R}))^n \) and \( 0 = T(\omega)\psi = \psi' - J^{-1}A(\cdot, \omega)\psi \) in \((L^1_{\text{loc}}(\mathbb{R}))^n\). This implies that for a.e. \( x \in \mathbb{R} \), \( \psi \) satisfies the equation

\[
\psi'(x) = J^{-1}A(x, \omega)\psi(x),
\]

or equivalently, the equation

\[
\mathcal{J}\psi'(x) = A(x, \omega)\psi(x).
\]

Therefore by Definition 18 this means \( \psi \) is a solution of the canonical ODEs in (4.13) with frequency \( \omega \in \Omega \). This completes the proof.

Let \((a, b) \subseteq \mathbb{R}\) be a bounded interval. Then we can consider \((W^{1,1}(a, b))^n\) and \((L^1(a, b))^n\) as a subspace of \((W^{1,1}_{\text{loc}}(\mathbb{R}))^n\) and \((L^1_{\text{loc}}(\mathbb{R}))^n\), respectively, by identifying them with the image of those latter spaces under the surjective map \( f \mapsto f|_{(a,b)} \). Then we can restrict the domain of \( T(\omega) \) to the subspace \((W^{1,1}(a, b))^n\) and it becomes a differential operator on \((W^{1,1}(a, b))^n\) with range in \((L^1(a, b))^n\). It is with this perspective in mind that we introduce and consider the following frequency dependent operator

\[
T|_{(a,b)}(\omega)\psi := \psi' - J^{-1}A(\cdot, \omega)|_{(a,b)}\psi, \quad \psi \in (W^{1,1}(a, b))^n, \quad \omega \in \Omega. \quad (4.48)
\]

**Proposition 53** For each \( \omega \in \Omega \) we have

\[
T|_{(a,b)}(\omega) : (W^{1,1}(a, b))^n \to (L^1(a, b))^n \quad (4.49)
\]
Moreover,

\[ T \big|_{(a,b)}(\omega) \in \mathcal{L}((W^{1,1}(a, b))^n, (L^1(a, b))^n) \quad (4.50) \]
\[ T \big|_{(a,b)} \in \mathcal{O}(\Omega, \mathcal{L}((W^{1,1}(a, b))^n, (L^1(a, b))^n)). \quad (4.51) \]

Moreover,

\[ (T(\omega)\psi) \big|_{(a,b)} = T \big|_{(a,b)}(\omega)\psi \big|_{(a,b)}; \quad (4.52) \]

for every \( \omega \in \Omega \), every \( \psi \in (W^{1,1}_{loc}(\mathbb{R}))^n \), and every bounded interval \( (a, b) \subseteq \mathbb{R} \).

**Proof.** To begin we note that by hypothesis \( A \in \mathcal{O}(\Omega, M_n(L^1(\mathbb{T}))) \). Let \( (a, b) \subseteq \mathbb{R} \) be any bounded interval. It follows from Lemma 88 that for each \( \omega \in \Omega \), \( A \big|_{(a,b)}(\cdot, \omega) := A(\cdot, \omega) \big|_{(a,b)} \in M_n(L^1(a, b)) \). It then follows from Lemma 93 that \( A \big|_{(a,b)} \in \mathcal{O}(\Omega, M_n(L^1(a, b))) \).

It then follows from Lemma 89 with \( p = \infty, q = s = 1 \) that we have \( \tilde{A} := \mathcal{J}^{-1}A \big|_{(a,b)} \in \mathcal{O}(\Omega, M_n(L^1(a, b))) \). But according to our definition we have

\[ T \big|_{(a,b)}(\omega)\psi = \psi' - \tilde{A}(\cdot, \omega)\psi, \quad \psi \in (W^{1,1}(a, b))^n, \quad \omega \in \Omega. \]

with \( \tilde{A} \in \mathcal{O}(\Omega, M_n(L^1(a, b))) \). It now follows from the fact \( (\cdot)' \in \mathcal{L}((W^{1,1}(a, b))^n, (L^1(a, b))^n) \) and Lemma 83 that \( T \big|_{(a,b)}(\omega) : (W^{1,1}(a, b))^n \to (L^1(a, b))^n \) is a well-defined linear operator and \( T \big|_{(a,b)}(\omega) \in \mathcal{L}((W^{1,1}(a, b))^n, (L^1(a, b))^n) \), for each \( \omega \in \Omega \). It follows now from both Lemma 79 and Lemma 90 that \( T \big|_{(a,b)} \in \mathcal{O}(\Omega, \mathcal{L}((W^{1,1}(a, b))^n, (L^1(a, b))^n)) \).

Now let \( \omega \in \Omega \), \( \psi \in (W^{1,1}_{loc}(\mathbb{R}))^n \), and \( (a, b) \subseteq \mathbb{R} \) be a bounded interval. Then by Lemma 69 we have \( \psi \big|_{(a,b)} \in (W^{1,1}(a, b))^n \) and so \( T \big|_{(a,b)}(\omega)\psi \big|_{(a,b)} \in (L^1(a, b))^n \). But \( (T(\omega)\psi) \big|_{(a,b)} \in (L^1(a, b))^n \) since \( T(\omega)\psi \in (L^1_{loc}(\mathbb{R}))^n \) and by Lemma 69 we have \( \psi' \big|_{(a,b)} = \psi' \big|_{(a,b)} \). Thus we conclude that

\[ (T(\omega)\psi) \big|_{(a,b)} = (\psi' - \mathcal{J}^{-1}A(\cdot, \omega)\psi) \big|_{(a,b)} = \psi' \big|_{(a,b)} - \mathcal{J}^{-1}A(\cdot, \omega) \big|_{(a,b)} \psi \big|_{(a,b)} \]
\[ T|_{(a,b)}(\omega) = T|_{(a,b)}(\omega). \]

And hence we have shown that \((T(\omega)\psi)|_{(a,b)} = T|_{(a,b)}(\omega)\psi|_{(a,b)}\) for every \(\omega \in \Omega\), every \(\psi \in (W^{1,1}_{\text{loc}}(\mathbb{R}))^n\), and every bounded interval \((a, b) \subseteq \mathbb{R}\). This completes the proof. ■

**Corollary 54** Let \(\omega \in \Omega\). Then the following statements are equivalent:

(i) \(\psi\) is a solution of the canonical ODEs in (4.13) with frequency \(\omega\).

(ii) \(\psi \in \ker(T(\omega))\).

(iii) \(\psi \in (W^{1,1}_{\text{loc}}(\mathbb{R}))^n\) and \(\psi|_{(a,b)} \in \ker(T|_{(a,b)}(\omega))\) every bounded interval \((a, b) \subseteq \mathbb{R}\).

**Proof.** We have already proven in Lemma 52 that statement (i) is equivalent to statement (ii). We now complete the proof of this corollary by proving statement (ii) is equivalent to statement (iii). Suppose \(\psi \in \ker(T(\omega))\) for some \(\omega \in \Omega\). Then \(\psi \in (W^{1,1}_{\text{loc}}(\mathbb{R}))^n\) and for any bounded interval \((a, b) \subseteq \mathbb{R}\) by Proposition 53 we have \(0 = (T(\omega)\psi)|_{(a,b)} = T|_{(a,b)}(\omega)\psi|_{(a,b)}\) implying \(\psi|_{(a,b)} \in \ker(T|_{(a,b)}(\omega))\). Conversely, if \(\psi \in (W^{1,1}_{\text{loc}}(\mathbb{R}))^n\) and \(\psi|_{(a,b)} \in \ker(T|_{(a,b)}(\omega))\) for every bounded interval \((a, b) \subseteq \mathbb{R}\) then by Proposition 53 this implies \(0 = T|_{(a,b)}(\omega)\psi|_{(a,b)} = (T(\omega)\psi)|_{(a,b)}\) for every bounded interval \((a, b) \subseteq \mathbb{R}\). But since \(T(\omega)\psi \in (L^1_{\text{loc}}(\mathbb{R}))^n\) this implies \(T(\omega)\psi = 0\) in \((L^1_{\text{loc}}(\mathbb{R}))^n\) and therefore \(\psi \in \ker(T(\omega))\). This completes the proof. ■

In the following proposition, we collect together some facts from [45, §II.2.5] that will be need in this chapter.

**Proposition 55** For every bounded interval \((a, b) \subseteq \mathbb{R}\) with \(0 \in (a, b)\) and for each \(\omega \in \Omega\), there exists a unique function \(\Psi|_{(a,b)}(\cdot, \omega) \in M_n(W^{1,1}(a,b))\) satisfying for a.e. \(x \in (a,b)\) the
matrix differential equation with initial condition

\[
\Psi'(x) = \mathcal{J}^{-1}A(x, \omega)\Psi(x), \quad \Psi(0) = I_n. \tag{4.53}
\]

Moreover, the following statements are true:

(i) If \( \psi \in \ker(T|_{(a,b)}(\omega)) \) then there exists a unique \( \gamma \in \mathbb{C}^n \) such that \( \psi = \Psi|_{(a,b)}(\cdot, \omega)\gamma \).

Conversely, if \( \gamma \in \mathbb{C}^n \) and we define \( \psi := \Psi|_{(a,b)}(\cdot, \omega)\gamma \) then \( \psi \in \ker(T|_{(a,b)}(\omega)) \).

(ii) For every \( (x, \omega) \in [a,b] \times \Omega \),

\[
\Psi|_{(a,b)}(x, \omega) = I_n + \int_0^x \mathcal{J}^{-1}A(t, \omega)\Psi|_{(a,b)}(t, \omega)dt. \tag{4.54}
\]

(iii) As a function of frequency, \( \Psi|_{(a,b)} \in \mathcal{O}(\Omega, M_n(W^{1,1}(a,b))) \).

(iv) For every \( (x, \omega) \in [a,b] \times \Omega \), \( \Psi^{-1}|_{(a,b)}(x, \omega) := \Psi|_{(a,b)}(x, \omega)^{-1} \) exists and as function of position \( \Psi^{-1}|_{(a,b)}(\cdot, \omega) \in M_n(W^{1,1}(a,b)) \).

**Proof.** Let us fix a bounded interval \( (a, b) \subseteq \mathbb{R} \) containing 0. We proceed by proving the existence and uniqueness portions of this proposition first and then we will prove statements (i)–(iv).

We begin by reminding the reader that, by the proof of Proposition 53, we have

\[
T|_{(a,b)}(\omega)\psi = \psi' - \tilde{A}(\cdot, \omega)\psi, \quad \psi \in (W^{1,1}(a, b))^n, \quad \omega \in \Omega.
\]

with \( \tilde{A} \in \mathcal{O}(\Omega, M_n(L^1(a,b))) \) where \( \tilde{A}(\cdot, \omega) := \mathcal{J}^{-1}A(\cdot, \omega)|_{(a,b)} \) for \( \omega \in \Omega \). Following [45, p. 69, §II.2.5, Definition 2.5.2], we introduce two new definitions.

**Definition 25** Let \( \omega_0 \in \Omega \). A matrix \( Y_0 \in M_n(W^{1,1}(a,b)) \) is called a **fundamental matrix** of \( T|_{(a,b)}(\omega_0) \psi = 0 \) if for each \( \psi \in \ker(T|_{(a,b)}(\omega_0)) \) there exists a \( \gamma \in \mathbb{C}^n \) such that \( \psi = Y_0\gamma \).
A matrix function \( Y : \Omega \to M_n(W^{1,1}(a, b)) \) is called a fundamental matrix function of \( T|_{(a,b)} \psi = 0 \) if \( Y(\omega) \) is a fundamental matrix of \( T|_{(a,b)}(\omega) \psi \) for each \( \omega \in \Omega \).

We now prove the existence portion of this proposition. By [45, p. 69, §II.2.5, Theorem 2.5.3] there exists a fundamental matrix function \( Y \in \mathcal{O}(\Omega, M_n(W^{1,1}(a, b))) \) of \( T|_{(a,b)} \psi = 0 \) with the property that for all \( \omega \in \Omega \), \( Y(a, \omega) = I_n \) and \( Y(\cdot, \omega) \) is invertible in \( M_n(W^{1,1}(a, b)) \).

Hence for each \( \omega \in \Omega \), there exists \( Y^{-1}(\cdot, \omega) \in M_n(W^{1,1}(a, b)) \) such that \( Y^{-1}(\cdot, \omega)Y(\cdot, \omega) = Y(\cdot, \omega)Y^{-1}(\cdot, \omega) = I_n \). But the functions \( Y(\cdot, \omega), Y^{-1}(\cdot, \omega) : [a, b] \to M_n(\mathbb{C}) \) are continuous implying \( Y(x, \omega)Y^{-1}(x, \omega) = I_n \) for all \( x \in [a, b] \). Thus \( Y(x, \omega) \) is invertible for all \( x \in [a, b] \) and \( Y^{-1}(x, \omega) = Y(x, \omega)^{-1} \).

Now for each \( \omega \in \Omega \), we define \( \Psi|_{(a,b)}(\cdot, \omega) := Y(\cdot, \omega)Y^{-1}(0, \omega) \). It follows by [45, p. 71, §II.2.5, Proposition 2.5.4] that \( \Psi|_{(a,b)}(\cdot, \omega) \in M_n(W^{1,1}(a, b)) \) is a fundamental matrix of \( T|_{(a,b)}(\omega) \psi = 0 \) for each \( \omega \in \Omega \). It then follows from [45, p. 72, §II.2.5, Corollary 2.5.5] that \( \Psi|_{(a,b)}(\cdot, \omega) \) is a solution of the equation

\[
\Psi' - \tilde{A}(\cdot, \omega)\Psi = 0
\]

in \( M_n(W^{1,1}(a, b)) \) for each \( \omega \in \Omega \). But this means for each \( \omega \in \Omega \), \( \Psi|_{(a,b)}(\cdot, \omega) \) satisfies for a.e. \( x \in (a, b) \) the matrix differential equation with initial condition

\[
\Psi'(x) = \mathcal{J}^{-1}A(x, \omega)\Psi(x), \quad \Psi(0) = I_n.
\]

This proves existence.

We now prove the uniqueness portion of this proposition. Let \( \omega \in \Omega \). Suppose \( \Phi \in M_n(W^{1,1}(a, b)) \) satisfies for a.e. \( x \in (a, b) \) the matrix differential equation with initial condi-
\[ \Psi'(x) = \mathcal{J}^{-1}A(x, \omega)\Psi(x), \quad \Psi(0) = I_n. \] (4.56)

It follows that \( \Phi \) is a solution of the equation

\[ \Psi' - \tilde{A}(\cdot, \omega)\Psi = 0 \]

in \( M_n(W^{1,1}(a, b)) \) with \( 0 \in [a, b] \) and \( \Phi(0) = I_n \). This implies by [45, p. 73, §II.2.5, Proposition 2.5.9] that \( \Phi \) is a fundamental matrix of \( T|_{(a, b)}(\omega)\psi = 0 \). But from the existence portion of this proof, \( Y(\cdot, \omega) \) is also a fundamental matrix of \( T|_{(a, b)}(\omega)\psi = 0 \) and so by [45, p. 71, §II.2.5, Proposition 2.5.4] there exists an invertible matrix \( C \in M_n(\mathbb{C}) \) such that \( \Phi = Y(\cdot, \omega)C \). Thus since \( I_n = \Phi(0) = Y(0, \omega)C \) this implies \( \Phi = Y(\cdot, \omega)Y^{-1}(0, \omega) = \Psi|_{(a, b)}(\cdot, \omega) \). This proves uniqueness.

Now we will prove statements (i)–(iv) of this proposition. We begin with statement (i). First, it follows from what we just proved that for each \( \omega \in \Omega \), \( \Psi|_{(a, b)}(\cdot, \omega) = Y(\cdot, \omega)Y^{-1}(0, \omega) \in M_n(W^{1,1}(a, b)) \) is a fundamental matrix of \( T|_{(a, b)}(\omega)\psi = 0 \). By Definition 25 this means if \( \psi \in \ker(T|_{(a, b)}(\omega)) \) then there exists \( \gamma \in (\mathbb{C})^n \) such that \( \psi = \Psi|_{(a, b)}(\cdot, \omega)\gamma \). But as discussed \( Y(\cdot, \omega) \) is invertible with inverse \( Y^{-1}(\cdot, \omega)^{-1} \in M_n(W^{1,1}(a, b)) \). Uniqueness of the representation \( \psi = \Psi|_{(a, b)}(\cdot, \omega)\gamma \) follows since \( \gamma = Y(0, \omega)Y^{-1}(\cdot, \omega)\psi \in (W^{1,1}(a, b))^n \).

Let \( \omega \in \Omega \), let \( \gamma \in (\mathbb{C})^n \), and define \( \psi := \Psi|_{(a, b)}(\cdot, \omega)\gamma \). Now because \( \psi = Y(\cdot, \omega)(Y^{-1}(0, \omega)\gamma) \) and \( Y \) is a fundamental matrix function of \( T|_{(a, b)}\psi = 0 \), it follows by Definition 25 that \( \psi \in \ker(T|_{(a, b)}(\omega)) \). This completes the proof of statement (i).
The proof of statement (ii) is straightforward. Let \((x, \omega) \in [a, b] \times \Omega\). Then \(\Psi := \Psi|_{(a,b)}(\cdot, \omega)\) satisfies (4.53) and so upon integrating and applying Lemma 70 we arrive at

\[
\Psi|_{(a,b)}(x, \omega) = I_n + \int_0^x \Psi'(t, \omega) dt = I_n + \int_0^x \mathcal{J}^{-1}A(t, \omega)\Psi(t, \omega) dt
\]

\[= I_n + \int_0^x \mathcal{J}^{-1}A(t, \omega)\Psi|_{(a,b)}(\cdot, \omega) dt.\]

This proves statement (ii).

We complete the proof of this proposition now by proving statements (iii) and (iv). We have already shown that there exists \(Y \in \mathcal{O}(\Omega, M_n(W^{1,1}(a, b)))\) such that for each \(\omega \in \Omega\), \(Y(\cdot, \omega)\) is invertible in \(M_n(W^{1,1}(a, b))\) and, denoting this inverse in \(M_n(W^{1,1}(a, b))\) by \(Y^{-1}(\cdot, \omega)\), we have for every \(x \in [a, b]\), \(Y(x, \omega)\) is invertible with \(Y^{-1}(x, \omega) = Y(x, \omega)^{-1}\) and \(\Psi|_{(a,b)}(\cdot, \omega) = Y(\cdot, \omega)Y^{-1}(0, \omega)\). This proves statement (iv) since for every \((x, \omega) \in [a, b] \times \Omega\), we have \(\Psi|_{(a,b)}(x, \omega)\) is invertible with inverse \(\Psi|_{(a,b)}^{-1}(x, \omega) := \Psi|_{(a,b)}(x, \omega)^{-1} = Y(0, \omega)Y^{-1}(x, \omega)\) and \(\Psi|_{(a,b)}^{-1}(\cdot, \omega) = Y(0, \omega)Y^{-1}(\cdot, \omega) \in M_n(W^{1,1}(a, b))\) by Lemma 72.

We will now show \(\Psi|_{(a,b)} \in \mathcal{O}(\Omega, M_n(W^{1,1}(a, b)))\). Since \(Y \in \mathcal{O}(\Omega, M_n(W^{1,1}(a, b)))\) and \(Y(\cdot, \omega)\) is invertible in \(M_n(W^{1,1}(a, b))\) with the inverse \(Y^{-1}(\cdot, \omega)\) for every \(\omega \in \Omega\), it follows from [45, p. 66, §II.2.3, Proposition 2.3.3] and [45, p. 7, §I.1.2, Proposition 1.2.5] that \(Y^{-1} \in \mathcal{O}(\Omega, M_n(W^{1,1}(a, b)))\). It follows from this and Lemma 91 that if we let \(\delta_0\) denote the evaluation map at \(x = 0\) then \(Y^{-1}(0, \cdot) = \delta_0 Y^{-1} \in \mathcal{O}(\Omega, M_n(\mathbb{C}))\). But the map \(\iota : M_n(\mathbb{C}) \to M_n(W^{1,1}(a, b))\) given by \((\iota B)(\cdot) := B\) is a continuous linear map and so by Lemma 78 we have \(Y^{-1}(0, \cdot) = \iota \delta_0 Y^{-1} \in \mathcal{O}(\Omega, M_n(W^{1,1}(a, b)))\). Therefore by Lemma 78 and Lemma 72 we have \(\Psi|_{(a,b)} = Y^{-1} \iota \delta_0 Y^{-1} \in \mathcal{O}(\Omega, M_n(W^{1,1}(a, b)))\). This proves statement (iii) and hence the proof of the proposition is complete.

From the functions \(\Psi|_{(a,b)}(\cdot, \omega)\) we define a matrix-valued function \(\Psi : \mathbb{R} \times \Omega \to M_n(\mathbb{C})\) by

\[
\Psi(x, \omega) := \Psi|_{(a,b)}(x, \omega) \tag{4.57}
\]
for each \((x, \omega) \in \mathbb{R} \times \Omega\) and for any interval \((a, b)\) containing 0 and \(x\).

**Lemma 56** The matrix-valued function \(\Psi : \mathbb{R} \times \Omega \to M_n(\mathbb{C})\) is well-defined. Moreover, for each \(\omega \in \Omega\), \(\Psi(\cdot, \omega) \in M_n(W^{1,1}(\mathbb{R}))\) and for any \(a, b \in \mathbb{R}\) with \(a < 0 < b\) we have

\[
\Psi(\cdot, \omega)|_{(a,b)} = \Psi|_{(a,b)}(\cdot, \omega).
\]

**Proof.** First, fix \(\omega \in \Omega\) and let \((u, v)\) be an interval such that \(0 \in (u, v) \subseteq (a, b)\). We will begin by showing

\[
\Psi|_{(a,b)}(\cdot, \omega)|_{(u,v)} = \Psi|_{(u,v)}(\cdot, \omega).
\]

We define \(Y := \Psi|_{(a,b)}(\cdot, \omega)|_{(u,v)}\) so that \(Y(0) = I_n\). Then by [45, §II.2.2, Proposition 2.2.1] we have \(Y \in M_n(W^{1,1}(u, v))\) and \(Y' = \Psi|_{(a,b)}(\cdot, \omega)'|_{(u,v)}\). This implies by the definition of \(\Psi|_{(a,b)}(\cdot, \omega)\) in Proposition 55 that \(Y\) is a function in \(M_n(W^{1,1}(u, v))\) satisfying the for a.e. \(x \in (u, v)\) the matrix differential equation with initial condition

\[
\Psi'(x) = \mathcal{J}^{-1}A(x, \omega)\Psi(x), \quad \Psi(0) = I_n.
\]

By the uniqueness portion of Proposition 55 we conclude that \(Y := \Psi|_{(u,v)}(\cdot, \omega)\). Thus we have shown that \(\Psi|_{(a,b)}(\cdot, \omega)|_{(u,v)} = \Psi|_{(u,v)}(\cdot, \omega)\).

We now prove the function \(\Psi\) defined by (4.57) is well-defined. Let \((x, \omega) \in \mathbb{R} \times \Omega\). Let \((a_1, b_1)\) and \((a_2, b_2)\) be any intervals containing 0 and \(x\). Then there exists \(u, v\) such that \((u, v) = (a_1, b_1) \cap (a_2, b_2)\). Now since 0 and \(x\) are in the interval \((u, v)\) then \(\Psi|_{(u,v)}(\cdot, \omega)\) and \(\Psi|_{(u,v)}(x, \omega)\) are well-defined by Proposition 55. And thus since \(0 \in (u, v) \subseteq (a_j, b_j)\) for \(j = 1, 2\), we get from (4.59) that

\[
\Psi|_{(a_1, b_1)}(x, \omega) = \Psi|_{(u,v)}(x, \omega) = \Psi|_{(a_2, b_2)}(x, \omega).
\]
This proves the function $\Psi$ is well-defined.

Next, to show that $\Psi(\cdot, \omega) \in M_n(W^{1,1}_{\text{loc}}(\mathbb{R}))$ for any $\omega \in \Omega$ its enough to show, by Lemma 71, that $\Psi(\cdot, \omega)|_{(a,b)} \in M_n(W^{1,1}(a,b))$ for any interval $(a, b)$ containing 0. But this follows from the definition (4.57) since for any $x \in (a, b)$

$$\Psi(x, \omega) = \Psi|_{(a,b)}(x, \omega)$$

which implies

$$\Psi(\cdot, \omega)|_{(a,b)} = \Psi|_{(a,b)}(\cdot, \omega) \in M_n(W^{1,1}(a,b)).$$

This completes the proof of the lemma.  

We are in a position now to prove Proposition 33.

**Proof.** [Proposition 33] To begin this proof we need to show that the matrix-valued function $\Psi : \mathbb{R} \times \Omega \to M_n(\mathbb{C})$, as defined by (4.57), for a fixed $\omega \in \Omega$ satisfies a.e. $x \in \mathbb{R}$ the matrix differential equation with initial condition

$$\mathcal{J}\Psi'(x) = A(x, \omega)\Psi(x), \quad \Psi(0) = I_n,$$

and $\Psi(\cdot, \omega)$ is the only function in $M_n(W^{1,1}_{\text{loc}}(\mathbb{R}))$ that does so. After which we will prove that it has the properties described in Proposition 33.

This first part follows now from the previous lemma since for any $\omega \in \Omega$ and interval $(a, b)$ containing 0 we have $\Psi(x, \omega)|_{(a,b)} = \Psi|_{(a,b)}(x, \omega)$ for all $x \in (a, b)$ and so by Proposition 55 the function $\Psi(\cdot, \omega)$ satisfies a.e. $x \in (a, b)$ the matrix differential equation with initial
condition

\[ \Psi'(x) = \mathcal{J}^{-1}A(x, \omega)\Psi(x), \quad \Psi(0) = I_n. \]

Hence this implies \( \Psi(\cdot, \omega) \) satisfies a.e. \( x \in \mathbb{R} \) the matrix differential equation with initial condition

\[ \mathcal{J}\Psi'(x) = A(x, \omega)\Psi(x), \quad \Psi(0) = I_n. \]

Let's now prove uniqueness. Fix \( \omega \in \Omega \). Suppose \( \Psi_1 \in M_n(W^{1,1}_{\text{loc}}(\mathbb{R})) \) satisfied a.e. \( x \in \mathbb{R} \) the matrix differential equation with initial condition

\[ \mathcal{J}\Psi'(x) = A(x, \omega)\Psi(x), \quad \Psi(0) = I_n. \]

Then for any interval \((a, b)\) containing 0, \( \Psi_1|_{(a,b)} \in M_n(W^{1,1}(a, b)) \) and satisfies a.e. \( x \in (a, b) \) the matrix differential equation with initial condition

\[ \Psi'(x) = \mathcal{J}^{-1}A(x, \omega)\Psi(x), \quad \Psi(0) = I_n. \]

and so by the uniqueness part of Proposition 55 this implies \( \Psi_1(\cdot)|_{(a,b)} = \Psi(\cdot, \omega)|_{(a,b)} \). By definition (4.57) this implies \( \Psi_1(\cdot)|_{(a,b)} = \Psi(\cdot, \omega)|_{(a,b)} \). Since this is true for any interval \((a, b)\) containing 0 we must have \( \Psi_1(\cdot) = \Psi(\cdot, \omega) \). This proves uniqueness.

Now we will prove Propositions 33.(i)–(vi). We start with property (i). Suppose that \( \psi \) is a solution of the canonical ODEs in (4.13) at the frequency \( \omega \in \Omega \). Then we must prove there exists a unique \( \gamma \in \mathbb{C}^n \) such that, as elements of \( (W^{1,1}_{\text{loc}}(\mathbb{R}))^n \), \( \psi = \Psi(\cdot, \omega)\gamma \). We begin by proving the uniqueness statement. Suppose that there did exist \( \gamma \in \mathbb{C}^n \) such that \( \psi = \Psi(\cdot, \omega)\gamma \). It follows from this, since \( \psi \in (W^{1,1}_{\text{loc}}(\mathbb{R}))^n \) and \( \Psi(\cdot, \omega) \in M_n(W^{1,1}_{\text{loc}}(\mathbb{R})) \) by Lemma
they have unique locally absolutely continuous representative functions \(\psi(\cdot) : \mathbb{R} \rightarrow \mathbb{C}^n\) and \(\Psi(\cdot, \omega) : \mathbb{R} \rightarrow M_n(\mathbb{C})\) which are continuous, that \(\psi(x) = \Psi(x, \omega)\gamma\) for all \(x \in \mathbb{R}\). In particular, since \(\Psi(0, \omega) = I_n\) this means \(\psi(0) = \Psi(0, \omega)\gamma = \gamma\). This proves uniqueness of \(\gamma\).

Let us now prove existence. By Lemma 52 we have \(\psi \in \ker(T(\omega))\). By Corollary 54 we have \(\psi|_{(a,b)} \in \ker(T|_{(a,b)}(\omega))\) for any bounded interval \((a, b) \subseteq \mathbb{R}\) containing 0 and so by Proposition 55.(i) there exists a \(\gamma|_{(a,b)} \in \mathbb{C}^n\) such that \(\psi|_{(a,b)} = \Psi|_{(a,b)}(\cdot, \omega)\gamma|_{(a,b)}\). But 0 \(\in (a, b)\) and so by (4.57) we have \(\gamma := \psi(0) = \psi(0)|_{(a,b)} = \psi|_{(a,b)}(0) = \Psi|_{(a,b)}(0, \omega)\gamma|_{(a,b)} = \Psi(0, \omega)\gamma|_{(a,b)} = \gamma|_{(a,b)}\). Thus from this and Lemma 56 we have \(\psi|_{(a,b)} = \Psi|_{(a,b)}(\cdot, \omega)\gamma = \Psi(\cdot, \omega)|_{(a,b)}\gamma = \Psi(\cdot, \omega)\gamma|_{(a,b)}\). But since this is true for every bounded interval \((a, b) \subseteq \mathbb{R}\) containing 0 this implies as elements of \((W^{1,1}_{\text{loc}}(\mathbb{R}))^n\) we have \(\psi = \Psi(\cdot, \omega)\gamma\). This proves existence.

Conversely, let \(\gamma \in (\mathbb{R})^n\) and define \(\psi := \Psi(\cdot, \omega)\gamma\). By Lemma 73 we have \(\psi \in (W^{1,1}_{\text{loc}}(\mathbb{R}))^n\). Then for any bounded interval \((a, b) \subseteq \mathbb{R}\) containing 0 we have by Lemma 56 and Proposition 55.(i) that \(\psi|_{(a,b)} = \Psi(\cdot, \omega)|_{(a,b)}\gamma = \Psi(\cdot, \omega)|_{(a,b)}(\cdot)\gamma \in \ker(T|_{(a,b)}(\omega))\). It follows from this and Proposition 53 that we have \((T(\omega)\psi)|_{(a,b)} = T|_{(a,b)}(\omega)\psi|_{(a,b)} = 0\) in \((L^1(a, b))^n\). But this is true for every bounded interval \((a, b) \subseteq \mathbb{R}\) containing 0 and so it follows this and Lemma 4.47 that \(T(\omega)\psi = 0\) in \((L^1_{\text{loc}}(\mathbb{R}))^n\) which means \(\psi \in \ker(T(\omega))\). Therefore by Corollary 54 this implies \(\psi\) is a solution of the canonical ODEs in (4.13) with frequency \(\omega\). This completes the proof of property (i).

Next we will prove property (ii). Let \((x, \omega) \in \mathbb{R} \times \Omega\). Take any interval \((a, b)\) containing 0 and \(x\). Then by Proposition 55 and the previous lemma we have

\[
\Psi(x, \omega) = \Psi|_{(a,b)}(x, \omega) = I_n + \int_0^x J^{-1} A(t, \omega)\Psi|_{(a,b)}(t, \omega)dt = I_n + \int_0^x J^{-1} A(t, \omega)\Psi(t, \omega)dt.
\]

This proves property (ii).
Next we will prove property (iii). That is, we must show for each fixed $x \in \mathbb{R}$, that $\Psi(x, \cdot) \in \mathcal{O}(\Omega, M_n(\mathbb{C}))$. To do this we fix $x \in \mathbb{R}$ and choose any bounded interval $(a, b) \subset \mathbb{R}$ with $0, x \in (a, b)$. Then by (4.57) we have $\Psi(x, \cdot) = \Psi|_{(a,b)}(x, \cdot)$. Let $\delta_x : M_n(W^{1,1}(a, b)) \to M_n(\mathbb{C})$ be the evaluation map at $x$ defined by $\delta_x B := B(x) - \text{the value of the unique absolutely continuous representative of } B \in M_n(W^{1,1}(a, b)) \text{ at } x$. Then by Proposition 55.(iii) we have $\Psi|_{(a,b)} \in \mathcal{O}(\Omega, M_n(\mathbb{C}))$ and so by Lemma 91 we have $\delta_x \Psi|_{(a,b)} \in \mathcal{O}(\Omega, M_n(\mathbb{C}))$. And thus we have $\Psi(x, \cdot) = \Psi|_{(a,b)}(x, \cdot) = \delta_x \Psi|_{(a,b)} \in \mathcal{O}(\Omega, M_n(\mathbb{C}))$ as desired. This proves property (iii).

Next we prove property (iv). To start, since $\Psi(x, \cdot) \in \mathcal{O}(\Omega, M_n(\mathbb{C}))$ for each $x \in \mathbb{R}$, we can define $\Psi_\omega : \mathbb{R} \times \Omega \to M_n(\mathbb{C})$ to be the partial derivative with respect to frequency in the $M_n(\mathbb{C})$ norm of the function $\Psi : \mathbb{R} \times \Omega \to M_n(\mathbb{C})$, i.e.,

$$\Psi_\omega(x, \omega_0) := \lim_{\omega \to \omega_0} (\omega - \omega_0)^{-1}(\Psi(x, \omega) - \Psi(x, \omega_0)), \quad \forall (x, \omega_0) \in \mathbb{R} \times \Omega.$$

Our goal is to prove that for any $(x, \omega_0) \in \mathbb{R} \times \Omega$, $\Psi_\omega(\cdot, \omega_0) \in M_n(W^{1,1}_{\text{loc}}(\mathbb{R}))$ and

$$\Psi_\omega(x, \omega_0) = \int_0^x \mathcal{J}^{-1}A_\omega(t, \omega_0)\Psi(t, \omega_0) + \mathcal{J}^{-1}A(t, \omega_0)\Psi_\omega(t, \omega_0)dt.$$

To do this we begin by choosing any bounded interval $(a, b) \subset \mathbb{R}$ with $0 \in (a, b)$. Next, let $\mathcal{I} : M_n(L^1(a, b)) \to M_n(W^{1,1}(a, b))$ be the integral map defined by

$$(\mathcal{I}B)(x) := \int_0^x B(t)dt, \quad B \in M_n(L^1(a, b)), \quad x \in (a, b).$$

Now by hypothesis we have $A \in \mathcal{O}(\Omega, L^1(\mathbb{T}))$ and so by Lemma 93 we have $A|_{(a,b)} \in \mathcal{O}(\Omega, L^1(a, b))$ and $(A|_{(a,b)})_\omega = A_\omega|_{(a,b)}$. It then follows from this and Lemma 89 with $p = \infty, q = s = 1$ that $\mathcal{J}^{-1}A|_{(a,b)} \in \mathcal{O}(\Omega, L^1(a, b))$ and $(\mathcal{J}^{-1}A|_{(a,b)})_\omega = \mathcal{J}^{-1}A_\omega|_{(a,b)}$. From this and the fact by Proposition 55.(iii) we have $\Psi|_{(a,b)} \in \mathcal{O}(\Omega, M_n(W^{1,1}(a, b)))$, it follows by Lemma 90 with $p = 1$ that $\mathcal{J}^{-1}A|_{(a,b)}\Psi|_{(a,b)} \in \mathcal{O}(\Omega, M_n(L^1(a, b)))$ whose derivative
is \((\mathcal{J}^{-1}A|_{(a,b)}\Psi|_{(a,b)})_\omega = \mathcal{J}^{-1}A\omega|_{(a,b)}\Psi|_{(a,b)} + \mathcal{J}^{-1}A|_{(a,b)}(\Psi|_{(a,b)})_\omega\). It follows from this and Lemma 92 that we must have \(I(\mathcal{J}^{-1}A|_{(a,b)}\Psi|_{(a,b)}) \in \mathcal{O}(\Omega, M_n(W^{1,1}(a,b)))\) and

\[
(I(\mathcal{J}^{-1}A|_{(a,b)}\Psi|_{(a,b)}))_\omega = I(\mathcal{J}^{-1}A\omega|_{(a,b)}\Psi|_{(a,b)} + \mathcal{J}^{-1}A|_{(a,b)}(\Psi|_{(a,b)})_\omega).
\]

This and Lemma 91 imply that for any \(x \in (a, b)\), \(\delta_xI(\mathcal{J}^{-1}A|_{(a,b)}\Psi|_{(a,b)}) \in \mathcal{O}(\Omega, M_n(\mathbb{C}))\) and

\[
(\delta_xI(\mathcal{J}^{-1}A|_{(a,b)}\Psi|_{(a,b)}))_\omega = \delta_xI(\mathcal{J}^{-1}A\omega|_{(a,b)}\Psi|_{(a,b)} + \mathcal{J}^{-1}A|_{(a,b)}(\Psi|_{(a,b)})_\omega). \tag{4.60}
\]

Now from Proposition 55.(ii) it follows that we have the identity

\[
\Psi(x, \cdot) = \Psi|_{(a,b)}(x, \cdot) = \delta_x\Psi|_{(a,b)} = I_n + \delta_xI(\mathcal{J}^{-1}A|_{(a,b)}\Psi|_{(a,b)}), \quad \forall x \in (a, b).
\]

From this and (4.60) it follows that

\[
\Psi_\omega(x, \cdot) = \delta_x(\Psi|_{(a,b)})_\omega = (I_n + \delta_xI(\mathcal{J}^{-1}A|_{(a,b)}\Psi|_{(a,b)}))_\omega
\]

\[
= \delta_xI(\mathcal{J}^{-1}A\omega|_{(a,b)}\Psi|_{(a,b)} + \mathcal{J}^{-1}A|_{(a,b)}(\Psi|_{(a,b)})_\omega), \quad \forall x \in (a, b). \tag{4.61}
\]

But this implies for any \((x, \omega_0) \in (a, b) \times \Omega\) we have \(\Psi_\omega(x, \omega_0) = \delta_x(\Psi|_{(a,b)})_\omega(\omega_0) = (\Psi|_{(a,b)})_\omega(x, \omega_0)\). And hence this shows that for each \(\omega_0 \in \Omega\),

\[
\Psi_\omega(\cdot, \omega_0)|_{(a,b)} = (\Psi|_{(a,b)})_\omega(\cdot, \omega_0) \in M_n(W^{1,1}(a,b)). \tag{4.62}
\]

But since this is true for any bounded interval \((a, b) \subseteq \mathbb{R}\) with \(0 \in (a, b)\), then by Lemma 71 for any \(\omega_0 \in \Omega\) we have \(\Psi_\omega(\cdot, \omega_0) \in M_n(W^{1,1}_{\text{loc}}(\mathbb{R}))\). Moreover, it follows that for any \((x, \omega_0) \in \mathbb{R} \times \Omega\) we can choose a bounded interval \((a, b) \subseteq \mathbb{R}\) with \(0, x \in (a, b)\) so that by
(4.57), (4.61), and (4.62) we have

\[
\Psi_\omega(x, \omega_0) = \delta_x \mathcal{I}(\mathcal{J}^{-1} A_\omega |_{(a,b)} \Psi |_{(a,b)} + \mathcal{J}^{-1} A |_{(a,b)} (\Psi |_{(a,b)})_\omega)(\omega_0) \\
= \int_0^x \mathcal{J}^{-1} A_\omega |_{(a,b)} (t, \omega_0) \Psi |_{(a,b)}(t, \omega_0) + \mathcal{J}^{-1} A |_{(a,b)} (\Psi |_{(a,b)})_\omega(t, \omega_0) dt \\
= \int_0^x \mathcal{J}^{-1} A_\omega(t, \omega_0) \Psi(t, \omega_0) + \mathcal{J}^{-1} A(t, \omega_0) \Psi_\omega(t, \omega_0) dt.
\]

This completes the proof of Proposition 33.(iv).

Next we prove property (v) of Proposition 33. The goal is to prove for every \((x, \omega) \in \mathbb{R} \times \Omega\) the matrix \(\Psi(x, \omega)\) is invertible and, denoting \(\Psi^{-1}(x, \omega) := \Psi(x, \omega)^{-1}\), that \(\Psi^{-1}(\cdot, \omega) \in M_n(W^{1,1}_\text{loc}(\mathbb{R}))\). Well, by Proposition 55.(iv) for any bounded interval \((a, b) \subseteq \mathbb{R}\) containing 0 and any \((x, \omega) \in (a, b) \times \Omega\) we have \(\Psi |_{(a,b)}(x, \omega)\) is invertible and denoting \(\Psi^{-1}_{(a,b)}(x, \omega) := \Psi^{-1}_{(a,b)}(\cdot, \omega) \in M_n(W^{1,1}(a, b))\). But by (4.57) and since \(\Psi(x, \omega) = \Psi |_{(a,b)}(x, \omega)\) this implies \(\Psi^{-1}(x, \omega) = \Psi^{-1}_{(a,b)}(x, \omega)\) for any \((x, \omega) \in (a, b) \times \Omega\). This proves for every \((x, \omega) \in \mathbb{R} \times \Omega\) the matrix \(\Psi(x, \omega)\) is invertible and for any \(\omega \in \Omega\), \(\Psi^{-1}(\cdot, \omega) |_{(a,b)} = \Psi^{-1}_{(a,b)}(\cdot, \omega) \in M_n(W^{1,1}(a, b))\) for any bounded interval \((a, b) \subseteq \mathbb{R}\) containing 0. And therefore by Lemma 71 this proves \(\Psi^{-1}(\cdot, \omega) \in M_n(W^{1,1}_\text{loc}(\mathbb{R}))\) for any \(\omega \in \Omega\). This completes the proof of property (v) of Proposition 33.

Finally, we will complete the proof of Proposition 33 by proving property (vi). To do this we must show that for every \((x, \omega) \in \mathbb{R} \times \Omega\) we have \(\Psi(x + d, \omega) = \Psi(x, \omega) \Psi(d, \omega)\). Fix \(\omega \in \Omega\) and define \(\Phi := \Psi(\cdot + d, \omega) \Psi(d, \omega)^{-1}\). We now show that \(\Phi \in M_n(W^{1,1}_\text{loc}(\mathbb{R}))\). First, since \(\Psi(\cdot, \omega) \in M_n(W^{1,1}_\text{loc}(\mathbb{R}))\), it follows from Lemma 74 that \(\Psi(\cdot + d, \omega) \in M_n(W^{1,1}_\text{loc}(\mathbb{R}))\). Second, since \(\Psi(d, \omega)^{-1}\) is just a constant matrix then \(\Psi(d, \omega)^{-1} \in M_n(W^{1,1}_\text{loc}(\mathbb{R}))\). It now follows from Lemma 73 that the product \(\Phi = \Psi(\cdot + d, \omega) \Psi(d, \omega)^{-1}\) satisfies a.e. the matrix differential equation with initial
condition

\[ \mathcal{J}\Psi'(x) = A(x, \omega)\Psi(x), \quad \Psi \in M_n(W^{1,1}_{\text{loc}}(\mathbb{R})), \quad \Psi(0) = I_n. \]

To do this we let \( \mathcal{L}_d : M_n(W^{1,1}_{\text{loc}}(\mathbb{R})) \to M_n(W^{1,1}_{\text{loc}}(\mathbb{R})) \) denote the translation operator as defined in Lemma 74. Then \( \Phi = \mathcal{L}_d(\Psi(\cdot, \omega)\Psi(d, \omega)^{-1}). \) Then by Lemma 74 and Lemma 73 we have

\[
\mathcal{J}\Phi' = \mathcal{J}(\mathcal{L}_d(\Psi(\cdot, \omega)\Psi(d, \omega)^{-1}))' = \mathcal{J}\mathcal{L}_d(\Psi(\cdot, \omega)\Psi(d, \omega)^{-1})' = \mathcal{J}\mathcal{L}_d(\Psi(\cdot, \omega)')\Psi(d, \omega)^{-1}.
\]

But Proposition 33 implies \( \mathcal{J}\Psi(\cdot, \omega)' = A(\cdot, \omega)\Psi(\cdot, \omega) \) and since by hypothesis \( A(\cdot + d, \omega) = A(\cdot, \omega) \) this implies \( \mathcal{L}_d(A(\cdot, \omega)\Psi(\cdot, \omega)) = A(\cdot + d, \omega)\Psi(\cdot + d, \omega) = A(\cdot, \omega)\mathcal{L}_d(\Psi(\cdot, \omega)). \) Hence these facts imply

\[
\mathcal{J}\Phi' = \mathcal{L}_d(\mathcal{J}\Psi(\cdot, \omega)'\Psi(d, \omega)^{-1}) = \mathcal{L}_d(A(\cdot, \omega)\Psi(\cdot, \omega))\Psi(d, \omega)^{-1} = A(\cdot, \omega)\Phi.
\]

But this is an equality of elements in \( M_n(L^1_{\text{loc}}(\mathbb{R})) \) and so together with the fact that \( \Phi(0) = \Psi(0 + d, \omega)\Psi(d, \omega)^{-1} = I_n \) this implies that \( \Phi \in M_n(W^{1,1}_{\text{loc}}(\mathbb{R})) \) satisfies a.e. the matrix differential equation with initial condition

\[ \mathcal{J}\Psi'(x) = A(x, \omega)\Psi(x), \quad \Psi \in M_n(W^{1,1}_{\text{loc}}(\mathbb{R})), \quad \Psi(0) = I_n. \]

Now though the uniqueness portion of Proposition 33 implies that we must have \( \Psi(\cdot, \omega) = \Phi. \) Thus \( \Psi(x + d, \omega) = \Psi(x, \omega)\Psi(d, \omega) \) for a.e. \( x \in \mathbb{R} \) and so, since Proposition 33.(ii) implies the function \( \Psi(x, \omega) \) is continuous as a function of \( x \in \mathbb{R} \), we must have \( \Psi(x + d, \omega) = \Psi(x, \omega)\Psi(d, \omega) \) for all \( x \in \mathbb{R} \). This proves Proposition 33.(vi) and therefore completes the

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We will use Proposition 33 now to prove the Floquet-Lyapunov Theorem.

**Proof.** [Theorem 34] We fix \( \omega \in \Omega \). To simplify notation we will denote the matricant by \( \Phi := \Psi(\cdot, \omega) \) and we will agree to treat, in the rest of this proof, all objects as depending implicitly on the frequency \( \omega \). Our goal here is to show that there exists a matrix \( K \in M_n(\mathbb{C}) \) and a function \( F \in M_n(W_{1,1}^{1,1}(\mathbb{R})) \) such that for every \( x \in \mathbb{R}, \)

\[
\Phi(x) = F(x)e^{ixK},
\]

where \( F(x + d) = F(x), \ F(0) = I_n, \ F^{-1}(x) := F(x)^{-1} \) exist, and \( F^{-1} \in M_n(W_{1,1}^{1,1}(\mathbb{R})) \).

To begin, let \( \lambda_1, \ldots, \lambda_s \) denote the distinct eigenvalues of the monodromy matrix \( \Phi(d) \). Let \( \log(z) \) be any branch of the logarithm which is analytic at each \( \lambda_j \), for \( j = 1, \ldots, s \). Using the Riesz-Dunford functional calculus as described for matrices in [35, pp. 304–334, §9], it follows that the matrix \( K := \frac{1}{id}\log(\Phi(d)) \) satisfies

\[ e^{idK} = \Phi(d), \]

where \( e^z \) is the exponential function. From the functional calculus it follows that the matrix function \( B(x) := e^{ixK} \), for \( x \in \mathbb{R} \) has the properties \( B \in M_n(W_{1,1}^{1,1}(\mathbb{R})) \), \( B(x) \) is invertible with inverse \( B^{-1}(x) := e^{-ixK} \) for \( x \in \mathbb{R} \), and \( B^{-1} \in M_n(W_{1,1}^{1,1}(\mathbb{R})) \).

We define the function \( F \) by \( F(x) := \Phi(x)e^{-ixK}, \ x \in \mathbb{R} \). It then follows from Lemma 73 that \( F \in M_n(W_{1,1}^{1,1}(\mathbb{R})) \). We also find that for every \( x \in \mathbb{R}, \ F(x) \) is invertible with inverse \( F^{-1}(x) := e^{ixK}\Phi(x)^{-1} \). It follows from Proposition 33.(v) and Lemma 73 that \( F^{-1} \in M_n(W_{1,1}^{1,1}(\mathbb{R})) \). Furthermore, since \( \Phi(0) = I_n = e^{-i0K} \) then \( F(0) = I_n \). Moreover, since \( e^{idK} = \Phi(d), \ e^{-idK} = \Phi(d)^{-1} \), and, by Proposition 33.(vi), \( \Phi(x + d) = \Phi(x)\Phi(d) \) for every
$x \in \mathbb{R}$, we have

\[
F(x + d) = \Phi(x + d)e^{-i(x+d)K} = \Phi(x)\Phi(d)e^{-idK}e^{-ixK} = \Phi(x)\Phi(d)^{-1}e^{-ixK}
\]

\[
= \Phi(x)e^{-ixK} = F(x), \quad \forall x \in \mathbb{R}.
\]

This completes the proof of Theorem 34. \hfill \blacksquare

Using the Floquet-Lyapunov Theorem and its proof we will now establish the validity of Theorem 35.

**Proof.** [Theorem 35] We first prove the statement: if $\psi$ is a nontrivial Floquet solution of the canonical ODEs in (4.13) with wavenumber-frequency pair $(k, \omega)$, Floquet multiplier $\lambda = e^{ikd}$, and order $m \in \mathbb{N} \cup \{0\}$ then $\psi = \Psi(\cdot, \omega)\gamma$ where $\gamma$ is a generalized eigenvector of the monodromy matrix $\Psi(d, \omega)$ of order $m + 1$ corresponding to the eigenvalue $\lambda$. We prove this by induction on the order $m$ of the Floquet solution. Before we begin we will prove the following lemma:

**Lemma 57** If $\{u_j\}_{j=0}^m \subseteq (W^{1,1}_\text{loc}(\mathbb{R}))^n \cap (L^1(\mathbb{T}))^n$ and for a.e. $x \in \mathbb{R},$

\[
\sum_{j=0}^m x^j u_j(x) = 0
\]

then $u_1 = \cdots = u_m = 0$.

**Proof.** We prove the statement by induction. The statement is obviously true for $m = 0$. Suppose the statement is true for $m \in \mathbb{N} \cup \{0\}$. Lets show its true for $m + 1$. Suppose $\{u_j\}_{j=0}^{m+1} \subseteq (W^{1,1}_\text{loc}(\mathbb{R}))^n \cap (L^1(\mathbb{T}))^n$ and satisfies for a.e. $x \in \mathbb{R},$

\[
\sum_{j=0}^{m+1} x^j u_j(x) = 0
\]
By Lemma 70 we can assume \( \{u_j\}_{j=0}^{m+1} \subseteq (AC_{\text{loc}}(\mathbb{R}))^n \) and satisfy \( u_j(x + d) = u_j(x) \) for all \( x \in \mathbb{R} \) and for \( j = 0, \ldots, m+1 \). Hence letting \( u_{j,i} \) denote the \( i \)th row entry of \( u_j \) we have \( u_{j,i} \in AC_{\text{loc}}(\mathbb{R}) \) is a \( d \)-periodic function and hence is bounded on \( \mathbb{R} \), for \( j = 0, \ldots, m+1, \ i = 0, \ldots, n \). This implies

\[
\lim_{x \to \infty} \frac{x^j u_{j,i}(x)}{x^{m+1}} = 0
\]

for \( j = 0, \ldots, m, \ i = 0, \ldots, n \) and there exists \( x_i \in [0, d) \) such that

\[
u_{m+1,i}(x_i) = \sup_{x \in \mathbb{R}} |u_{m+1,i}(x)|
\]

for \( i = 0, \ldots, n \). Now it follows from our hypothesis that

\[
\sum_{j=0}^{m+1} x^j u_{j,i}(x) = 0, \quad \forall x \in \mathbb{R}, \ i = 1, \ldots, n.
\]

Thus we conclude for \( l \in \mathbb{N} \),

\[
\sup_{x \in \mathbb{R}} |u_{m+1,i}(x)| = u_{m+1,i}(x_i) = \lim_{l \to \infty} u_{m+1,i}(x_i + ld) = \lim_{l \to \infty} - \sum_{j=0}^{m} \frac{(x_i + ld)^j u_{j,i}(x_i + ld)}{(x_i + ld)^{m+1}} = 0
\]

for \( i = 0, \ldots, n \). This implies \( u_{m+1} = 0 \). But now the sequence \( \{u_j\}_{j=0}^{m} \) satisfies the hypotheses of this lemma and so by the induction hypotheses we conclude that \( u_m = u_{m-1} = \cdots = u_0 = 0 \). Therefore by induction the statement is true for all \( m \in \mathbb{N} \cup \{0\} \). This completes the proof. \( \blacksquare \)

We now begin the proof of the above statement. Let us show the base case \( m = 0 \) is true. Suppose \( \psi \) is a Floquet solution of the canonical ODEs in (4.13) with wavenumber-frequency pair \((k, \omega)\), Floquet multiplier \( \lambda = e^{ikd} \), and order \( m = 0 \). Then by Definition
19, \( \psi(x) = e^{ikx}u_0(x) \) for a.e. \( x \in \mathbb{R} \), for some \( u_0 \in (W^{1,1}_{\text{loc}}(\mathbb{R}))^n \cap (L^1(\mathbb{T}))^n \) with \( u_0 \neq 0 \). By Lemma 57 we conclude \( \psi \neq 0 \) and so by Proposition 33.(i), \( \psi = \Psi(\cdot, \omega)\gamma \) for some nonzero \( \gamma \in \mathbb{C}^n \). These facts and Proposition 33.(vi) imply for a.e. \( x \in \mathbb{R} \), \( \Psi(x, \omega)\Psi(d, \omega)\gamma = \psi(x + d) = e^{ikx}e^{ikd}u_0(x + d) = \lambda e^{ikd}u_0(x) = \lambda \psi(x) = \Psi(x, \omega)\lambda \gamma \). Hence Proposition 33.(v) implies \( \Psi(d, \omega)\gamma = \lambda \gamma \). But this implies, since \( \gamma \neq 0 \), that \( \gamma \) is a generalized eigenvector of \( \Psi(d, \omega) \) of order 1 corresponding to the eigenvalue \( \lambda \). This proves the base case.

Suppose now the statement is true for \( m \in \mathbb{N} \cup \{0\} \). We will now show its true for \( m + 1 \). Let \( \psi \) be a Floquet solution of the canonical ODEs in (4.13) with wavenumber-frequency pair \((k, \omega)\), Floquet multiplier \( \lambda = e^{ikd} \), and order \( m + 1 \). Then by Definition 19, \( \psi(x) = e^{ikx} \sum_{j=0}^{m+1} x^j u_j(x) \) for a.e. \( x \in \mathbb{R} \), for some \( \{u_j\}_{j=0}^{m+1} \subseteq (W^{1,1}_{\text{loc}}(\mathbb{R}))^n \cap (L^1(\mathbb{T}))^n \) with \( u_{m+1} \neq 0 \). By Lemma 57 we conclude \( \psi \neq 0 \) and so by Proposition 33.(i), \( \psi = \Psi(\cdot, \omega)\gamma \) for some nonzero \( \gamma \in \mathbb{C}^n \). We define

\[
\tilde{\gamma} := (\Psi(d, \omega) - \lambda I_n)\gamma, \quad \tilde{\psi} := \Psi(\cdot, \omega)\tilde{\gamma}.
\]

We will now prove \( \tilde{\psi} \) is a Floquet solution of the canonical ODEs in (4.13) with wavenumber-frequency pair \((k, \omega)\), Floquet multiplier \( \lambda = e^{ikd} \), and order \( m \). First, it follows by Proposition 33.(i) that \( \tilde{\psi} \) is a solution of the canonical ODEs in (4.13) at the frequency \( \omega \in \Omega \). Next, by Proposition 33.(vi), the Floquet representation of \( \psi \), and the fact \((x+d)^j = \sum_{l=0}^{j} \binom{j}{l} d^{j-l} x^l \), it follows for a.e. \( x \in \mathbb{R} \) that

\[
\tilde{\psi}(x) = \Psi(x, \omega)(\Psi(d, \omega) - e^{ikd} I_n)\gamma = \psi(x + d) - e^{ikd} \psi(x)
\]

\[
= e^{i(k+d)x} \sum_{j=0}^{m+1} (x+d)^j u_j(x + d) - e^{ikd} e^{ikx} \sum_{j=0}^{m+1} x^j u_j(x)
\]

\[
= e^{ikx} \sum_{j=0}^{m+1} ((x+d)^j - x^j) e^{ikd} u_j(x) = e^{ikx} \sum_{j=1}^{m+1} \sum_{l=0}^{j-1} \binom{j}{l} d^{j-l} x^l e^{ikd} u_j(x)
\]

\[
= e^{ikx} \sum_{l=0}^{m} x^l \left( \sum_{j=l}^{m} \binom{j}{l} d^{j+1-l} e^{ikd} u_{j+1}(x) \right) = e^{ikx} \sum_{l=0}^{m} x^l \tilde{u}_j(x),
\]

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where \( \tilde{u}_j = \sum_{j=1}^{m} (j) d^{j+1-l} e^{jkd} u_{j+1} \in (W^{1,1}_{loc}(\mathbb{R}))^n \cap (L^1(\mathbb{T}))^n \) for \( j = 0, \ldots, m \) and \( \tilde{u}_m = de^{ikd} u_{m+1} \neq 0 \). But this proves \( \tilde{\psi} \) is a Floquet solution of the canonical ODEs in (4.13) with wavenumber-frequency pair \((k, \omega)\), Floquet multiplier \( \lambda = e^{ikd} \), and order \( m \). Hence by the induction hypothesis and the uniqueness portion of Proposition 33.(i) we conclude that \( \tilde{\gamma} \) is a generalized eigenvector of \( \Psi(d, \omega) \) of order \( m + 1 \) corresponding to the eigenvalue \( \lambda \). But since \( \tilde{\gamma} = (\Psi(d, \omega) - \lambda I_n) \gamma \) this implies \( \gamma \) is a generalized eigenvector of \( \Psi(d, \omega) \) of order \( m + 2 \) corresponding to the eigenvalue \( \lambda \). This proves the statement for \( m + 1 \). Therefore by induction the statement is true for all \( m \in \mathbb{N} \cup \{0\} \).

To complete the proof of this theorem we must show that if \( \gamma \) is a generalized eigenvector of \( \Psi(d, \omega) \) of order \( m + 1 \) corresponding to the eigenvalue \( \lambda \) then for any \( k \in \mathbb{C} \) such that \( \lambda = e^{ikd} \), \( \psi = \Psi(\cdot, \omega) \gamma \) is a Floquet solution of the canonical ODEs in (4.13) with wavenumber-frequency pair \((k, \omega)\), Floquet multiplier \( \lambda = e^{ikd} \), and order \( m \).

Let \( \gamma \) be a generalized eigenvector of \( \Psi(d, \omega) \) of order \( m + 1 \) corresponding to the eigenvalue \( \lambda \) where \( m \in \mathbb{N} \cup \{0\} \). Define \( \psi := \Psi(\cdot, \omega) \gamma \). By Proposition 33.(i), \( \psi \) is a nontrivial solution of the canonical ODEs in (4.13) at the frequency \( \omega \in \Omega \). We will now show that \( \psi \) is a Floquet solution with the desired properties.

To simplify notation we will denote the matricant by \( \Phi := \Psi(\cdot, \omega) \). Then by the Floquet-Lyapunov Theorem there exists a matrix \( K \in M_n(\mathbb{C}) \) and a function \( F \in M_n(W^{1,1}_{loc}(\mathbb{R})) \) such that for every \( x \in \mathbb{R} \),

\[
\Phi(x) = F(x)e^{ixK},
\]

where \( F(x + d) = F(x) \), \( F(0) = I_n \), \( F^{-1}(x) := F(x)^{-1} \) exist, and \( F^{-1} \in M_n(W^{1,1}_{loc}(\mathbb{R})) \). Moreover, by the proof of the Floquet-Lyapunov Theorem we can assume that

\[
K = \frac{1}{id} \log(\Phi(d))
\]

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where, letting \( \lambda_1, \ldots, \lambda_s \) denote the distinct eigenvalues of the monodromy matrix \( \Phi(d) \), \( \log(z) \) is a branch of the logarithm which is analytic at each \( \lambda_l \), for \( l = 1, \ldots, s \). Fix an \( x \in \mathbb{R} \). We define the function

\[
f(z) := e^{\frac{ix}{z} \log(z)}
\]

with the same domain as the function \( \log(z) \). It follows that \( f \) is analytic at eigenvalues of the monodromy matrix \( \Phi(d) \) as well. By the Riesz-Dunford functional calculus as described for matrices in [35, pp. 304–334, §9] we can define the matrix \( f(\Phi(d)) \). But \( f(z) = e^{ix\left( \frac{1}{z} \log(z) \right)} \) and so it follows by [35, pp. 324, §9.7, Theorem 2] that

\[
f(\Phi(d)) = e^{ixK}.
\]

For each \( l = 1, \ldots, s \), let \( P_l \in M_n(\mathbb{C}) \) denote the projection onto the generalized eigenspace of the monodromy matrix \( \Phi(d) \) corresponding to the eigenvalue \( \lambda_l \). Without loss of generality we may assume \( \lambda_1 = \lambda \). Let \( m_l \) denote the index of the eigenvalue \( \lambda_l \) for \( l = 1, \ldots, s \). Let \( f_{l,j} \) denote the value of the \( j \)th derivative of \( f(z) \) at the eigenvalue \( \lambda_l \) for \( l = 1, \ldots, s \) and \( j = 0, \ldots, m_k - 1 \). By spectral resolution of \( f(\Phi(d)) \) (see [35, p. 314, §9.5, Theorem 1], [35, p. 319, §9.5, Theorem 3], and [35, p. 321, §9.6, Theorem 1]) we have

\[
f(\Phi(d)) = \sum_{l=1}^{s} \sum_{j=0}^{m_l-1} \frac{f_{l,j}}{j!} (\Phi(d) - \lambda l I_n)^j P_l
\]

Now since by hypothesis \( \gamma \) is a generalized eigenvector of \( \Phi(d) \) of order \( m+1 \) corresponding to the eigenvalue \( \lambda_1 \) this implies

\[
P_1 \gamma = \gamma, \quad P_l \gamma = 0, \quad \text{for } l \neq 1,
\]

\[
(\Phi(d) - \lambda_1 I_n)^m \gamma \neq 0, \quad (\Phi(d) - \lambda_1 I_n)^j \gamma = 0, \quad \text{for } j > m.
\]
Thus we have
\[ f(\Phi(d))\gamma = \sum_{j=0}^{m} \frac{f_{1,j}}{j!} (\Phi(d) - \lambda_1 I_n)^j \gamma, \quad f_{1,j} = \left. \frac{d^j f}{dz^j} \right|_{z=\lambda_1}. \]

But we have
\[ \left. \frac{d^j f}{dz^j} \right|_{z=\lambda_1} = \left( \frac{x}{d} \right) \left( \frac{x}{d} - 1 \right) \cdots \left( \frac{x}{d} - (j - 1) \right) e^{\left( \frac{x}{d} - (j-1) \right) \log(\lambda_1)}, \quad j = 1, \ldots, m. \]

We define the polynomials in the variable \( x \) by
\[ p_0(x) := 1, \quad p_j(x) := \left( \frac{x}{d} \right) \left( \frac{x}{d} - 1 \right) \cdots \left( \frac{x}{d} - (j - 1) \right) e^{-(j-1) \log(\lambda_1)}. \]

Choose any \( k \in \mathbb{C} \) such that
\[ \lambda_1 = e^{ikd}. \]

Then, since \( \lambda_1 = e^{\log(\lambda_1)} \) which follows from the fact \( \log(z) \) is a branch of the logarithm analytic at \( \lambda_1 \), there exist \( q \in \mathbb{Z} \) such that
\[ ikd = \log(\lambda_1) + i2\pi q. \]

From these facts it follows that
\[ e^{ikx} e^{i\kappa K} \gamma = f(\Phi(d))\gamma = \sum_{j=0}^{m} \frac{f_{1,j}}{j!} (\Phi(d) - \lambda_1 I_n)^j \gamma = \sum_{j=0}^{m} e^{\frac{x}{d} \log(\lambda_1)} \frac{p_j(x)}{j!} (\Phi(d) - \lambda_1 I_n)^j \gamma = e^{ikx} \sum_{j=0}^{m} \frac{p_j(x)}{j!} e^{-i2\pi q/d} (\Phi(d) - \lambda_1 I_n)^j \gamma. \]
And from this we conclude

\[ \psi(x) = \Phi(x)\gamma = F(x)e^{ixK}\gamma = e^{ix}\sum_{j=0}^{m} \frac{p_j(x)}{j!} e^{-ix2\pi q/d} F(x)(\Phi(d) - \lambda I_n)^j \gamma, \] (4.63)

for every \( x \in \mathbb{R} \). But \( v_j(\cdot) := \frac{1}{j!}e^{-i\cdot 2\pi q/d} F(\cdot)(\Phi(d) - \lambda I_n)^j \gamma \in (W^{1,1}_{loc}(\mathbb{R}))^n \cap (L^1(\mathbb{T}))^n \) and the polynomial \( p_j(\cdot) \) has degree \( j \), for \( j = 0, \ldots, m \). Moreover, \( v_m(0) = \frac{1}{m!}(\Phi(d) - \lambda I_n)^m \gamma \neq 0 \) and \( \lambda = e^{ikd} \). From these facts and the representation (4.63) we conclude that for any \( k \in \mathbb{C} \) such that \( \lambda = e^{ikd} \), \( \psi \) is Floquet solution of the canonical ODEs in (4.13) with wavenumber-frequency pair \((k, \omega)\), Floquet multiplier \( \lambda = e^{ikd} \), and order \( m \). This completes the proof of Proposition 35. \( \blacksquare \)

**Proof.** [Corollary 36] Fix an \( \omega \in \Omega \). Let \( \{\gamma_j\}_{j=1}^n \) be a Jordan basis for the matrix \( \Psi(d, \omega) \). Then for \( j = 1, \ldots, n \), we can define \( l_j, \lambda_j, \) and \( k_j \) to be such that \( \gamma_j \) is a generalized eigenvalue of \( \Psi(d, \omega) \) of order \( l_j \) corresponding to the eigenvalue \( \lambda_j = e^{ik_jd} \). We define \( \{\psi_j\}_{j=1}^n \) by \( \psi_j := \Psi(\cdot, \omega)\gamma_j \) for \( j = 1, \ldots, n \). Then since \( \{\gamma_j\}_{j=1}^n \) is a basis for \( (\mathbb{C})^n \) it follows by 33.(i) and 33.(v) that the set of solutions of the canonical ODEs in (4.13) at the frequency \( \omega \) is a vector space over \( \mathbb{C} \) and \( \{\psi_j\}_{j=1}^n \) is a basis for that space. Moreover, by Theorem 35, \( \psi_j \) is a Floquet solution of the canonical ODEs in (4.13) with wavenumber-frequency pair \((k_j, \omega)\), Floquet multiplier \( \lambda_j = e^{ik_jd} \), and order \( l_j - 1 \), for \( j = 1, \ldots, n \). This completes the proof. \( \blacksquare \)

**Proof.** [Corollary 37 & Corollary 38] Let \( B \) denote the Bloch variety of the canonical ODEs in (4.13), i.e.,

\[ B := \{(k, \omega) \in \mathbb{C} \times \Omega \mid (k, \omega) \text{ is the wavenumber-frequency pair of some nontrivial Bloch solution of the canonical ODEs in (4.13)}\}. \]
Define the function $D : \mathbb{C} \times \Omega \to \mathbb{C}$ by

$$D(k, \omega) := \det \left( e^{i k d} I_n - \Psi(d, \omega) \right)$$

and the set $\mathcal{C} \subseteq \mathbb{C} \times \Omega$ by

$$\mathcal{C} := \{ (k, \omega) \in \mathbb{C} \times \Omega \mid D(k, \omega) = 0 \}.$$ 

Now Corollary 37 follows from Corollary 38 since $D(k + 2\pi/d, \omega) = D(k, \omega)$. Hence to prove these corollaries we need only show the function $D$ is a nonconstant holomorphic function and $\mathcal{B} = \mathcal{C}$.

We will now prove $\mathcal{B} = \mathcal{C}$. Let $(k, \omega) \in \mathcal{B}$. Then there exists a $\psi \neq 0$ which is a Bloch solution of the canonical ODEs in (4.13) with wavenumber-frequency pair $(k, \omega)$. But this implies by Definition 19 that $\psi$ is a Floquet solution of the canonical ODEs in (4.13) with wavenumber-frequency pair $(k, \omega)$, Floquet multiplier $\lambda = e^{i k d}$, and order 1. By Theorem 35 this implies $\psi = \Psi(\cdot, \omega) \gamma$ where $\gamma$ is an eigenvector of the monodromy matrix $\Psi(\cdot, \omega)$ with $\lambda = e^{i k d}$ as its corresponding eigenvalue. Thus we must have $\det \left( e^{i k d} I_n - \Psi(d, \omega) \right) = 0$ and hence $(k, \omega) \in \mathcal{C}$. This proves $\mathcal{B} \subseteq \mathcal{C}$.

To prove the reverse inclusion, let $(k, \omega) \in \mathcal{C}$. Then $\det \left( e^{i k d} I_n - \Psi(d, \omega) \right) = 0$ and this implies $\lambda = e^{i k d}$ is an eigenvalue of the monodromy matrix $\Psi(d, \omega)$. Let $\gamma$ be an eigenvector of the monodromy matrix $\Psi(d, \omega)$ corresponding to the eigenvalue $\lambda = e^{i k d}$. Then Theorem 35 implies $\psi := \Psi(\cdot, \omega) \gamma$ is a Floquet solution of the canonical ODEs in (4.13) with wavenumber-frequency pair $(k, \omega)$, Floquet multiplier $\lambda = e^{i k d}$, and order 1. This and Lemma 57 imply $\psi$ is a nontrivial Bloch solution of the canonical ODEs in (4.13) with wavenumber-frequency pair $(k, \omega)$ and hence $(k, \omega) \in \mathcal{B}$. This proves $\mathcal{C} \subseteq \mathcal{B}$. Therefore $\mathcal{B} = \mathcal{C}$. 

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We will now prove the function $D : \mathbb{C} \times \Omega \to \mathbb{C}$ is a nonconstant function. First, $f(\lambda, \omega) := \det (\lambda I_n - \Psi(d, \omega))$ is a monic polynomial of degree $n$ in the variable $\lambda$ for each $\omega \in \mathbb{C}$. Thus $f : \mathbb{C} \times \Omega \to \mathbb{C}$ is a nonconstant function. Now $D(k, \omega) = f(e^{ikd}, \omega)$ and this implies $D$ must be a nonconstant function for otherwise we could fix an $\omega_0 \in \Omega$ and then there would exist $c \in \mathbb{C}$ such that $f(e^{ikd}, \omega_0) - c = 0$ for every $k \in \mathbb{C}$ implying, since $f(\lambda, \omega_0) - c$ is a monic polynomial of degree $n$ in the variable $\lambda$, the function $g(k) := e^{ikd}$ maps into a finite set of values in $\mathbb{C}$ which is a contradiction. Thus we have proven the function $D$ is a nonconstant function.

We complete the proof of these corollaries now by proving the function $D : \mathbb{C} \times \Omega \to \mathbb{C}$ is a holomorphic function. By Lemma 80 we have $O(\Omega, M_n(\mathbb{C})) = M_n(O(\Omega, \mathbb{C}))$. Thus by Proposition 33.(iii) we have $\Psi(\cdot, \cdot) \in M_n(O(\Omega, \mathbb{C}))$, i.e., the entries of the monodromy matrix function $\Psi(\cdot, \cdot)$ are holomorphic functions of frequency. This implies the entries of the matrix $M(k, \omega) := e^{ikd}I_n - \Psi(d, \omega)$ are holomorphic functions of the variable $(k, \omega)$ in the domain $\mathbb{C} \times \Omega$. And since the function $F(B) := \det(B)$ is a polynomial in the entries of the matrix $B \in M_n(\mathbb{C})$ this implies $D(k, \omega) = F(M(k, \omega))$ is a holomorphic functions of the variable $(k, \omega)$ in the domain $\mathbb{C} \times \Omega$. This completes the proof.

Proofs for Section 4.2.2

We begin with proving a key result in the study of the energy flux for canonical ODEs.

Proof. [Proposition 39] Fix an $\omega \in \Omega_{\mathbb{R}}$. Then by Proposition 33 we have $\Psi(\cdot, \omega) \in M_n(W^{1,1}_{\text{loc}}(\mathbb{R}))$, by Lemma 75 we have $\Psi(\cdot, \omega)^* \in M_n(W^{1,1}_{\text{loc}}(\mathbb{R}))$, and hence by Lemma 73 we have $\Psi(\cdot, \omega)^*i j \Psi(\cdot, \omega) \in M_n(W^{1,1}_{\text{loc}}(\mathbb{R}))$. For simplification of notation let $\Phi := \Psi(\cdot, \omega)$.

To complete the proof we must show $\Phi^* i j \Phi = i j$ in $M_n(W^{1,1}_{\text{loc}}(\mathbb{R}))$. First, it follows from Proposition 33 that $\Phi^* = j^{-1}A(\cdot, \omega)\Phi$ in $M_n(L^1_{\text{loc}}(\mathbb{R}))$. Second, by our assumptions we
have \((\mathcal{J}^{-1})^* = -\mathcal{J}^{-1}\) and \(A(\cdot, \omega)^* = A(\cdot, \omega)\) in \(M_n(L^1_{\text{loc}}(\mathbb{R}))\). Thus by these facts and by Lemmas 75 and 73 we have

\[(\Phi^* i \mathcal{J} \Phi)' = (\Phi')^* i \mathcal{J} \Phi + \Phi^* i \mathcal{J} \Phi' = (\mathcal{J}^{-1} A(\cdot, \omega) \Phi)^* i \mathcal{J} \Phi + \Phi^* i A(\cdot, \omega) \Phi = -i \Phi^* A(\cdot, \omega) \Phi + i \Phi^* A(\cdot, \omega) \Phi = 0.\]

It follows from this that there exists a constant matrix \(C \in M_n(\mathbb{C})\) such that

\[\Phi^* i \mathcal{J} \Phi = C.\]

But since \(\Phi(0) = \Psi(0, \omega) = I_n\) this implies \(C = i \mathcal{J}\) and hence

\[\Phi^* i \mathcal{J} \Phi = i \mathcal{J}.\]

This completes the proof.  

\textbf{Proof.} \textbf{[Lemma 41]} Let \(\omega_0 \in \Omega_{\mathbb{R}}\). By Lemma 76 and the hypotheses the Hamiltonian satisfies \(A(\cdot, \omega)^* = A(\cdot, \omega)\) as an element of \(M_n(L^1(\mathbb{T}))\) for every \(\omega \in \Omega_{\mathbb{R}}\) and \(A \in \mathcal{O}(\Omega, M_n(L^1(\mathbb{T})))\) as a function of frequency. By definition since \(A_\omega\) is derivative of the \(A\) with respect to frequency in the Banach space \(M_n(L^1(\mathbb{T}))\) we have \(A_\omega(\cdot, \omega) \in M_n(L^1(\mathbb{T}))\) for every \(\omega \in \Omega\) and

\[\lim_{\omega \to \omega_0} \|(\omega - \omega_0)^{-1} (A(\cdot, \omega) - A(\cdot, \omega_0)) - A_\omega(\cdot, \omega_0)\|_{L^1(\mathbb{T})} = 0.\]

It then follows from these facts and Lemma 76 that

\[0 = \lim_{\omega \to \omega_0} \|(\omega - \omega_0)^{-1} (A(\cdot, \omega) - A(\cdot, \omega_0)) - A_\omega(\cdot, \omega_0)\|_{L^1(\mathbb{T})}\]

\[= \lim_{\omega \to \omega_0} \|(\omega - \omega_0)^{-1} (A(\cdot, \omega) - A(\cdot, \omega_0)) - A_\omega(\cdot, \omega_0)\|^*\|_{L^1(\mathbb{T})}\]
\[
= \lim_{\omega \to \omega_0} \|(\omega - \omega_0)^{-1}(A(\cdot, \omega) - A(\cdot, \omega_0)) - A_\omega(\cdot, \omega_0)^*\|_{L^1(\mathbb{T})}.
\]

From which it follows that \(A_\omega(\cdot, \omega_0)^* = A_\omega(\cdot, \omega_0)\) in \(M_n(L^1(\mathbb{T}))\)). This completes the proof.

**Proof.** [Theorem 42] Let \(\omega_0 \in \Omega_\mathbb{R}\). Then Proposition 33 implies \(\Psi(\cdot, \omega_0), \Psi_\omega(\cdot, \omega_0) \in M_n(W^{1,1}_{\text{loc}}(\mathbb{R}))\) and by Lemma 75 have \(\Psi(\cdot, \omega_0)^* \in M_n(W^{1,1}_{\text{loc}}(\mathbb{R}))\). Thus by Lemma 73 we have \(\Psi(\cdot, \omega_0)^* \mathcal{J} \Psi_\omega(\cdot, \omega_0) \in M_n(W^{1,1}_{\text{loc}}(\mathbb{R}))\). Thus by notation 4.5.(xxix) we must have \((\Psi(\cdot, \omega_0)^* \mathcal{J} \Psi_\omega(\cdot, \omega_0))' \in M_n(L^1_{\text{loc}}(\mathbb{R}))\). Now since \(A_\omega(\cdot, \omega_0) \in M_n(L^1(\mathbb{T})) \subseteq M_n(L^1_{\text{loc}}(\mathbb{R}))\) then it follows from Lemma 84 that we also have \(\Psi(\cdot, \omega_0)^* A_\omega(\cdot, \omega_0) \Psi(\cdot, \omega_0) \in M_n(L^1_{\text{loc}}(\mathbb{R}))\).

Our next goal is to show that as elements of \(M_n(L^1_{\text{loc}}(\mathbb{R}))\),

\[
\Psi(\cdot, \omega_0)^* A_\omega(\cdot, \omega_0) \Psi(\cdot, \omega_0) = (\Psi(\cdot, \omega_0)^* \mathcal{J} \Psi_\omega(\cdot, \omega_0))'.
\]

We begin by defining two operators that will play a key role in our proof. The first is the integral map defined by

\[
(\mathcal{I}B)(x) := \int_0^x B(t)dt, \quad B \in (L^1_{\text{loc}}(\mathbb{R}))^n, \quad x \in \mathbb{R}.
\]

By Lemma 87, the map \(\mathcal{I} : (L^1_{\text{loc}}(\mathbb{R}))^n \to (W^{1,1}_{\text{loc}}(\mathbb{R}))^n\) is well-defined and

\[
(\mathcal{I}f)' = f,
\]

for every \(f \in (W^{1,1}_{\text{loc}}(\mathbb{R}))^n\). The second map is defined for each \(\omega \in \Omega\) by

\[
(S(\omega)f)(x) := \Psi(x, \omega) \int_0^x \Psi(t, \omega)^{-1} f(t)dt, \quad f \in (L^1_{\text{loc}}(\mathbb{R}))^n, \quad x \in \mathbb{R}.
\]
Lemma 58  For each $\omega \in \Omega$, the map $S(\omega) : (L^{1}_{\text{loc}}(\mathbb{R}))^{n} \to (W^{1,1}_{\text{loc}}(\mathbb{R}))^{n}$ is a right inverse of the map $T(\omega) : (W^{1,1}_{\text{loc}}(\mathbb{R}))^{n} \to (L^{1}_{\text{loc}}(\mathbb{R}))^{n}$ defined in (4.47), i.e., for every $f \in (L^{1}_{\text{loc}}(\mathbb{R}))^{n}$ we have

$$T(\omega)S(\omega)f = f.$$ 

Proof. Let $\omega \in \Omega$. Let first prove the map $S(\omega) : (L^{1}_{\text{loc}}(\mathbb{R}))^{n} \to (W^{1,1}_{\text{loc}}(\mathbb{R}))^{n}$ is well-defined. By Proposition 33.(v) it follows that $\Psi^{-1}(x,\omega) := \Psi(x,\omega)^{-1}$ exists for every $x \in \mathbb{R}$ and $\Psi^{-1}(\cdot,\omega) \in M_{n}(W^{1,1}_{\text{loc}}(\mathbb{R}))$. Let $f \in (L^{1}_{\text{loc}}(\mathbb{R}))^{n}$. Then it follows by Lemma 84 that $\Psi^{-1}(\cdot,\omega)f \in (L^{1}_{\text{loc}}(\mathbb{R}))^{n}$ and applying the integral map to this yields $\mathcal{I}(\Psi^{-1}(\cdot,\omega)f) \in (W^{1,1}_{\text{loc}}(\mathbb{R}))^{n}$. By Lemma 73 and the definition of $S(\omega)$ we find that

$$S(\omega)f = \Psi(\cdot,\omega)\mathcal{I}(\Psi^{-1}(\cdot,\omega)f) \in (W^{1,1}_{\text{loc}}(\mathbb{R}))^{n}.$$ 

Thus the map is well-defined.

Next, recall that the map $T(\omega) : (W^{1,1}_{\text{loc}}(\mathbb{R}))^{n} \to (L^{1}_{\text{loc}}(\mathbb{R}))^{n}$ in (4.47) is

$$T(\omega)\psi := \psi' - \mathcal{J}^{-1}A(\cdot,\omega)\psi, \quad \psi \in (W^{1,1}_{\text{loc}}(\mathbb{R}))^{n}.$$ 

We now show $S(\omega)$ is a right inverse of $T(\omega)$. Let $f \in (L^{1}_{\text{loc}}(\mathbb{R}))^{n}$ then by Lemma 73 we have

$$(S(\omega)f)' = (\Psi(\cdot,\omega)\mathcal{I}(\Psi^{-1}(\cdot,\omega)f))' = \Psi(\cdot,\omega)'\mathcal{I}(\Psi^{-1}(\cdot,\omega)f) + \Psi(\cdot,\omega)\mathcal{I}(\Psi^{-1}(\cdot,\omega)f)'$$

$$= \mathcal{J}^{-1}A(\cdot,\omega)\Psi(\cdot,\omega)\mathcal{I}(\Psi^{-1}(\cdot,\omega)f) + \Psi(\cdot,\omega)\Psi^{-1}(\cdot,\omega)f = \mathcal{J}^{-1}A(\cdot,\omega)S(\omega)f + f.$$ 

Hence it follows that

$$T(\omega)S(\omega)f = (S(\omega)f)' - \mathcal{J}^{-1}A(\cdot,\omega)S(\omega)f = f.$$
Thus $S(\omega)$ is a right inverse of $T(\omega)$. This completes the proof.

Let $\gamma \in \mathbb{C}^n$. Define $\phi := \Psi_{\omega}(\cdot, \omega_0)\gamma$ and $f := \mathcal{J}^{-1}A_{\omega}(\cdot, \omega_0)\Psi(\cdot, \omega_0)\gamma$. We will now show that $\phi = S(\omega_0)f$. It follows from Lemma 84 that $f \in (L^1_{\text{loc}}(\mathbb{R}))^n$. It follows by Proposition 33.(iv) that $\phi \in (W^{1,1}_{\text{loc}}(\mathbb{R}))^n$ and satisfies

$$\phi = \Psi_{\omega}(\cdot, \omega_0)\gamma = \mathcal{I} \left( \mathcal{J}^{-1}A_{\omega}(\cdot, \omega_0)\Psi(\cdot, \omega_0)\gamma + \mathcal{J}^{-1}A(\cdot, \omega_0)\Psi_{\omega}(\cdot, \omega_0)\gamma \right) = \mathcal{I} \left( f + \mathcal{J}^{-1}A(\cdot, \omega_0)\phi \right)$$

since $\mathcal{J}^{-1}A_{\omega}(\cdot, \omega_0)\Psi(\cdot, \omega_0)\gamma \in M_n(L^1_{\text{loc}}(\mathbb{R}))$ by Lemma 84. Hence by Lemma 87 it follows that

$$T(\omega_0)\phi = \phi' - \mathcal{J}^{-1}A(\cdot, \omega_0)\phi = f.$$ 

But by Lemma 58 we also have $T(\omega_0)S(\omega_0)f = f$ with $S(\omega_0)f \in (W^{1,1}_{\text{loc}}(\mathbb{R}))^n$. Hence $\psi := \phi - S(\omega_0)f \in (W^{1,1}_{\text{loc}}(\mathbb{R}))^n$ and satisfies

$$T(\omega_0)\psi = 0,$$

that is, $\psi \in \ker(T(\omega_0))$. Thus by Corollary 54 and Proposition 33.(i) there exists a unique $\tilde{\gamma} \in \mathbb{C}^n$ such that $\psi = \Psi(\cdot, \omega_0)\tilde{\gamma}$. In fact, $\tilde{\gamma} = \psi(0) = \phi(0) - (S(\omega_0)f)(0)$. But $(S(\omega)f)(0) = 0$ and $\phi(0) = \Psi_{\omega}(0, \omega_0)\gamma = 0$. Thus we have $\psi = 0$ and so $\phi = S(\omega_0)f$ as desired.

Now it follows from what we have just shown that for every $\gamma \in \mathbb{C}^n$,

$$\Psi_{\omega}(\cdot, \omega_0)\gamma = S(\omega_0)(\mathcal{J}^{-1}A_{\omega}(\cdot, \omega_0)\Psi(\cdot, \omega_0)\gamma)$$
from which it follows by the definition of $S(\omega_0)$ and Lemma 87 that

$$\Psi_\omega(x, \omega_0) = \Psi(x, \omega_0) \int_0^x \Psi(t, \omega_0)^{-1} J^{-1} A_\omega(t, \omega_0) \Psi(t, \omega_0) dt$$

$$= \Psi(x, \omega_0) \int_0^x \Psi(t, \omega_0)^{-1} J^{-1} A_\omega(t, \omega_0) \Psi(t, \omega_0) dt, \quad \forall x \in \mathbb{R}.$$

But since this is true for every $\gamma \in \mathbb{C}^n$ this implies

$$\Psi_\omega(x, \omega_0) = \Psi(x, \omega_0) \int_0^x \Psi(t, \omega_0)^{-1} J^{-1} A_\omega(t, \omega_0) \Psi(t, \omega_0) dt, \quad \forall x \in \mathbb{R}.$$

Now if we multiply both sides by $\Psi(x, \omega)^* J$ and apply Proposition 39 we get

$$\Psi(x, \omega_0)^* J \Psi_\omega(x, \omega_0) = \Psi(x, \omega)^* J \Psi(x, \omega_0) \int_0^x \Psi(t, \omega_0)^{-1} J^{-1} A_\omega(t, \omega_0) \Psi(t, \omega_0) dt$$

$$= J \int_0^x \Psi(t, \omega_0)^{-1} J^{-1} A_\omega(t, \omega_0) \Psi(t, \omega_0) dt = \int_0^x \Psi(t, \omega_0)^* A_\omega(t, \omega_0) \Psi(t, \omega_0) dt$$

for every $x \in \mathbb{R}$. It follows from this and Lemma 87 that

$$(\Psi(\cdot, \omega_0)^* J \Psi_\omega(\cdot, \omega_0))' = \Psi(\cdot, \omega_0)^* A_\omega(\cdot, \omega_0) \Psi(\cdot, \omega_0)$$

and

$$\frac{1}{d} \int_0^d \Psi(t, \omega_0)^* A_\omega(t, \omega_0) \Psi(t, \omega_0) dt = \frac{1}{d} \Psi(d, \omega)^* J \Psi_\omega(d, \omega_0).$$

This completes the proof.

**Proof.** [Corollary 43] Let $\omega \in \Omega_\mathbb{R}$ and $\gamma_1, \gamma_2 \in \mathbb{C}^n$. Then by Theorem 42 we have

$$\Psi(\cdot, \omega)^* J \Psi_\omega(\cdot, \omega) \gamma_1 \in (W^{1,1}_{loc}(\mathbb{R}))^n$$

$$\Psi(\cdot, \omega)^* A_\omega(\cdot, \omega) \Psi(\cdot, \omega) \gamma_1 \in (L^1_{loc}(\mathbb{R}))^n$$

$$(\Psi(\cdot, \omega)^* J \Psi_\omega(\cdot, \omega) \gamma_1)' = \Psi(\cdot, \omega)^* A_\omega(\cdot, \omega) \Psi(\cdot, \omega) \gamma_1$$
Hence it follows from this and Lemma 75 that \( \langle \Psi(\cdot, \omega)^* \mathcal{J} \Psi(\cdot, \omega) \gamma_1, \gamma_2 \rangle_{\mathbb{C}} \in W^{1,1}_{\text{loc}}(\mathbb{R}) \), \( \langle A_\omega(\cdot, \omega) \psi_1, \psi_2 \rangle_{\mathbb{C}} \in L^1_{\text{loc}}(\mathbb{R}) \), and \( \langle \Psi(\cdot, \omega)^* \mathcal{J} \Psi(\cdot, \omega) \gamma_1, \gamma_2 \rangle_{\mathbb{C}}' = \langle A_\omega(\cdot, \omega) \psi_1, \psi_2 \rangle_{\mathbb{C}} \). And by Theorem 42 we conclude that

\[
\frac{1}{d} \int_0^d \langle A_\omega(x, \omega) \psi_1(x), \psi_2(x) \rangle_{\mathbb{C}} dx = \frac{1}{d} \langle \Psi(d, \omega)^* \mathcal{J} \Psi(d, \omega) \gamma_1, \gamma_2 \rangle_{\mathbb{C}}.
\]

This completes the proof. \( \square \)

**Proofs for Section 4.2.3**

We begin by proving the sesquilinear form defined in (4.31) is a Hermitian form.

**Proof.** [Lemma 44] Let \((k, \omega) \in B_{\mathbb{R}}\). By Lemma 41, for any \(\gamma_1, \gamma_2 \in \ker(e^{ikd}I_n - \Psi(d, \omega))\), if we let \(\psi_1 := \Psi(\cdot, \omega) \gamma_1\), \(\psi_2 := \Psi(\cdot, \omega) \gamma_2\) then we have

\[
q_{(k, \omega)}(\gamma_1, \gamma_2) = \frac{1}{d} \int_0^d \langle A_\omega(x, \omega) \psi_1(x), \psi_2(x) \rangle_{\mathbb{C}} dx = \frac{1}{d} \int_0^d \langle A_\omega(x, \omega) \psi_1(x), \psi_2(x) \rangle_{\mathbb{C}} dx = \frac{1}{d} \int_0^d \langle A_\omega(x, \omega) \psi_1(x), \psi_2(x) \rangle_{\mathbb{C}} dx = \frac{1}{d} \int_0^d \langle A_\omega(x, \omega) \psi_1(x), \psi_2(x) \rangle_{\mathbb{C}} dx = \overline{q_{(k, \omega)}(\gamma_2, \gamma_1)},
\]

implying \(q_{(k, \omega)}(\gamma_1, \gamma_1) \in \mathbb{R}\). This completes the proof. \( \square \)

**Proof.** [Proposition 45] By Definition 23, the canonical ODEs in (4.13) are of definite type at a point \((k_0, \omega_0) \in B_{\mathbb{R}}\) if and only if

\[
\frac{1}{d} \int_0^d \langle A_\omega(x, \omega_0) \psi(x), \psi(x) \rangle_{\mathbb{C}} dx \neq 0
\]

for any nontrivial Bloch solution \(\psi\) of the canonical ODEs in (4.13) with wavenumber-frequency \((k_0, \omega_0)\). But by Theorem 35, \(\psi\) is a nontrivial Bloch solution \(\psi\) of the canonical ODEs in (4.13) with wavenumber-frequency \((k_0, \omega_0)\) if and only if \(\psi = \Psi(\cdot, \omega_0) \gamma\) where \(\gamma\)
is an eigenvector of the monodromy matrix \( \Psi(d, \omega_0) \) corresponding to the eigenvalue \( e^{ik_0d} \). And hence the canonical ODEs in (4.13) are of definite type at a point \((k_0, \omega_0) \in \mathcal{B}_\mathbb{R}\) if and only if \( \ker(e^{ik_0d}I_n - \Psi(d, \omega_0)) \neq \{0\} \) and

\[
q_{(k_0, \omega_0)}(\gamma, \gamma) \neq 0, \quad \text{for every nonzero } \gamma \in \ker(e^{ik_0d}I_n - \Psi(d, \omega_0)).
\] (4.64)

Thus to complete the proof we need only show that (4.64) implies \( q_{(k_0, \omega_0)} \), as defined in (4.31), is a definite sesquilinear form which is bounded.

Suppose that (4.64) is true. Then it follows from Lemma 44 that \( q_{(k_0, \omega_0)}(\gamma, \gamma) \in \mathbb{R}/\{0\} \) for every nonzero \( \gamma \in \ker(e^{ik_0d}I_n - \Psi(d, \omega_0)) \). Suppose there existed nonzero \( \gamma_-, \gamma_+ \in \ker(e^{ik_0d}I_n - \Psi(d, \omega_0)) \) such that \( q_{(k_0, \omega_0)}(\gamma_-, \gamma_-) < 0 < q_{(k_0, \omega_0)}(\gamma_+, \gamma_+). \) Then we can define a continuous real-valued function \( f : [0, 1] \to \mathbb{R} \) by

\[
f(x) := q_{(k_0, \omega_0)}(x \gamma_+ + (1-x) \gamma_-, x \gamma_+ + (1-x) \gamma_-).
\]

Now \( f(0) < 0 < f(1) \) and so by the intermediate value theorem that there exists \( x_0 \in (0, 1) \) such that \( f(x_0) = 0 \) implying by hypothesis that \( x_0 \gamma_+ + (1-x_0) \gamma_- = 0. \) But this implies \( 0 < x_0^2 q_{(k_0, \omega_0)}(\gamma_+, \gamma_+) = (1-x_0)^2 q_{(k_0, \omega_0)}(\gamma_-, \gamma_-) < 0, \) a contradiction. Thus we conclude that \( \text{sgn}(q_{(k_0, \omega_0)}(\gamma, \gamma)) =: \text{sgn}(q_{(k_0, \omega_0)}) \) is independent of the choice of nonzero \( \gamma \) in \( \ker(e^{ik_0d}I_n - \Psi(d, \omega_0)). \)

We define a sesquilinear form by

\[
\langle \gamma_1, \gamma_2 \rangle_{(k_0, \omega_0)} := \text{sgn}(q_{(k_0, \omega_0)}) q_{(k_0, \omega_0)}(\gamma_1, \gamma_2), \quad \gamma_1, \gamma_2 \in \ker(e^{ik_0d}I_n - \Psi(d, \omega_0)).
\]

It follows from what we have just shown that \( \langle , \rangle_{(k_0, \omega_0)} \) is an inner product on the finite-
dimensional vector space $\ker (e^{ik_0d}I_n - \Psi(d, \omega_0))$ and so

$$||\gamma||_{(k_0, \omega_0)} := \langle \gamma, \gamma \rangle_{(k_0, \omega_0)}^{\frac{1}{2}} = |q_{(k_0, \omega_0)}(\gamma, \gamma)|^{\frac{1}{2}}, \quad \gamma \in \ker (e^{ik_0d}I_n - \Psi(d, \omega_0)),$$

defines a norm on this space as does the Euclidean norm $|| \cdot ||_\mathbb{C}$. But since any two norms on a finite dimensional vector space are equivalent this implies there exists $C_1, C_2 > 0$ such that

$$C_1||\gamma||_\mathbb{C}^2 \leq ||\gamma||_{(k_0, \omega_0)}^2 \leq C_2||\gamma||_\mathbb{C}^2$$

for every $\gamma \in \ker (e^{ik_0d}I_n - \Psi(d, \omega_0))$. This completes the proof. \hfill \blacksquare

**Proof.** [Lemma 46] Let \{$(k_j, \omega_j)$\}$_{j \in \mathbb{N}}$ be a sequence in $\mathcal{B}_\mathbb{R}$ and \{$(\gamma_j)\}_{j \in \mathbb{N}}$ a sequence in $\mathbb{C}^n$ such that $\gamma_j \in \ker (e^{ik_jd}I_n - \Psi(d, \omega_j)), ||\gamma_j||_\mathbb{C} = 1$ for all $j \in \mathbb{N}$ and

$$(k_j, \omega_j) \xrightarrow{|| \cdot ||_\mathbb{C}} (k_0, \omega_0) \quad \text{and} \quad \gamma_j \xrightarrow{|| \cdot ||_\mathbb{C}} \gamma_0, \quad \text{as} \quad j \rightarrow \infty,$$

for some $k_0, \omega_0 \in \mathbb{C}$ and $\gamma_0 \in \mathbb{C}^n$. We will now show that $||\gamma_0||_\mathbb{C} = 1$, $\gamma_0 \in \ker (e^{ik_0d}I_n - \Psi(d, \omega_0))$, and $(k_0, \omega_0) \in \mathcal{B}_\mathbb{R}$.

First, since $\gamma_j \xrightarrow{|| \cdot ||_\mathbb{C}} \gamma_0$ as $j \rightarrow \infty$, then

$$1 = \lim_{j \rightarrow \infty} ||\gamma_j||_\mathbb{C} = ||\gamma_0||_\mathbb{C}.$$

Second, if we define a matrix-valued function by $M(k, \omega) := e^{ikd}I_n - \Psi(d, \omega), (k, \omega) \in \mathbb{C} \times \Omega$ then this function is continuous since

$$||M(k, \omega) - M(k_0, \omega_0)||_\mathbb{C} \leq ||M(k, \omega) - M(k_0, \omega)||_\mathbb{C} + ||M(k_0, \omega) - M(k_0, \omega_0)||_\mathbb{C}$$

$$= |e^{ikd} - e^{ik_0d}|n^{\frac{1}{2}} + ||\Psi(d, \omega) - \Psi(d, \omega_0)||_\mathbb{C}$$
and by Proposition 33.(iii) we have $\Psi(d, \cdot) \in \mathcal{O}(\Omega, M_n(\mathbb{C}))$ implying $||\Psi(d, \omega) - \Psi(d, \omega_0)||_C \to 0$ as $\omega \to \omega_0$. Thus we conclude from this that

$$||M(k_0, \omega_0)\gamma_0||_C = \lim_{j \to \infty} ||M(k_j, \omega_j)\gamma_j - M(k_0, \omega_0)\gamma_0||_C$$

$$\leq \lim_{j \to \infty} ||M(k_j, \omega_j)\gamma_j - M(k_0, \omega_0)\gamma_j||_C + \lim_{j \to \infty} ||M(k_0, \omega_0)\gamma_j - M(k_0, \omega_0)\gamma_0||_C$$

$$\leq \lim_{j \to \infty} ||M(k_j, \omega_j) - M(k_0, \omega_0)||_C ||\gamma_j||_C + \lim_{j \to \infty} ||M(k_0, \omega_0)||_C ||\gamma_j - \gamma_0||_C$$

$$= 0$$

and hence $\gamma_0 \in \ker(e^{ik_0d}I_n - \Psi(d, \omega_0))$. It now follows from this and Corollary 38 that $(k_0, \omega_0) \in \mathcal{B}_R$.

Now we complete the proof by showing

$$q_{(k_j, \omega_j)}(\gamma_j, \gamma_j) \xrightarrow[j \to \infty]{} q_{(k_0, \omega_0)}(\gamma_0, \gamma_0), \quad \text{as } j \to \infty. \quad (4.65)$$

First, it follows from Corollary 43 that for any $(k, \omega) \in \mathcal{B}_R$ and for any $\beta_1, \beta_2 \in \ker(e^{ikd}I_n - \Psi(d, \omega))$ we have

$$q_{(k, \omega)}(\beta_1, \beta_2) = \frac{1}{d} \langle \Psi(d, \omega)^* \mathcal{J} \Psi(d, \omega) \beta_1, \beta_2 \rangle_C.$$ 

Thus we extend the definition of $q_{(k, \omega)}$ to all of $\mathbb{C}^n \times \mathbb{C}^n$ in the obvious way by

$$q_{(k, \omega)}(\beta_1, \beta_2) := \frac{1}{d} \langle \Psi(d, \omega)^* \mathcal{J} \Psi(d, \omega) \beta_1, \beta_2 \rangle_C, \quad \beta_1, \beta_2 \in \mathbb{C}^n.$$ 

It follows from this definition that $q_{(k, \omega)} : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ is a continuous sesquilinear form since

$$|q_{(k, \omega)}(\beta_1, \beta_2)| \leq d^{-1} ||\Psi(d, \omega)^* \mathcal{J} \Psi(d, \omega)||_C ||\beta_1||_C ||\beta_2||_C, \quad \forall \beta_1, \beta_2 \in \mathbb{C}^n.$$
Hence for any \((k, \omega) \in B_\mathbb{R}\) and any \(\beta \in \mathbb{C}^n\) we have the inequality

\[
|q_{(k, \omega)}(\beta, \beta) - q_{(k_0, \omega_0)}(\gamma, \gamma)| \leq |q_{(k, \omega)}(\beta, \beta) - q_{(k, \omega)}(\beta, \gamma)| \\
+ |q_{(k, \omega)}(\beta, \gamma) - q_{(k_0, \omega_0)}(\beta, \gamma_0)| \\
+ |q_{(k_0, \omega_0)}(\beta, \gamma_0) - q_{(k_0, \omega_0)}(\gamma_0, \gamma_0)|
\]

and the inequalities

\[
|q_{(k, \omega)}(\beta, \beta) - q_{(k, \omega)}(\beta, \gamma)| = |q_{(k, \omega)}(\beta, \beta - \gamma)| \\
\leq d^{-1}||\Psi(d, \omega)|J|\Psi_\omega(d, \omega)||c||\beta - \gamma_0||c,
\]

\[
|q_{(k_0, \omega_0)}(\beta, \gamma_0) - q_{(k_0, \omega_0)}(\gamma_0, \gamma_0)| = |q_{(k_0, \omega_0)}(\beta, \gamma)| \\
\leq d^{-1}||\Psi(d, \omega)|J|\Psi_\omega(d, \omega)||c||\beta - \gamma_0||c, \gamma_0||c,
\]

\[
|q_{(k, \omega)}(\beta, \gamma) - q_{(k_0, \omega_0)}(\beta, \gamma)| = d^{-1}||\Psi(d, \omega)|J|\Psi_\omega(d, \omega)\|
\]

\[
- \Psi(d, \omega_0)|J|\Psi_\omega(d, \omega_0)||c||\beta - \gamma_0||c,
\]

And thus it follows from these inequalities that (4.65) is true if

\[
\Psi(d, \omega)|J|\Psi_\omega(d, \omega) \xrightarrow{\| \cdot \|_C} \Psi(d, \omega_0)|J|\Psi_\omega(d, \omega_0), \quad \text{as} \quad \omega \xrightarrow{\|} \omega_0.
\]

But by Lemma 77, multiplication of matrices in \(M_n(\mathbb{C})\) is a continuous bilinear map and so in order to prove the latter statement we need only show

\[
\Psi(d, \omega)|J|\Psi_\omega(d, \omega) \xrightarrow{\| \cdot \|_C} \Psi(d, \omega_0)|J|\Psi_\omega(d, \omega_0), \quad \text{as} \quad \omega \xrightarrow{\|} \omega_0.
\]
Convergence of the first limit follows from the fact that the adjoint map \((\cdot)^*: M_n(\mathbb{C}) \to M_n(\mathbb{C})\) is a norm preserving map, i.e., \(\|B^*\|_C = \|B\|_C\) for every \(B \in M_n(\mathbb{C})\), since \(\|\Psi(d, \omega)^* - \Psi(d, \omega_0)^*\|_C = \|(\Psi(d, \omega) - \Psi(d, \omega_0))^*\|_C = \|\Psi(d, \omega) - \Psi(d, \omega_0)\|_C \to 0\) as \(\omega \to \omega_0\). Now for the other limit, by Lemma 81 since \(\Psi(d, \cdot) \in \mathcal{O}(\Omega, M_n(\mathbb{C}))\) and, by Proposition 33.(iii), \(\Psi_\omega(d, \cdot)\) is its derivative in the \(\|\cdot\|_C\) norm then \(\Psi_\omega(d, \cdot) \in \mathcal{O}(\Omega, M_n(\mathbb{C}))\). But this implies \(\Psi_\omega(d, \cdot)\) is a continuous matrix-valued function on \(\Omega\) in the \(\|\cdot\|_C\) norm and so \(\Psi_\omega(d, \cdot) \xrightarrow{|\cdot|_C} \Psi_\omega(d, \omega_0)\) as \(\omega \xrightarrow{\|\cdot\|_C} \omega_0\). Therefore (4.65) is true and this completes the proof.

**Proof.** [Theorem 47] Suppose \((k_0, \omega_0) \in \mathcal{B}_\mathbb{R}\) is a point of definite type for the canonical ODEs in (4.13). We will first show that there exists an \(r > 0\) such that every \((k, \omega) \in B((k_0, \omega_0), r) \cap \mathcal{B}_\mathbb{R}\) is a point of definite type for the canonical ODEs in (4.13). We will prove this by contradiction. Suppose the statement was not true. Then we can find a sequence of points \(\{(k_j, \omega_j)\}_{j=1}^\infty \subseteq \mathcal{B}_\mathbb{R}\) and a sequence \(\{\psi_j\}_{j=1}^\infty \subseteq (W^{1,1}_{\text{loc}}(\mathbb{R}))^n\) such that \(\psi_j\) is a nontrivial Bloch solution of the canonical ODEs in (4.13) with wavenumber-frequency pair \((k_j, \omega_j)\) and

\[
\frac{1}{d} \int_0^d \langle A_\omega(x, \omega) \psi_j(x), \psi_j(x) \rangle_C dx = 0,
\]

for every \(j \in \mathbb{N}\). By Theorem 35 there exists a nonzero \(\gamma_j \in \ker(e^{ik_j d} I_n - \Psi(d, \omega_j))\) such that \(\psi_j = \Psi(d, \omega_j) \gamma_j\), for every \(j \in \mathbb{N}\). Let \(\beta_j := \frac{\gamma_j}{|\gamma_j|_C}\), for \(j \in \mathbb{N}\). Then \(\{\beta_j\}_{j=1}^\infty\) is a sequence of vectors with unit norm in \(\mathbb{C}^n\) and hence since \(\{\beta \in \mathbb{C}^n : \|\beta\|_C = 1\}\) is a compact set this implies there exists a convergent subsequence \(\{\beta_{j_l}\}_{l=1}^\infty\) such that \(\beta_{j_l} \xrightarrow{|\cdot|_C} \beta_0\) as \(l \to \infty\) for some \(\beta_0 \in \mathbb{C}^n\) with \(\|\beta_0\|_C = 1\). Thus since \(\{(k_{j_l}, \omega_{j_l})\}_{l \in \mathbb{N}} \subseteq \mathcal{B}_\mathbb{R}, \{\beta_{j_l}\}_{j \in \mathbb{N}} \subseteq \mathbb{C}^n\) such that \(\beta_{j_l} \in \ker(e^{ik_{j_l} d} I_n - \Psi(d, \omega_{j_l}))\) with \(\|\beta_{j_l}\|_C = 1\) for all \(l \in \mathbb{N}\) and

\[
(k_{j_l}, \omega_{j_l}) \xrightarrow{|\cdot|_C} (k_0, \omega_0) \text{ and } \beta_{j_l} \xrightarrow{|\cdot|_C} \beta_0, \text{ as } l \to \infty
\]
then by Lemma 46, $\beta_0 \in \ker(e^{ik_0d}I_n - \Psi(d, \omega_0))$ with $\|\beta_0\|_C = 1$ and

$$q_{(k_j, \omega_j)}(\beta_j, \beta_j) \xrightarrow{l \to \infty} q_{(k_0, \omega_0)}(\beta_0, \beta_0)$$

But for every $l \in \mathbb{N}$ we have

$$q_{(k_j, \omega_j)}(\beta_j, \beta_j) = \|\gamma_j\|_C^2 q_{(k_j, \omega_j)}(\gamma_j, \gamma_j)$$

$$= \|\gamma_j\|_C^2 \frac{1}{d} \int_0^d \langle A_\omega(x, \omega) \psi_j(x), \psi_j(x) \rangle_C dx = 0$$

implying

$$q_{(k_0, \omega_0)}(\beta_0, \beta_0) = 0, \quad \|\beta_0\|_C = 1, \quad \beta_0 \in \ker(e^{ik_0d}I_n - \Psi(d, \omega_0)).$$

By Proposition 45 this contradicts that $(k_0, \omega_0) \in \mathcal{B}_\mathbb{R}$ is a point of definite type for the canonical ODEs in (4.13). Thus we conclude that there exists an $r > 0$ such that every $(k, \omega) \in B((k_0, \omega_0), r) \cap \mathcal{B}_\mathbb{R}$ is a point of definite type for the canonical ODEs in (4.13).

Now let $\{(k_j, \omega_j)\}_{j=1}^\infty \subseteq \mathcal{B}_\mathbb{R}$ be any sequence converging to $(k_0, \omega_0)$ in the Euclidean norm $\|\cdot\|_C$. Then since the set $\{\beta \in \mathbb{C}^n : \|\beta\|_C = 1\}$ is compact we can find a sequence $\{\gamma_j\}_{j=1}^\infty$ such that $\gamma_j \in \ker(e^{ik_jd}I_n - \Psi(d, \omega_j))$ with $\|\gamma_j\|_C = 1$ for all $j \in \mathbb{N}$ and

$$\gamma_j \xrightarrow{\|\cdot\|_C} \gamma_0, \quad \text{as } j \to \infty$$

for some $\gamma_0 \in \{\beta \in \mathbb{C}^n : \|\beta\|_C = 1\}$. From what we just proved there exists a $J \in \mathbb{N}$ such that if $j \geq J$ then $(k_j, \omega_j)$ is a point of definite type for the canonical ODEs in (4.13). Thus by Proposition 45 and Lemma 46 it follows that $\gamma_0 \in \ker(e^{ik_0d}I_n - \Psi(d, \omega_0))$ with $\|\gamma_0\|_C = 1$ and

$$\text{sgn}(q_{(k_j, \omega_j)}) = \frac{q_{(k_j, \omega_j)}(\gamma_j, \gamma_j)}{|q_{(k_j, \omega_j)}(\gamma_j, \gamma_j)|} \xrightarrow{j \to \infty} \frac{q_{(k_0, \omega_0)}(\gamma_0, \gamma_0)}{|q_{(k_0, \omega_0)}(\gamma_0, \gamma_0)|} = \text{sgn}(q_{(k_0, \omega_0)}), \quad \text{as } j \to \infty.$$
From which we conclude that for \( j \) sufficiently large we must have \( \text{sgn}(q_{(k_j, \omega_j)}) = \text{sgn}(q_{(k_0, \omega_0)}) \).

Therefore, since this is true for an arbitrary sequence in \( \mathcal{B}_\mathbb{R} \), we conclude that there exists an \( r > 0 \) such that every \( (k, \omega) \in B((k_0, \omega_0), r) \cap \mathcal{B}_\mathbb{R} \) is a point of definite type for the canonical ODEs in (4.13) and \( \text{sgn}(q_{(k, \omega)}) = \text{sgn}(q_{(k_0, \omega_0)}) \). This completes the proof.

**Proofs for Section 4.2.4**

We now begin proving our main results on the perturbation theory for canonical ODEs. Our proof relies heavily on the spectral perturbation theory for holomorphic matrix functions and the reader may wish to review chapter 3 before preceding.

**Proof.** [Theorem 48] Let \( (k_0, \omega_0) \in \mathcal{B}_\mathbb{R} \) be a point of definite type for the canonical ODEs in (4.13). Let \( g \) be the number of Jordan blocks (geometric multiplicity) in the Jordan form of the monodromy matrix \( \Psi(d, \omega_0) \) corresponding to the eigenvalue \( \lambda_0 = e^{ik_0d} \) and \( m_1 \geq \cdots \geq m_g \geq 1 \) the dimensions of each of those Jordan blocks (partial multiplicities). Define \( m := m_1 + \cdots + m_g \) (algebraic multiplicity). Let \( \epsilon > 0 \) be given.

Choose an \( r_0 > 0 \) such that \( 0 < r_0 \leq \epsilon \) and \( B(\omega_0, r_0) \in \Omega \). Define the function \( L : U \rightarrow M_n(\mathbb{C}) \) by

\[
L(\omega, k) := \Psi(d, \omega) - e^{ikd}I_n. \tag{4.66}
\]

where \( U := B(\omega_0, r_0) \times B(k_0, r_0) \). By definition of a holomorphic matrix-valued function of two variables, see 3.2.(i), and the fact \( \Psi(d, \cdot) \in \mathcal{O}(B(\omega_0, r_0), M_n(\mathbb{C})) \), it follows that

\[
L \in \mathcal{O}(U, M_n(\mathbb{C})). \tag{4.67}
\]

We will now use the results from section 3.2 and give a spectral perturbation analysis of the
holomorphic matrix function $L(\cdot, k)$ near the point $(\omega_0, k_0) \in \sigma(L) = \mathcal{B}^{-1}$.

First, since $B(\omega_0, r_0)^* = B(\omega_0, r_0)$, the adjoint of the holomorphic function $\Psi(d, \omega)$, see Definition 14, is

$$\Psi^*(d, \omega) = \Psi(d, \overline{\omega})^*, \quad \omega \in B(\omega_0, r_0),$$

(4.68)

and by Lemma 24 we have

$$\Psi^*(d, \cdot) \in \mathcal{O}(B(\omega_0, r_0), M_n(\mathbb{C})).$$

(4.69)

Next, since $U^* = U$, the adjoint $L^*$ of the holomorphic function of two variables $L$, see Definition 15, is

$$L^*(\omega, k) = L(\overline{\omega}, \overline{k})^*, \quad (\omega, k) \in U,$$

(4.70)

and by Lemma 26 we have

$$L^* \in \mathcal{O}(U, M_n(\mathbb{C})).$$

(4.71)

We now prove series of lemmas which relate the spectrum of $L$ to the spectrum of its adjoint $L^*$.

**Lemma 59** For every $(\omega, k) \in U$ we have

$$L^*(\omega, k) \mathcal{J} = -e^{-ikd}\Psi^*(d, \omega) \mathcal{J} L(\omega, k).$$

(4.72)
Proof. By Proposition 39 we have
\[
\Psi^*(d, \omega) \mathcal{J} \Psi(d, \omega) = \Psi(d, \omega)^* \mathcal{J} \Psi(d, \omega) = \mathcal{J}
\]
for every \( B(\omega_0, r_0) \cap \mathbb{R} \subseteq \Omega \). Thus if \((\omega, k) \in U \cap \mathbb{R}^2\) then
\[
-e^{-ikd} \Psi^*(d, \omega) \mathcal{J} L(\omega, k) = -e^{-ikd} \Psi(d, \omega)^* \mathcal{J} (\Psi(d, \omega) - e^{ikd} I_n)
\]
\[
= \Psi(d, \omega)^* \mathcal{J} - e^{-ikd} \mathcal{J} = (\Psi(d, \omega)^* - e^{-ikd}) \mathcal{J}
\]
\[
= L^*(\omega, k) \mathcal{J}.
\]

But for the matrix-valued function \( f(\omega, k) := -e^{-ikd} \Psi^*(d, \omega) \mathcal{J} L(\omega, k) - L^*(\omega, k) \mathcal{J} \) we have \( f \in \mathcal{O}(U, M_n(\mathbb{C})) \) and its identically equal to zero on \( U \cap \mathbb{R}^2 \) where \( U = B(\omega_0, r_0) \times B(k_0, r_0) \).

By Lemma 17, \( f \) is an analytic function of two variables identically equal to zero on \( U \cap \mathbb{R}^2 \) hence using the power series expansion of \( f \) at the point \((\omega_0, k_0)\) and the fact \( f = 0 \) on \( U \cap \mathbb{R}^2 \) we can conclude \( f = 0 \) in an open connected set of \( \mathbb{C}^2 \) in \( U \) which contains \((\omega_0, k_0)\).

Thus, since \( U \) is an open connected set on which \( f \) is analytic, by analytic continuation we conclude that \( f = 0 \) on \( U \). This completes the proof of the lemma. \( \blacksquare \)

The following two lemmas are the key to the proof of this theorem.

**Lemma 60** If \( \omega(k) \) is an eigenvalue Puiseux series of \( L(\cdot, k) \) expanded about \( k_0 \) with limit point \( \omega_0 \) then \( \omega(k) \) and \( \omega^*(k) \), the adjoint of the Puiseux series \( \omega(k) \), both are eigenvalue Puiseux series of \( L^*(\cdot, k) \) and \( L(\cdot, k) \) expanded about \( k_0 \) with limit point \( \omega_0 \).

**Proof.** By Lemma 59, the fact \( \mathcal{J} \) and \( \Psi(d, \omega) \) are invertible matrices for every \( \omega \in \Omega \), implies the holomorphic matrix function \( L(\cdot, k) \) and its adjoint \( L^*(\cdot, k) \) are equivalent, i.e, \( L(\cdot, k) \sim L^*(\cdot, k) \), at \( \omega \) in the sense of holomorphic matrix functions from chapter 3 for every \((\omega, k) \in U \). In particular, \( L(\cdot, k) \) and its adjoint \( L^*(\cdot, k) \) have the same eigenvalues for every
$k \in B(k_0, r_0)$, i.e., for every $k \in B(k_0, r_0)$

$$\sigma(L^*(\cdot, k)) = \sigma(L(\cdot, k)).$$

This implies if $\omega(k)$ is an eigenvalue Puiseux series of $L(\cdot, k)$ expanded about $k_0$ with limit point $\omega_0$ then by Lemma 28 its adjoint $\omega^*(k)$ is an eigenvalue Puiseux series of $L^*(\cdot, k)$ expanded about $\overline{k}_0 = k_0$ with limit point $\overline{\omega}_0 = \omega_0$. But since $\sigma(L^*(\cdot, k)) = \sigma(L(\cdot, k))$ then all the values of the branches of the Puiseux series $\omega^*(k)$ and $\omega(k)$ are eigenvalues of both $L^*(\cdot, k)$ and $L(\cdot, k)$ which, by the definition of eigenvalue Puiseux series, means $\omega^*(k)$ and $\omega(k)$ are also eigenvalue Puiseux series of $L^*(\cdot, k)$ expanded about $k_0$ with limit point $\omega_0$. This completes the proof. ■

**Lemma 61** If $\omega(k)$ is an eigenvalue Puiseux series of $L(\cdot, k)$ expanded about $k_0$ with limit point $\omega_0$ and $\beta$ is generating eigenvector of $L(\cdot, k)$ at the point $(\omega_0, k_0)$ and associated with $\omega(k)$ then $\mathcal{J}\beta$ is a generating eigenvector of $L^*(\cdot, k)$ at the point $(\omega_0, k_0)$ and associated with $\omega(k)$. Moreover,

$$\langle L_\omega(\omega_0, k_0)\beta, \mathcal{J}\beta \rangle_C \neq 0. \quad (4.73)$$

**Proof.** Let $\omega(k)$ be an eigenvalue Puiseux series of $L(\cdot, k)$ expanded about $k_0$ with limit point $\omega_0$ and $\beta$ is generating eigenvector of $L(\cdot, k)$ at the point $(\omega_0, k_0)$ and associated with $\omega(k)$. By definition of generating eigenvector, there exists a eigenpair Puiseux series $(\omega(k), \gamma(k))$ of $L(\cdot, k)$ expanded about $k_0$ such that $\omega(k)$ has limit point $\omega_0$ and $\gamma(k)$ has limit point $\beta$. By Lemma 59 and the fact $\det(\mathcal{J}) \neq 0$ it follows that $(\omega(k), \mathcal{J}\gamma(k))$ is an eigenpair Puiseux series of $L^*(\cdot, k)$ expanded about $k_0$. Hence $\mathcal{J}\gamma(k_0) = \mathcal{J}\beta_0$ is the limit point of the eigenvector Puiseux series $\mathcal{J}\gamma(k)$. By definition of generating eigenvector this means $\mathcal{J}\beta$ is a generating eigenvector of $L^*(\cdot, k)$ at the point $(\omega_0, k_0)$ and associated with $\omega(k)$. Moreover, since $\beta$ is a generating eigenvector of $L(\cdot, k)$ at the point $(\omega_0, k_0)$ this
means $\beta \neq 0$ and $\beta \in \ker L(\omega_0, k_0) = \ker(e^{ik_0dI_n} - \Psi(d, \omega_0))$. This means by Lemma 45 and Corollary 43 that

$$0 \neq q_{(k_0, \omega_0)}(\beta, \beta) = \frac{1}{d} \langle \Psi(d, \omega_0)^* \mathcal{J} \Psi(d, \omega_0) \beta, \beta \rangle_C = \frac{1}{d} \langle \mathcal{J} \Psi(d, \omega_0) \beta, \Psi(d, \omega_0) \beta \rangle_C$$

$$= \frac{1}{d} \langle \mathcal{J} \Psi(d, \omega_0) \beta, e^{ik_0d} \beta \rangle_C = -\frac{e^{ik_0d}}{d} \langle \Psi(d, \omega_0) \beta, \mathcal{J} \beta \rangle_C.$$

Therefore,

$$\langle L_\omega(\omega_0, k_0) \gamma_0, \mathcal{J} \beta_0 \rangle_C = \langle \Psi(d, \omega_0) \beta, \mathcal{J} \beta \rangle_C \neq 0.$$

This completes the proof of the lemma. □

**Proposition 62** If $\omega(k)$ is an eigenvalue Puiseux series of $L(\cdot, k)$ expanded about $k_0$ with limit point $\omega_0$ then $\omega(k)$ is a single-valued real analytic nonconstant function of $k$.

**Proof.** Let $\omega(k)$ is an eigenvalue Puiseux series of $L(\cdot, k)$ expanded about $k_0$ with limit point $\omega_0$. Then by Lemma 60 we know that $\omega^*(k)$ is also an eigenvalue Puiseux series of $L(\cdot, k)$ expanded about $k_0$ with limit point $\omega_0$. Let $\omega_1(k) := \omega(k)$ and $\omega_2(k) := \omega^*(k)$. By definition of the adjoint of a convergent Puiseux series we have $\omega_2^*(k) = (\omega^*)^*(k) = \omega(k) = \omega_1(k)$.

Suppose that for some branch $\omega_{1,h_1}(k)$ of $\omega_1(k)$ and some branch $\omega_{2,h_2}(k)$ of $\omega_2(k)$ there existed an $r > 0$ such that $\omega_{1,h_1}(k) \neq \omega_{2,h_2}(k)$ for $|k - k_0| < r$. Then this implies by Proposition 29 that for every generating eigenvector $\beta$ of $L(\cdot, k)$ at the point $(\omega_0, k_0)$ and associated with $\omega_1(k)$ and for any generating eigenvector $\gamma$ of $L^*(\cdot, k)$ at the point $((\omega_0, k_0))$ and associated with $\omega_2^*(k)(= \omega_1(k))$ we have

$$\langle L_\omega(\omega_0, k_0) \beta, \gamma \rangle_C = 0.$$

But by Lemma 61 we know that $\mathcal{J} \beta$ is a generating eigenvector of $L^*(\cdot, k)$ at the point
Thus we have a contradiction. This contradiction implies that there must exist an \( r > 0 \) such that if \( \omega_{1,h_1}(k) \) is any branch of \( \omega_1(k) \) and if \( \omega_{2,h_2}(k) \) is any branch of \( \omega_2(k) \) then \( \omega_{1,h_1}(k) = \omega_{2,h_2}(k) \) for \( |k - k_0| < r \). Thus if we fix the branch \( \omega_{2,h_2}(k) \) then all branches of \( \omega_1(k) \) are equal to \( \omega_{2,h_2}(k) \) for \( |k - k_0| < r \). By Lemma 30 this implies \( \omega_1(k) \) is a single-valued analytic function of \( k \) for \( k \in B(k_0, r) \). For the exact same reasoning this implies \( \omega_2(k) \) is a single-valued analytic function of \( k \) and we have \( \omega_1(k) = \omega_2(k) \) for all \( k \in B(k_0, r) \). But since \( \omega(k) = \omega_1(k) = \omega_2(k) = \omega^*(k) \) where \( \omega^*(k) \) is the adjoint of the analytic function \( \omega(k) \), this implies by definition of adjoint of a Puiseux series, that \( \omega(k) = \omega^*(k) = \overline{\omega(k)} \) for all \( k \in B(k_0, r) \). And so \( \omega(k) \) is a single-valued analytic function in \( B(k_0, r) \) and is real-valued for real \( k \), i.e., is a real analytic function of \( k \) in \( B(k_0, r) \).

To complete the proof we need only show that \( \omega(k) \) is a nonconstant function of \( k \), i.e., \( \omega(k) = \omega(k_0) \) for every \( k \) in \( B(k_0, r) \) is a contradiction. By hypothesis we have \( \omega(k_0) = \omega_0 \) and since \( \omega(k) \) is an eigenvalue Puiseux series of \( L(\cdot, k) \) this means \( 0 = \det L(\omega(k), k) \) for every \( k \) in \( B(k_0, r) \). If \( \omega(k) \) was a constant function then this would mean \( 0 = \det L(\omega(k), k) = \det L(\omega_0, k) = \det(e^{ikd}I_n - \Psi(d, \omega_0)) \) for every \( k \) in \( B(k_0, r) \). But \( \det(\lambda I_n - \Psi(d, \omega_0)) \) is a monic polynomial in \( \lambda \) and so has a nonzero but finite number of zeros where as \( f(k) := e^{ikd} \) being a nonconstant analytic function is an open map and hence \( f(B(k_0, r)) \) is a nonempty open set in \( \mathbb{C} \) and hence contains an uncountable number of elements. But then because \( 0 = \det (f(k)I_n - \Psi(d, \omega_0)) \) for every \( k \) in \( B(k_0, r) \) this implies \( f(B(k_0, r)) \) has only a finite number of elements since they are all roots of \( \det(\lambda I_n - \Psi(d, \omega_0)) \). A contradiction. Therefore \( \omega(k) \) is a nonconstant function. This completes the proof. \[ \blacksquare \]

**Lemma 63** Let \( \omega_1(k) \) and \( \omega_2(k) \) are two different eigenvalue Puiseux series of \( L(\cdot, k) \) ex-
panded about \( k_0 \) with limit point \( \omega_0 \). If \( \beta_1, \beta_2 \) are generating eigenvectors of \( L(\cdot, k) \) at the point \((\omega_0, k_0)\) corresponding to \( \omega_1(k) \) and \( \omega_2(k) \), respectively, then

\[
\langle L_\omega(\omega_0, k_0)\beta_1, J\beta_2 \rangle_\mathbb{C} = 0. \tag{4.74}
\]

**Proof.** Let \( \omega_1(k) \) and \( \omega_2(k) \) are two different eigenvalue Puiseux series of \( L(\cdot, k) \) expanded about \( k_0 \) with limit point \( \omega_0 \). Let \( \beta_1, \beta_2 \) are generating eigenvectors of \( L(\cdot, k) \) at the point \((\omega_0, k_0)\) corresponding to \( \omega_1(k) \) and \( \omega_2(k) \), respectively. Then by Proposition 62 \( \omega_1(k) \) and \( \omega_2(k) \) are single-valued real analytic functions and hence there adjoints satisfy \( \omega_j^*(k) = \overline{\omega_j(k)} \) for \( j = 1, 2 \). By Lemma 61 we know \( J\beta_2 \) is a generating eigenvector of \( L^*(\cdot, k) \) at the point \((\overline{\omega_0}, \overline{k_0}) = (\omega_0, k_0)\) and associated with \( \omega_2(k) \). This implies by Proposition 29, since \( \omega_1(k) \) and \( \omega_2^*(k) = \omega_2(k) \) are different analytic functions, that

\[
\langle L_\omega(\omega_0, k_0)\beta_1, J\beta_2 \rangle_\mathbb{C} = 0. \tag{4.75}
\]

This completes the proof. \( \blacksquare \)

**Proposition 64** The holomorphic matrix function \( L(\cdot, k_0) \) has \( \omega_0 \) an eigenvalue of finite algebraic multiplicity. Furthermore, it is a semisimple eigenvalue of \( L(\cdot, k_0) \). Moreover, its geometric multiplicity as an eigenvalue of \( L(\cdot, k_0) \) is \( g \) and this is the order of the analytic function \( \det L(\omega, k_0) = D(k_0, \omega) \) at \( \omega = \omega_0 \).

**Proof.** First, since \((k_0, \omega_0) \in \mathcal{B}_\mathbb{R}\) then \( 0 = D(k_0, \omega_0) = \det(e^{ik_0d}I_n - \Psi(d, \omega_0)) = (-1)^n \det(L(\omega_0, k_0)) \) so that \( \omega_0 \in \sigma(L(\cdot, k_0)) \) and hence by definition \( \omega_0 \) an eigenvalue of the holomorphic matrix function \( L(\cdot, k_0) \). To show that it is an eigenvalue of finite algebraic multiplicity and a semisimple eigenvalue, it suffices to show, by Proposition 15, that

\[
L_\omega(\omega_0, k_0)\gamma_0 = -L(\omega_0, k_0)\gamma_1
\]
has no solution for \( \gamma_0 \in \ker(L(\omega_0, k_0))/\{0\} \) and \( \gamma_1 \in \mathbb{C}^n \).

Suppose there existed vectors \( \gamma_0 \in \ker(L(\omega_0, k_0))/\{0\} \), \( \gamma_1 \in \mathbb{C}^n \) such that \( L(\omega_0, k_0)\gamma_0 = -L(\omega_0, k_0)\gamma_1 \). It follows from this, the fact \( \ker(L(\omega_0, k_0)) = \ker(e^{ik_0d}I_n - \Psi(d, \omega_0)) \), Proposition 45 along with the fact \((k_0, \omega_0)\) is a point of definite type, that we have

\[
0 \neq \frac{1}{d}(\Psi(d, \omega_0)^* \mathcal{J}\Psi(d, \omega_0)\gamma_0, \gamma_0) \\mathbb{C} \\
= \frac{1}{d}(\mathcal{J}\Psi(d, \omega_0)\gamma_0, \Psi(d, \omega_0)\gamma_0) \\mathbb{C} = \frac{1}{d}(\mathcal{J}\Psi(d, \omega_0)\gamma_0, e^{ik_0d}\gamma_0) \\mathbb{C} \\
= -\frac{e^{ik_0d}}{d}(\Psi(d, \omega_0)\gamma_0, \mathcal{J}\gamma_0) \\mathbb{C} = -\frac{e^{ik_0d}}{d}(L(\omega_0, k_0)\gamma_0, \mathcal{J}\gamma_0) \\mathbb{C}.
\]

But \( L(\omega_0, k_0)\gamma_0 = 0 \) and so by Lemma 59 this implies \( 0 = L^*(\omega_0, k_0)\mathcal{J}\gamma_0 = L(\omega_0, k_0)^*\mathcal{J}\gamma_0 \) which implies

\[
(L(\omega_0, k_0)\gamma_1, \mathcal{J}\gamma_0) \\mathbb{C} = (\gamma_1, L(\omega_0, k_0)\mathcal{J}\gamma_0) \\mathbb{C} = 0
\]

which implies

\[
0 \neq -\frac{e^{ik_0d}}{d}(L(\omega_0, k_0)\gamma_0, \mathcal{J}\gamma_0) \\mathbb{C} = -\frac{e^{ik_0d}}{d}(L(\omega_0, k_0)\gamma_0 + L(\omega_0, k_0)\gamma_1, \mathcal{J}\gamma_0) \\mathbb{C} = 0.
\]

This is a contradiction. Thus \( \omega_0 \) is an eigenvalue of \( L(\cdot, k_0) \) of finite algebraic multiplicity and a semisimple eigenvalue. And so its geometric multiplicity equals its algebraic multiplicity and hence by Proposition 10, its geometric multiplicity is \( \dim \ker(L(\omega_0, k_0)) = \dim \ker(e^{ik_0d}I_n - \Psi(d, \omega_0)) = g \) and its algebraic multiplicity again \( g \) is equal to the order of the analytic function \( \det(L(\omega, k_0)) \) at \( \omega = \omega_0 \). And since \( \det(L(\omega, k_0)) = (-1)^n \det(e^{ik_0d}I_n - \Psi(d, \omega_0)) = D(k_0, \omega) \) for \( \omega \in B(\omega_0, r_0) \) this implies \( g \) is the order of the zero of the analytic function \( D(k_0, \omega) \) at \( \omega = \omega_0 \). This completes the proof of the proposition.

Proposition 65 The holomorphic matrix function \( L(\omega, \cdot) \) has \( k_0 \) an eigenvalue of finite algebraic multiplicity. Moreover, \( k_0 \) as an eigenvalue of \( L(\omega, \cdot) \) has \( g, m_1 \geq \ldots \geq m_g, m \)
as its geometric, partial, and algebraic multiplicities, respectively. In particular, the order of the zero of the analytic function \( \det L(\omega_0, k) = D(k, \omega_0) \) at \( k = k_0 \) is \( m \).

**Proof.** We have \( L(\omega_0, k) = \Psi(d, \omega_0) - e^{ikd}I_n \). If we define the holomorphic matrix functions \( F(\lambda) := \Psi(d, \omega_0) - \lambda I_n \) and \( G(k) := \Psi(d, \omega_0) - e^{ikd}I_n \) then it follows that \( G(k) = F(e^{ikd}) \). It thus follows that \( k_0 \) is an eigenvalue of \( G \), \( \lambda_0 := e^{ik_0d} \) is an eigenvalue of \( F \), and there is a one-to-one correspondence with their geometric, partial, and algebraic multiplicities. But as we know by Example 1 in chapter 3 and by definition of the numbers \( g, m_1 \geq \ldots \geq m_g \), and \( m \), it follows that the geometric, partial, and algebraic multiplicities of the eigenvalue \( \lambda_0 \) of \( F \) are just, in the Jordan normal form of the matrix \( \Psi(d, \omega_0) \) corresponding to the eigenvalue \( \lambda_0 \), \( g \) the number of Jordan blocks, their orders \( m_1, \ldots, m_g \), and \( m = m_1 + \cdots + m_g \), respectively. Thus \( g, m_1, \ldots, m_g \), and \( m \) are the geometric, partial, and algebraic multiplicities of the eigenvalue \( k_0 \) of \( L(\omega_0, k) \). By Proposition 10, since \( m \) being the algebraic multiplicity of the eigenvalue \( k_0 \) of \( L(\omega_0, \cdot) \), we know that \( m \) is the order of the zero of the function \( \det L(\omega_0, k) \) at \( k = k_0 \). And since \( \det L(\omega_0, k) = D(k, \omega_0) \), the proof is complete. \( \blacksquare \)

**Theorem 66** There exists a \( \delta > 0 \) such that \( B(k_0, \delta) \times B(\omega_0, \epsilon) \subseteq U \) and \( \mathcal{B} \cap B(k_0, \delta) \times B(\omega_0, \epsilon) \) is the union of the graphs of \( g \) nonconstant real analytic functions \( \omega_1(k), \ldots, \omega_g(k) \) given by convergent power series

\[
\omega_j(k) = \omega_0 + \nu_{j,m_j}(k-k_0)^{m_j} + \sum_{l=m_j+1}^{\infty} \nu_{j,l}(k-k_0)^l, \quad |k-k_0| < \delta \quad (4.76)
\]

where

\[
\nu_{j,m_j} \neq 0, \quad (4.77)
\]

for \( j = 1, \ldots, g \). Moreover, there exists analytic functions \( \varphi_1(k), \ldots, \varphi_g(k) \) belonging to
\( \mathcal{O}(B(k_0, \delta), \mathbb{C}^n) \) such that

\[
\varphi_j(k) \neq 0, \quad L(\omega_j(k), k) \varphi_j(k) = 0, \quad |k - k_0| < \delta, \quad j = 1, \ldots, g
\]

and the vectors \( \varphi_1(k), \ldots, \varphi_g(k) \) are linearly independent for \( |k - k_0| < \delta \).

**Proof.** The spectrum \( \sigma(L) \) of the holomorphic matrix-valued function \( L \) is given by

\[
\sigma(L) = \{ (\omega, k) \in U \mid \det L(\omega, k) = 0 \}.
\]

It follows from the definition of \( L \) and the set \( U = B(\omega_0, r_0) \times B(k_0, r_0) \subseteq \Omega \times \mathbb{C} \) that the inverse set \( \sigma(L)^{-1} \) is a subset of the Bloch variety \( \mathcal{B} \) since

\[
\sigma(L)^{-1} = \{ (k, \omega) \in B(k_0, r_0) \times B(\omega_0, r_0) \mid \det L(\omega, k) = 0 \}
= \{ (k, \omega) \in B(k_0, r_0) \times B(\omega_0, r_0) \mid \det D(k, \omega) = 0 \}
= \mathcal{B} \cap B(k_0, r_0) \times B(\omega_0, r_0).
\]

Now by Proposition 64 the algebraic multiplicity of the eigenvalue \( \omega_0 \) of \( L(\cdot, k_0) \) is \( g \) and so by Theorem 21, there exists an \( \delta > 0 \), which we may assume without loss of generality \( \delta \leq r_0 \), such that \( \sigma(L)^{-1} \cap B(k_0, \delta) \times B(\omega_0, \epsilon) \) is the union of the graphs of \( l \) convergent Puiseux series \( \omega_1(k), \ldots, \omega_l(k) \) expanded about \( k_0 \) with limit point \( \omega_0 \), domain \( B(k_0, \delta) \), and periods \( q_1, \ldots, q_l \), respectively, which satisfy \( g = q_1 + \cdots + q_l \). This implies each \( \omega_j(k) \) is an eigenvalue Puiseux series of \( L(\cdot, k) \) expanded about \( k_0 \) with limit point \( \omega_0 \) and so by Proposition 62, \( \omega_j(k) \) is a single-valued nonconstant real analytic function of \( k \) in the domain \( B(k_0, \delta) \), for \( j = 1, \ldots, l \). Then, since \( \delta \leq r_0 \leq \epsilon \) and \( \sigma(L)^{-1} = \mathcal{B} \cap B(k_0, r_0) \times B(\omega_0, r_0) \), we have \( \sigma(L)^{-1} \cap B(k_0, \delta) \times B(\omega_0, \epsilon) = \mathcal{B} \cap B(k_0, \delta) \times B(\omega_0, \epsilon) \) and it is the union of the graphs of \( g \) convergent Puiseux series \( \omega_1(k), \ldots, \omega_l(k) \) single-valued nonconstant real analytic function of \( k \) in the domain \( B(k_0, \delta) \) with \( \omega_j(k_0) = \omega_0 \), for \( j = 1, \ldots, l \).
Let \( \omega_1(k), \ldots, \omega_{\hat{g}}(k) \) denote all those analytic functions which are not identically equal on \( B(k_0, \delta) \).

This implies now that they have a convergent series expansion

\[
\omega_j(k) = \omega_0 + \nu_{j,q_j}(k - k_0)^{q_j} + \sum_{l=q_j+1}^{\infty} \nu_{j,l}(k - k_0)^l, \quad |k - k_0| < \delta
\]

where

\[
\nu_{j,q_j} \neq 0,
\]

for \( j = 1, \ldots, \hat{g} \). Without loss of generality we may assume \( q_1 \geq \cdots \geq q_{\hat{g}} \).

Now by the Weierstrass Preparation Theorem, see Theorem 18 in the chapter 3, since \( \det L \in \mathcal{O}((\omega_0, k_0)), \det L(\omega_0, k_0) = 0, \det L(\cdot, k_0) \neq 0 \) which follows since \( L(\cdot, k_0) \) has \( \omega_0 \) as a semisimple eigenvalue, then

\[
\det L = f_0 f_1, \quad f_0 \in \mathcal{O}_0((\omega_0, k_0)), \quad f_1 \in \mathcal{O}((\omega_0, k_0))
\]

where \( f_1(\omega, k) \neq 0 \) for \( (\omega, k) \in B(\omega_0, r) \times B(k_0, r) \) with \( 0 < r \ll 1 \) and \( \deg f_0 = g \), since \( g \) is the order of the zero of \( \det L(\omega, k_0) = f_0(\omega, k_0) f_1(\omega, k_0) \) because by Proposition 64 we know \( \omega_0 \) is a semisimple eigenvalue of \( L(\cdot, k_0) \), which means its geometric multiplicity \( g \) equals its algebraic multiplicity equals the order of the zero of \( \det L(\omega, k_0) \) at \( \omega = \omega_0 \) which thus equals the order of the zero of \( f_0(\omega, k_0) \) at \( \omega = \omega_0 \) which is by definition the \( \deg f_0 \). Now by Theorem 19 since \( f_0 \neq 1 \) because \( \deg f_0 = g > 1 \) there exists unique and distinct irreducibles \( p_1, \ldots, p_s \in \mathcal{O}_0((\omega_0, k_0)) \) such that

\[
f_0 = \prod_{j=1}^{s} p_j^{m(j)}
\]
where $m(j)$ is the unique number representing the multiplicity of the irreducible factor $p_j$, for $j = 1, \ldots, s$ and $g = \deg f_0 = \sum_{j=1}^s m(j) \deg p_j$. But for $\hat{p}_j(\omega, k) := \omega - \omega_j(k) = (\omega - \omega_0) - (\omega_j(k) - \omega_0)$ we have $\hat{p}_j \in \mathcal{O}_0((\omega_0, k_0))$ and it irreducible, for $j = 1, \ldots, \hat{g}$. Now though as we have shown $\sigma(L)^{-1} \cap B(k_0, \delta) \times B(\omega_0, \epsilon)$ is the graph of the $\omega_1(k), \ldots, \omega_{\hat{g}}(k)$ and hence since $\det L = f_0 f_1$ then the zeros of $f_0$ are exactly the zeros of $\hat{p}_j \in \mathcal{O}_0((\omega_0, k_0))$ implying by Theorem 20 and the uniqueness of the factorization into irreducibles we must have after possible reordering of the indices, $\hat{g} = s$, $p_j = \hat{p}_j$ for $j = 1, \ldots, \hat{g}$, and

$$f_0 = \prod_{j=1}^{\hat{g}} p_j^{m(j)}.$$

This of course implies, since $\deg \hat{p}_j = 1$ for each $j$, that

$$g = \deg f_0 = \sum_{j=1}^{\hat{g}} \deg \hat{p}_j^{m(j)} = \sum_{j=1}^{\hat{g}} m(j) \deg \hat{p}_j = \sum_{j=1}^{\hat{g}} m(j).$$

Now it follows from what we have just shown that

$$\det L(\omega_0, k) = f_0(\omega_0, k)f_1(\omega_0, k) = f_1(\omega_0, k) \prod_{j=1}^{\hat{g}} p_j^{m(j)}(\omega_0, k)$$

$$= f_1(\omega_0, k) \prod_{j=1}^{\hat{g}} (\omega_0 - \omega_j(k))^{m(j)} = f_1(\omega_0, k) \prod_{j=1}^{\hat{g}} (-\nu_{j,q_j}(k-k_0)^q_j + o((k-k_0)^q_j))^{m(j)}$$

$$= \left( f_1(\omega_0, k_0) \prod_{j=1}^{\hat{g}} (-\nu_{j,q_j})^{m(j)} \right) (k-k_0)^{\sum_{j=1}^{\hat{g}} q_j m(j)} + o \left( (k-k_0)^{\sum_{j=1}^{\hat{g}} q_j m(j)} \right)$$

$$= \left( f_1(\omega_0, k_0) \prod_{j=1}^{\hat{g}} (-\nu_{j,q_j})^{m(j)} \right) (k-k_0)^{\sum_{j=1}^{\hat{g}} q_j m(j)} + o \left( (k-k_0)^{\sum_{j=1}^{\hat{g}} q_j m(j)} \right)$$

as $k \to k_0$. The leading term $f_1(\omega_0, k_0) \prod_{j=1}^{\hat{g}} (-\nu_{j,q_j})^{m(j)} \neq 0$ implying the order of the zero of $\det L(\omega_0, k)$ at $k = k_0$ is $\sum_{j=1}^{\hat{g}} q_j m(j)$, but by Proposition 65 the order of the zero of

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\[ \det L(\omega_0, k) \text{ at } k = k_0 \text{ is } m. \] This implies
\[ m = \sum_{j=1}^{\hat{g}} q_j m(j). \]

**Lemma 67** For \(0 < |k - k_0| \ll 1\), \(\omega_j(k)\) is a semisimple eigenvalue of \(L(\cdot, k)\) and
\[ m(j) = \dim \ker L(\omega_j(k), k), \quad j = 1, \ldots, \hat{g}. \tag{4.79} \]

**Proof.** For \(0 < |k - k_0| \ll 1\) we have \(\omega_{j_1}(k) \neq \omega_{j_2}(k)\) for \(j_1 \neq j_2\) with \(1 \leq j_1, j_2 \leq \hat{g}\), since by assumption the functions \(\omega_1(k), \ldots, \omega_{\hat{g}}(k)\) where not identically equal on \(B(k_0, \delta)\). And since
\[ \det L(\omega, k) = f_0(\omega, k)f_1(\omega, k) = f_1(\omega, k) \prod_{j=1}^{\hat{g}} (\omega - \omega_j(k))^{m(j)}, \quad f_1(\omega, k) \neq 0 \]
for all \((\omega, k) \in B(\omega_0, r) \times B(k_0, r)\) for \(0 < r \ll 1\), these fact together with \(\omega_j(k_0) = \omega_0\) imply that if \(0 < |k - k_0| \ll 1\) then the order of the zero of the function \(\det L(\cdot, k)\) at \(\omega = \omega_j(k)\) is \(m(j)\) which by 10 this is the algebraic multiplicity of the eigenvalue \(\omega_j(k)\) of the holomorphic matrix function \(L(\cdot, k)\), for \(j = 1, \ldots, \hat{g}\). Now it follows by the local Smith form (see Theorem 9) of the holomorphic matrix function \(L(\omega_j(k), k)\) at the eigenvalue \(k_0\) that
\[ \mu(j) := \dim \ker L(\omega_j(k), k), \quad j = 1, \ldots, \hat{g}, \quad 0 < |k - k_0| \ll 1, \]
are constant. Our goal is to now show that \(\mu(j) = m(j)\), for \(j = 1, \ldots, \hat{g}\). Notice first that by definition of \(\mu(j)\), it is the geometric multiplicity of the eigenvalue of \(\omega_j(k)\) of the holomorphic matrix function \(L(\cdot, k)\) and since \(m(j)\) is the algebraic multiplicity of the eigenvalue \(\omega_j(k)\) of
the holomorphic matrix function $L(\cdot, k)$ for $0 < |k - k_0| \ll 1$. But it follows from Theorem 47 that if $|k - k_0| \ll 1$ and $k$ is real then $(k, \omega_j(k))$ is point of definite type for the canonical ODEs in (4.13) and hence by Proposition 64 we know $\omega_j(k)$ is a semisimple eigenvalue of the holomorphic matrix function $L(\cdot, k)$ which by definition of semisimple eigenvalue means its geometric multiplicity $\mu(j)$ equals its algebraic multiplicity $m(j)$, i.e., $\mu(j) \leq m(j)$, for $j = 1, \ldots, \hat{g}$. And therefore this implies for all $k$ satisfying $0 < |k - k_0| \ll 1$ we have

$$m(j) = \mu(j) = \dim \ker L(\omega_j(k), k), \quad j = 1, \ldots, \hat{g},$$

which means $\omega_j(k)$ is a semisimple eigenvalue of $L(\cdot, k)$. This completes the proof.

Now it follows this lemma and the local Smith form, Theorem 9, of the holomorphic matrix function $L(\omega_j(k), k)$ in the variable $k$ with eigenvalue $k_0$, that there exists an $r > 0$ and there exists function $N_j, M_j \in \mathcal{O}(B(k_0, r), M_n(\mathbb{C}))$ with $N_j(k), M_j(k)$ invertible matrices and

$$N_j(k)L(\omega_j(k), k)M_j(k) = \text{diag}\{0, \ldots, 0, (k - k_0)^{m_j, m_j+1}, \ldots, (k - k_0)^{m_j, n}\}$$

(4.80)

for every $k \in B(k_0, r)$, where the number of zeros down the diagonal is given by $m(j)$ and $\{m_j, i\} = m(j) + 1$ is a decreasing sequence of nonnegative integers, for $j = 1, \ldots, \hat{g}$.

Let $e_1, \ldots, e_n$ denote the standard orthonormal vectors from $\mathbb{C}^n$. We now define the vector-valued functions

$$\varphi_{j,i}(k) := M_j(k)e_i, \quad i = 1, \ldots, m(j), \quad j = 1, \ldots, \hat{g}, \quad |k - k_0| < r.$$

It follows, since $M_j \in \mathcal{O}(B(k_0, r), M_n(\mathbb{C}))$, that $\varphi_{j,i} \in \mathcal{O}(B(k_0, r), \mathbb{C}^n)$ for all $i, j$ and

$$L(\omega_j(k), k)\varphi_{j,i}(k) = 0, \quad i = 1, \ldots, m(j), \quad j = 1, \ldots, \hat{g}, \quad |k - k_0| < r.$$
Lemma 68 For every $k$ with $|k - k_0| \ll 1$, the vectors $\varphi_{j,i}(k)$, $i = 1, \ldots, m(j)$, $j = 1, \ldots, \hat{g}$ are linear independent, $\{\varphi_{j,i}(k)\}_{i=1}^{m(j)}$ is a basis for $\ker L(\omega_j(k), k)$, for $j = 1, \ldots, \hat{g}$, and $\bigcup_{j=1}^{\hat{g}} \{\varphi_{j,i}(k_0)\}_{i=1}^{m(j)}$ is a basis for $\ker L(\omega_0, k_0)$.

Proof. For the first part of the theorem it suffices to prove that their limit points $\varphi_{j,i}(k_0)$, $i = 1, \ldots, m(j)$, $j = 1, \ldots, \hat{g}$ are linearly independent. Suppose they were not. Then there exists constants $c_{j,i}$, $i = 1, \ldots, m(j)$, $j = 1, \ldots, \hat{g}$ with $\sum_{j=1}^{\hat{g}} \sum_{i=1}^{m(j)} |c_{j,i}| > 0$ such that

$$0 = \sum_{j=1}^{\hat{g}} \sum_{i=1}^{m(j)} c_{j,i} \varphi_{j,i}(k_0).$$

But if $j_1, j_2 \in \{1, \ldots, \hat{g}\}$, $i_1 \in \{1, \ldots, m(j_1)\}$, and $i_2 \in \{1, \ldots, m(j_2)\}$ then $(\omega_{j_1}(k), \varphi_{i_1,j_1}(k))$ and $(\omega_{j_2}(k), \varphi_{i_2,j_2}(k))$ are eigenpair Puiseux series of $L(\cdot, k)$ expanded about $k_0$. By the definition of generating eigenvector, $\varphi_{j_1,i_1}(k_0)$ and $\varphi_{j_2,i_2}(k_0)$ are generating eigenvectors of $L(\cdot, k)$ at the point $(\omega_0, k_0)$ and associated with $\omega_1(k)$ and $\omega_2(k)$, respectively. Now if $j_1 \neq j_2$ then $\omega_{j_1}(k)$ and $\omega_{j_2}(k)$ are two different eigenvalue Puiseux series of $L(\cdot, k)$ expanded about $k_0$ with the same limit point $\omega_0$. This implies by Lemma 63 that

$$0 = \langle L_\omega(\omega_0, k_0) \varphi_{j_1,i_1}(k_0), \mathcal{I} \varphi_{j_2,i_2}(k_0) \rangle_C = -\langle \mathcal{I} \Psi_\omega(d, \omega_0) \varphi_{j_1,i_1}(k_0), \varphi_{j_2,i_2}(k_0) \rangle_C$$

$$= -e^{ik_0 d} e^{-ik_0 d} \langle \mathcal{I} \Psi_\omega(d, \omega_0) \varphi_{j_1,i_1}(k_0), \varphi_{j_2,i_2}(k_0) \rangle_C$$

$$= -e^{-ik_0 d} \langle \mathcal{I} \Psi_\omega(d, \omega_0) \varphi_{j_1,i_1}(k_0), \Psi(d, \omega_0) \varphi_{j_2,i_2}(k_0) \rangle_C$$

$$= -e^{-ik_0 d} \langle \Psi(d, \omega_0)^* \mathcal{I} \Psi_\omega(d, \omega_0) \varphi_{j_1,i_1}(k_0), \varphi_{j_2,i_2}(k_0) \rangle_C$$

$$= -de^{-ik_0 d} q_{(k_0, k_0)}(\varphi_{j_1,i_1}(k_0), \varphi_{j_2,i_2}(k_0))$$

$$= -de^{-ik_0 d} \text{sgn}(q_{(k_0, k_0)})(\varphi_{j_1,i_1}(k_0), \varphi_{j_2,i_2}(k_0)).$$
Thus we have the orthogonality relations
\[
\langle \varphi_{j_1,i_1}(k_0), \varphi_{j_2,i_2}(k_0) \rangle_{(k_0,\omega_0)} = 0, \quad j_1 \neq j_2.
\]

Now if we let \( \psi_j := \sum_{i=1}^{m(j)} c_{j,i} \varphi_{j,i}(k_0) \) then it follows from these orthogonality relations that
\[
\langle \psi_{j_1}, \psi_{j_2} \rangle_{(k_0,\omega_0)} = 0, \quad j_1 \neq j_2.
\]

And hence since \( 0 = \sum_{j=1}^{\hat{g}} \sum_{i=1}^{m(j)} c_{j,i} \varphi_{j,i}(k_0) = \sum_{j=1}^{\hat{g}} \psi_j \) it follows that
\[
0 = \left\langle \sum_{j=1}^{\hat{g}} \psi_j, \psi_j \right\rangle_{(k_0,\omega_0)} = \langle \psi_{j_1}, \psi_{j_1} \rangle_{(k_0,\omega_0)}.
\]

But by Proposition 45 we know that \( \langle \ , \ \rangle_{(k_0,\omega_0)} \) is an inner product on ker\( L(\omega_0, k_0) \) and so we conclude \( \psi_j = 0 \) for every \( j \). But \( M_j(k_0) \) is an invertible matrix and hence \( 0 = M_j(k_0)^{-1} \psi_j = \sum_{i=1}^{m(j)} c_{j,i} e_i \) which implies that \( \sum_{i=1}^{m(j)} |c_{j,i}| = 0 \) for every \( j \). Thus we conclude \( \sum_{j=1}^{\hat{g}} \sum_{i=1}^{m(j)} |c_{j,i}| = 0 \) a contradiction. Therefore the limit points \( \varphi_{j,i}(k_0), i = 1, \ldots, m(j), j = 1, \ldots, \hat{g} \) are linearly independent. This completes the proof. \( \blacksquare \)

Without loss of generality we may take \( 0 < r \leq \delta \) such that the consequences of Lemma 68 is true for the functions \( \varphi_{j,i}(k) \), \( i = 1, \ldots, m(j), j = 1, \ldots, \hat{g} \). Define the functions \( \omega_{j,i}(k) := \omega_j(k), i = 1, \ldots, m(j), j = 1, \ldots, \hat{g} \) on \( B(k_0, r) \). It follows that there is \( \sum_{j=1}^{\hat{g}} \sum_{i=1}^{m(j)} = g \) of them, they are nonconstant real analytic functions with the union of there graphs being \( B \cap B(k_0, r) \times B(\omega_0, \varepsilon) \). The order of the zero of \( \omega_j(k) - \omega_0 \) at \( k = k_0 \) is \( q_{j,i} := q_j, i = 1, \ldots, m(j), j = 1, \ldots, \hat{g} \) such that \( \sum_{j=1}^{\hat{g}} \sum_{i=1}^{m(j)} q_{j,i} = \sum_{j=1}^{\hat{g}} m(j) q_{i,j} = m \). Moreover, the functions \( \varphi_{j,i}(k), i = 1, \ldots, m(j), j = 1, \ldots, \hat{g} \) are analytic functions belonging to \( \mathcal{O}(B(k_0, r), \mathbb{C}^n) \) such that for each \( k \in B(k_0, r) \) the vectors \( \varphi_{j,i}(k), i = 1, \ldots, m(j), j = 1, \ldots, \hat{g} \) are linear independent, \( \{ \varphi_{j,i}(k) \}_{i=1}^{m(i)} \) are a basis for ker\( L(\omega_{j,i}(k), k) \), for \( j = 1, \ldots, \hat{g} \), and \( \bigcup_{j=1}^{\hat{g}} \{ \varphi_{j,i}(k_0) \}_{i=1}^{m(i)} \) is a basis for ker\( L(\omega_0, k_0) \). Thus the proof this theorem will be complete if we can show that \( q_{j,i}, \)
\[ i = 1, \ldots, m(j), \ j = 1, \ldots, \hat{g} \] are the dimensions of the Jordan blocks in the Jordan normal form of \( \Psi(d, \omega_0) \) corresponding to the eigenvalue \( e^{ik_0d} \).

It follows from what we just stated that \( (\omega_{i,j}(k), \varphi_{j,i}(k)), \ i = 1, \ldots, m(j), \ j = 1, \ldots, \hat{g} \) are eigenpair Puiseux series of \( L(\cdot, k) \). And the order of the zero of \( \omega_{j,i}(k) - \omega_0 \) at \( k = k_0 \) is \( q_{j,i} \), \( i = 1, \ldots, m(j), \ j = 1, \ldots, \hat{g} \). We denote any one of them by \( (\omega(k), \varphi(k)) \). Then both \( \omega(k) \) and \( \varphi(k) \) are analytic at \( k = k_0 \) with \( \omega(k_0) = \omega_0 \). Let \( q \) be the order of the zero of \( \omega_j(k) - \omega_0 \) at \( k = k_0 \). It follows from Proposition 32 that \( \varphi(k) \) is a generating function of order \( q \) for \( L(\omega_0, \cdot) \) at the eigenvalue \( k_0 \). It follows by Proposition 11 that

\[ \varphi_l := \frac{1}{j!} \varphi^{(l)}(\lambda_0), \ l = 0, \ldots, q - 1 \]

is a Jordan chain of \( L(\omega_0, \cdot) \) of length \( q \) corresponding to the eigenvalue \( k_0 \). By definition of a Jordan chain for a holomorphic function we have

\[ \varphi_0 \neq 0 \quad \text{and} \quad \sum_{h=0}^{l} \frac{1}{h!} L^{(0,h)}(\omega_0, k_0) \varphi_{l-h} = 0, \ l = 0, \ldots, q - 1. \]

But \( L^{(0,0)}(\omega_0, k_0) = (\Psi(d, \omega_0) - e^{ik_0d}I_n) \) and \( L^{(0,h)}(\omega_0, k_0) = -(id)^he^{ik_0d}I_n \) for \( h \geq 0 \). Hence for \( l = 0, \ldots, q - 1 \) we have

\[
0 = (\Psi(d, \omega_0) - e^{ik_0d}I_n) \varphi_0, \quad \varphi_0 \neq 0, \\
0 = \sum_{h=0}^{l} \frac{1}{h!} L^{(0,h)}(\omega_0, k_0) \varphi_{l-h} = (\Psi(d, \omega_0) - e^{ik_0d}I_n) \varphi_l + \sum_{h=1}^{l} -(id)^he^{ik_0d} \varphi_{l-h}, \ l \geq 1
\]

Thus it follow that

\[
(\Psi(d, \omega_0) - e^{ik_0d}I_n) \varphi_0 = 0, \quad \varphi_0 \neq 0, \\
(\Psi(d, \omega_0) - e^{ik_0d}I_n)^l \varphi_l = (id e^{ik_0d})^l \varphi_0, \ 1 \leq l \leq q - 1
\]
This implies that \( \gamma_{q-1} := \frac{\varphi_{q-1}}{(d - \omega_0^i)^{q-1}} \) is a generalized eigenvector of \( \Psi(d, \omega) \) of order \( q \) corresponding to the eigenvalue \( e^{ik_0 d} \). Thus if we define

\[
\gamma_l := (\Psi(d, \omega_0) - e^{ik_0 d}I_n)^{q-1-l} \gamma_{q-1}, \quad l = 0, \ldots, q - 1
\]

then \( \{\gamma_l\}_{l=0}^{q-1} \) is a Jordan chain of \( \Psi(d, \omega_0) \) of length \( q \) corresponding to the eigenvalue \( e^{ik_0 d} \) with \( \gamma_0 = \varphi_0 \) the eigenvector.

Thus to each pair \((\omega_{i,j}(k), \varphi_{j,i}(k))\) we get a Jordan chain, \( \{\gamma_{j,i,0}\}_{i=0}^{q_{j,i}-1} \), of \( \Psi(d, \omega_0) \) of length \( q_{j,i} - 1 \) corresponding to the eigenvalue \( e^{ik_0 d} \) with \( \gamma_{j,i,0} = \varphi_{j,i}(k_0) \) the eigenvector. Thus since the elements in a Jordan chain are always linearly independent and, since \( \bigcup_{j=1}^{\hat{g}} \{\varphi_{j,i}(k_0)\}_{i=1}^{m(i)} \) is a basis for \( \ker L(\omega_0, k_0) \), the eigenvectors of these chains \( \gamma_{j,i,0} = \varphi_{j,i}(k_0) \), \( i = 1, \ldots, m(j) \), \( j = 1, \ldots, \hat{g} \) are linearly independent and this implies the collection of Jordan chains \( \bigcup_{j=1}^{\hat{g}} \{\varphi_{j,i}(k_0)\}_{i=1}^{m(i)} \) are linearly independent. But the dimension of the generalized eigenspace for \( \Psi(d, \omega_0) \) corresponding to the eigenvalue \( e^{ik_0 d} \) is equal to its algebraic multiplicity which is \( m \). And the collection of Jordan chains \( \bigcup_{j=1}^{\hat{g}} \{\varphi_{j,i}(k_0)\}_{i=1}^{m(i)} \) are linearly independent and there are a total of \( \sum_{j=1}^{\hat{g}} \sum_{i=1}^{m(j)} q_{i,j} = m \) of them and they belong to the generalized eigenspace for \( \Psi(d, \omega_0) \) corresponding to the eigenvalue \( e^{ik_0 d} \). Therefore the collection of Jordan chains \( \bigcup_{j=1}^{\hat{g}} \{\varphi_{j,i}(k_0)\}_{i=1}^{m(i)} \) are a basis for the generalized eigenspace for \( \Psi(d, \omega_0) \) corresponding to the eigenvalue \( e^{ik_0 d} \), and in particular, they are a Jordan basis for that space for matrix \( \Psi(d, \omega_0) \). This implies in the Jordan normal form of \( \Psi(d, \omega_0) \) corresponding to the eigenvalue \( e^{ik_0 d} \) it has \( \sum_{j=1}^{\hat{g}} \sum_{i=1}^{m(j)} = g \) Jordan blocks whose orders are exactly \( q_{i,j} \), \( i = 1, \ldots, m(j) \), \( j = 1, \ldots, \hat{g} \), but by hypothesis these orders where \( m_1, \ldots, m_g \) implying by the uniqueness of the orders of the Jordan blocks these lists have the same numbers in them. This therefore completes the proof. 

\( \blacksquare \)

The proof of Theorem 48.(i) now follows from Proposition 64 and Proposition 65. The proof of Theorem 48.(ii) follows from Theorem 66. This completes the proof of Theorem 48.
Proof. [Corollary 49] Suppose the conditions

\[ D(k_0, \omega_0) = 0, \quad \frac{\partial D}{\partial \omega}(k_0, \omega_0) \neq 0, \quad (k_0, \omega_0) \in \mathbb{R} \times \Omega \]

are satisfied where \( D(k, \omega) = \det(e^{ikd}I_n - \Psi(d, \omega)) \), defined in Corollary 38. By Corollary 38 the first condition implies \((k_0, \omega_0) \in \mathcal{B}_\mathbb{R}\). Choose \( r > 0 \) such that \( B(\omega_0, r) \subseteq \Omega \). Let \( \lambda_0 := e^{ik_0d} \) and \( M(\varepsilon) := \Psi(d, \varepsilon + \omega_0) \) for \( \varepsilon \in B(0, r) \). By Proposition 33.(iii) we have \( \Psi(d, \cdot) \in \mathcal{O}(B(\omega_0, r), M_n(\mathbb{C})) \) and hence \( M(\cdot) \in \mathcal{O}(B(0, r), M_n(\mathbb{C})) \) which implies \( M(\varepsilon) \) is an analytic matrix function of the perturbation parameter \( \varepsilon \). Furthermore, \( \lambda_0 \) is an eigenvalue of the unperturbed matrix \( M(0) = \Psi(d, \omega_0) \) since \( 0 = D(\omega_0, k_0) = \det(e^{ik_0d}I_n - \Psi(d, \omega_0)) = \det(\lambda_0 I_n - \Psi(d, \omega_0)) \). Moreover, we have

\[
\frac{\partial}{\partial \varepsilon} \det(\lambda I_n - M(\varepsilon))|_{(0, \lambda_0)} = \frac{\partial}{\partial \varepsilon} \det(\lambda I_n - \Psi(d, \varepsilon + \omega_0))|_{(0, \lambda_0)} \\
= \frac{\partial}{\partial \omega} \det(\lambda I_n - \Psi(d, \omega))|_{(\omega_0, \lambda_0)} = \frac{\partial}{\partial \omega} \det(e^{ik_0d}I_n - \Psi(d, \omega))|_{(\omega_0, k_0)} \\
= \frac{\partial D}{\partial \omega}(k_0, \omega_0) \neq 0
\]

Thus the generic condition, defined in [61, see (1.1)], is satisfied for the matrix function \( M(\varepsilon) \) for the unperturbed matrix \( M(0) \) with the eigenvalue \( \lambda_0 \). By [61, Theorem 2.1.(iv)] the Jordan normal form of \( M(0) = \Psi(d, \omega_0) \) corresponding to the eigenvalue \( \lambda_0 = e^{ik_0d} \) consists of a single \( m \times m \) Jordan block and there exists an eigenvector \( \gamma_0 \) of \( \Psi(d, \omega_0) \) and an eigenvector \( \beta_0 \) of \( \Psi(d, \omega_0)^* \) corresponding to the eigenvalue \( \overline{\lambda_0} = e^{-ik_0d} \) such that

\[
\langle \Psi_\omega(d, \omega_0)\gamma_0, \beta_0 \rangle_C \neq 0.
\]

But since the Jordan normal form of \( \Psi(d, \omega_0) \) corresponding to the eigenvalue \( \lambda_0 = e^{ik_0d} \) consists of a single \( m \times m \) Jordan block, this implies that \( \dim \ker(\lambda_0 I_n - \Psi(d, \omega_0)) = 1 \).
But because $\Psi(d, \omega_0)^*$ is a adjoint of $\Psi(d, \omega_0)$ this implies the Jordan normal form corresponding to the eigenvalue $\bar{\lambda}_0 = e^{-ik_0d}$ also consists of a single $m \times m$ Jordan block so that $\dim \ker(\bar{\lambda}_0 I_n - \Psi(d, \omega_0)) = 1$ as well. And therefore, since any eigenvector $\gamma$ of $\Psi(d, \omega_0)$ corresponding to the eigenvalue $\lambda_0$ must be a scalar multiple of $\gamma_0$ and similarly any eigenvector $\beta$ of $\Psi(d, \omega_0)^*$ corresponding to the eigenvalue $\bar{\lambda}_0$ must be a scalar multiple of $\beta_0$, we must have

$$\langle \Psi_\omega(d, \omega_0) \gamma, \beta \rangle_\mathbb{C} \neq 0$$

for any eigenvector $\gamma$ of $\Psi(d, \omega_0)$ corresponding to the eigenvalue $\lambda_0 = e^{ik_0d}$ and for any eigenvector $\beta$ of $\Psi(d, \omega_0)^*$ corresponding to the eigenvalue $\bar{\lambda}_0 = e^{-ik_0d}$. That is the key statement.

Indeed, Proposition 39 implies

$$\Psi(d, \omega_0)^* \mathcal{J} \Psi(d, \omega_0) = \mathcal{J}$$

and by Proposition 33.(v) we know that $\Psi(d, \omega_0)$ is an invertible matrix and hence for any eigenvector $\gamma$ of $\Psi(d, \omega_0)$ corresponding to the eigenvalue $\lambda_0 = e^{ik_0d}$ we have

$$\Psi(d, \omega_0)^* \mathcal{J} \gamma = \mathcal{J} \Psi(d, \omega_0)^{-1} \gamma = e^{-ik_0d} \mathcal{J} \gamma.$$ 

Then since $\mathcal{J}$ is invertible this implies $\mathcal{J} \gamma$ is an eigenvector of $\Psi(d, \omega_0)^*$ corresponding to the eigenvalue $e^{-ik_0d} = \bar{\lambda}_0$. But this means

$$0 \neq \langle \Psi_\omega(d, \omega_0) \gamma, \mathcal{J} \gamma \rangle_\mathbb{C} = -e^{ik_0d} e^{-ik_0d} \langle \mathcal{J} \Psi_\omega(d, \omega_0) \gamma, \gamma \rangle_\mathbb{C}$$

$$= -e^{-ik_0d} \langle \mathcal{J} \Psi_\omega(d, \omega_0) \gamma, \Psi(d, \omega_0) \gamma \rangle_\mathbb{C}$$

$$= -e^{-ik_0d} \langle \Psi(d, \omega_0)^* \mathcal{J} \Psi_\omega(d, \omega_0) \gamma, \gamma \rangle_\mathbb{C}$$
\[-de^{-ik_0 d} \frac{1}{d} \int_0^d \langle A_\omega(x, \omega) \psi(x), \psi(x) \rangle \, dx, \quad \psi := \Psi(\cdot, \omega_0) \gamma, \]

where the last equality follows from Corollary 43. Thus we have shown for any $\gamma \in \ker(e^{ik_0 d} I_n - \Psi(d, \omega_0))$ with $\gamma \neq 0$ we have

\[
\frac{1}{d} \int_0^d \langle A_\omega(x, \omega) \psi(x), \psi(x) \rangle \, dx \neq 0, \quad \psi := \Psi(\cdot, \omega_0) \gamma.
\]

Therefore if $\psi$ is any nontrivial Bloch solution of the canonical ODEs in 4.13 with $(k_0, \omega_0)$ as its wavenumber-frequency pair, the by Theorem 35 there exists an eigenvector $\gamma$ of the monodromy matrix $\Psi(d, \omega_0)$ corresponding to the eigenvalue $e^{ik_0 d}$ such that $\psi = \Psi(\cdot, \omega_0) \gamma$, i.e., $\gamma \in \ker(e^{ik_0 d} I_n - \Psi(d, \omega_0))$ with $\gamma \neq 0$, and thus from what we just proved we must have

\[
\frac{1}{d} \int_0^d \langle A_\omega(x, \omega) \psi(x), \psi(x) \rangle \, dx \neq 0.
\]

This proves $(k_0, \omega_0) \in B_\mathbb{R}$ is a point of definite type for the canonical ODEs in 4.13. Moreover, from what we have shown the Jordan normal form of $\Psi(d, \omega_0)$ corresponding to the eigenvalue $\lambda_0 = e^{ik_0 d}$ consists of a single $m \times m$ Jordan block.

We will now prove the converse. Suppose that $(k_0, \omega_0) \in B_\mathbb{R}$ is a point of definite type for the canonical ODEs in (4.13) such that the Jordan normal form of $\Psi(d, \omega_0)$ corresponding to the eigenvalue $\lambda_0 = e^{ik_0 d}$ consists of a single $m \times m$ Jordan block, for some $m \in \mathbb{N}$. By Theorem 48.(i), since $g = 1$ in this case, we know that the order of the zero of the function $D(k_0, \omega)$ at $\omega = \omega_0$ is 1. But this is equivalent to $D(k_0, \omega_0) = 0$ and $\frac{\partial D}{\partial \omega}(k_0, \omega_0) \neq 0$. Therefore the converse is true. This completes the proof. \qed
4.5 Auxiliary Material

In this section we give some notation, convention, and background material frequently used throughout this chapter.

Notation and Convention

(i) \( M_{m,n}(E) := \) the set of all \( m \times n \) matrices with entries in a space \( E \).

(ii) \( M_m(E) := M_{m,m}(E) \).

(iii) \( E^m := M_{m,1}(E) \).

(iv) As convention we identify \( E \) with \( E^1 \) and the set of \( m \)-tuples whose entries are in \( E \) with \( M_{1,m}(E) \).

(v) \( A^T := \) the transpose of the matrix \( A \in M_{m,n}(E) \).

(vi) \( (M_{m,n}(E), \| \cdot \|_{M_{m,n}(E)}) := \) the Banach space with norm

\[
\|A\|_{M_{m,n}(E)} := \left( \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}, \quad \text{for any } A := [a_{ij}]_{i=1,j=1}^{m,n} \in M_{m,n}(E)
\]

where \( (E, \| \cdot \|) \) is a Banach space.

(vii) \( \mathcal{L}(E, F) := \) the Banach space of all continuous linear operators from a Banach space \( (E, \| \cdot \|_E) \) to a Banach space \( (F, \| \cdot \|_F) \) with the operator norm

\[
\|T\|_{\mathcal{L}(E,F)} := \sup_{e \in E, \|e\|_E \neq 0} \frac{\|Te\|_F}{\|e\|_E}
\]

(viii) \( B(a, r) := \{ e \in E : \|e - a\| < r \} \), where \( (E, \| \cdot \|) \) is a Banach space, \( a \in E \), and \( r > 0 \).
\( \mathcal{O}(\Omega, E) := \{ f : \Omega \rightarrow E \mid f \text{ is holomorphic on } \Omega \} \) where \((E, \|\cdot\|)\) is a Banach space, \(\Omega\) an open connected set in \(\mathbb{C}\), and \(f : \Omega \rightarrow E\) is said to be holomorphic on \(\Omega\) provided \(f\) is differentiable at each point in \(\Omega\), i.e., the limit

\[
f_\omega(\omega_0) := \lim_{\omega \to \omega_0} (\omega - \omega_0)^{-1}(f(\omega) - f(\omega_0))
\]

exists for every \(\omega_0 \in \Omega\). The function \(f_\omega : \Omega \rightarrow E\) is called the derivative of \(f\).

\( (\mathbb{C}, |\cdot|) := \) complex numbers with the standard norm \(|a + ib| := \sqrt{a^2 + b^2}\) and complex conjugation \(\overline{a + ib} := a - ib\), for \(a, b \in \mathbb{R}\).

\( ||\cdot||_\mathbb{C} := ||\cdot||_{M_{m,n}(\mathbb{C})} \)

\( I_m := \) the identity matrix in \(M_m(\mathbb{C})\).

\( \overline{A} := \) the matrix formed from the matrix \(A \in M_{m,n}(\mathbb{C})\) by replacing each entry by its complex conjugate.

\( A^* := \overline{A}^T\), the adjoint of the matrix \(A \in M_{m,n}(\mathbb{C})\).

\( \langle u, v \rangle_\mathbb{C} := u^*v\), the standard inner product on \(\mathbb{C}^m\).

\( L^p(a, b), ||\cdot||_p \) := the Banach space of measurable functions \(f\) on \((a, b)\) (modulo functions which vanish almost everywhere (a.e.)) such that

\[
||f||_p := \left( \int_a^b |f(x)|^p dx \right)^{1/p} < \infty \quad (1 \leq p < \infty),
\]

\[
||f||_\infty := \text{ess sup}_{x \in (a,b)} |f(x)| < \infty \quad (p = \infty).
\]

\( ||\cdot||_p := ||\cdot||_{M_{m,n}(L^p(a,b))} \)

\( L^p_{\text{loc}}(\mathbb{R}) := \) the space of all measurable functions \(f\) on \(\mathbb{R}\) (modulo functions which vanish a.e.) such that \(f|_{(a,b)} \in L^p(a,b)\) for all \(a < b\).
(xix) \( L^p(\mathbb{T}) := \{f \in L^p_{\text{loc}}(\mathbb{R}) : f(x + d) = f(x) \text{ for a.e. } x \in \mathbb{R}\} \), the Banach space with norm
\[
\|f\|_{L^p(\mathbb{T})} := \left( \frac{1}{d} \int_0^d |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \quad (1 \leq p < \infty),
\]
\[
\|f\|_{L^\infty(\mathbb{T})} := \text{ess sup}_{x \in (0,d)} |f(x)| < \infty \quad (p = \infty).
\]

(xx) \( \| \cdot \|_{L^p(\mathbb{T})} := \| \cdot \|_{M_{m,n}(L^p(\mathbb{T}))} \)

(xx) As convention we identify an element \( A \) of \( M_{m,n}(L^p(a,b)) \) or \( M_{m,n}(L^p_{\text{loc}}(\mathbb{R})) \) with any one of its representative functions \( A(\cdot) \) (and visa versa). And hence any identities, inequalities, etc. throughout this chapter are to be understood as being true almost everywhere unless otherwise stated.

(xxii) As convention we identify an element \( A \in \mathcal{O}(\Omega, M_{m,n}(L^p(a,b))) \) with the function \( A : (a,b) \times \Omega \rightarrow M_{m,n}(\mathbb{C}) \) defined by \( A(\cdot, \omega) := A(\omega) \). Similarly, a function \( A : (a,b) \times \Omega \rightarrow M_{m,n}(\mathbb{C}) \) is said to be in \( \mathcal{O}(\Omega, M_{m,n}(L^p(a,b))) \) as a function of frequency and we write \( A \in \mathcal{O}(\Omega, M_{m,n}(L^p(a,b))) \) provided \( A(\cdot, \omega) \) is \( A(\cdot) \in \mathcal{O}(\Omega, M_{m,n}(L^p(a,b))) \) for each \( \omega \in \Omega \) and \( A(\cdot) \in \mathcal{O}(\Omega, M_{m,n}(L^p(a,b))) \).

(xxiii) As convention we identify an element \( A \in \mathcal{O}(\Omega, M_{m,n}(L^p(\mathbb{T}))) \) with the function \( A : \mathbb{R} \times \Omega \rightarrow M_{m,n}(\mathbb{C}) \) defined by \( A(\cdot, \omega) := A(\omega) \). Similarly, a function \( A : \mathbb{R} \times \Omega \rightarrow M_{m,n}(\mathbb{C}) \) is said to be in \( \mathcal{O}(\Omega, M_{m,n}(L^p(\mathbb{T}))) \) as a function of frequency and we write \( A \in \mathcal{O}(\Omega, M_{m,n}(L^p(\mathbb{T}))) \) provided \( A(\cdot, \omega) \in A(\cdot) \in \mathcal{O}(\Omega, M_{m,n}(L^p(\mathbb{T}))) \) for each \( \omega \in \Omega \) and \( A(\cdot) \in \mathcal{O}(\Omega, M_{m,n}(L^p(\mathbb{T}))) \).

(xxiv) \( \int_U A(x)dx := \left[ \int_U a_{ij}(x)dx \right]_{i=1,j=1}^{m,n} \), for any measurable set \( U \) contained in \( [a,b] \) and any \( A := [a_{ij}]_{i=1,j=1}^{m,n} \in M_{m,n}(L^p(a,b)) \cup M_{m,n}(L^p_{\text{loc}}(\mathbb{R})) \).

(xxv) As convention we set \( \int_c^x A(t)dt := \int_{[c,x]} A(t)dt \) if \( x \leq c \) and \( \int_c^x A(t)dt := -\int_x^c A(t)dt \) if \( c < a \), whenever \( c, x \in [a,b] \) and \( A \in M_{m,n}(L^p(a,b)) \cup M_{m,n}(L^p_{\text{loc}}(\mathbb{R})) \).
\( W^{1,1}(a, b) := \{ f \in L^1(a, b) \mid f' \in L^1(a, b) \} \), the standard Sobolev space with norm

\[ \| f \|_{1,1} := \| f \|_1 + \| f' \|_1 \]

where \( f' \) denotes the weak derivative of \( f \).

\[ \| \cdot \|_{1,1} := \| \cdot \|_{M_{m,n}(W^{1,1}(a, b))} \]

\[ W^{1,1}_{\text{loc}}(\mathbb{R}) := \{ f \in L^1_{\text{loc}}(\mathbb{R}) \mid f' \in L^1_{\text{loc}}(\mathbb{R}) \} \].

\( A' := \left[ a'_{ij} \right]_{i=1,j=1}^{m,n}, \) for any \( A := [a_{ij}]_{i=1,j=1}^{m,n} \) in \( M_{m,n}(W^{1,1}(a, b)) \) or \( M_{m,n}(W^{1,1}_{\text{loc}}(\mathbb{R})) \).

As convention we always identify an element \( A \) of \( M_{m,n}(W^{1,1}(a, b)) \) or \( M_{m,n}(W^{1,1}_{\text{loc}}(\mathbb{R})) \) with its unique absolutely continuous or unique locally absolutely continuous representative, respectively, (see Lemma 70 below) so that, according to this identification, \( A(x) = A(u) + \int_u^x A'(t)dt \) for all \( u, x \in [a, b] \), if \( A \in M_{m,n}(W^{1,1}(a, b)) \), and for all \( u, x \in \mathbb{R} \), if \( A \in M_{m,n}(W^{1,1}_{\text{loc}}(\mathbb{R})) \).

As convention we identify an element \( A \in \mathcal{O}(\Omega, M_{m,n}(W^{1,1}(a, b))) \) with the function \( A : [a, b] \times \Omega \to M_{m,n}(\mathbb{C}) \) defined by \( A(x, \omega) := A(\omega)(x) - \) the value of the function \( A : \Omega \to M_{m,n}(W^{1,1}(a, b)) \) at \( \omega \in \Omega \) and the value of the unique absolutely continuous representative of \( A(\omega) \) at \( x \in [a, b] \). Similarly, a function \( A : I \times \Omega \to M_{m,n}(\mathbb{C}) \), where \( I := (a, b) \) or \( [a, b] \), is said to be in \( \mathcal{O}(\Omega, M_{m,n}(W^{1,1}(a, b))) \) and we write \( A \in \mathcal{O}(\Omega, M_{m,n}(W^{1,1}(a, b))) \) provided \( A(\omega) := A(\cdot, \omega) \in M_{m,n}(W^{1,1}(a, b)) \) for each \( \omega \in \Omega \) and \( A(\cdot) \in \mathcal{O}(\Omega, M_{m,n}(W^{1,1}(a, b))) \).

**Background Material**

The following is background material for this chapter.

In what follows we will need notation for the derivative of a function in the classical sense. Let
$f : I \to \mathbb{C}$ be a complex-valued function defined on an interval $I \subseteq \mathbb{R}$. If $f$ is differentiable (in the classical sense) everywhere on $I$ we denote its derivative by $df/dx$, otherwise we define the function $df/dx : I \to \mathbb{C}$ by

$$\frac{df}{dx}\bigg|_{x=t} := \begin{cases} \lim_{x \to t} \frac{f(x) - f(t)}{x-t}, & \text{if the limit exists} \\ 0, & \text{else.} \end{cases}$$

**Lemma 69** If $A \in M_{m,n}(W^{1,1}_{\text{loc}}(\mathbb{R}))$ then $A|_{(a,b)} \in M_{m,n}(W^{1,1}(a,b))$ and $A'|_{(a,b)} = A'|_{(a,b)}$ for any bounded interval $(a,b) \subseteq \mathbb{R}$.

**Proof.** We first prove the statement in the case $m = n = 1$. Suppose $f \in W^{1,1}_{\text{loc}}(\mathbb{R})$. Then by the definition of the weak derivative $(\cdot)'$ since $f' \in L^1_{\text{loc}}(\mathbb{R})$ we have

$$\int_\mathbb{R} \varphi'(x)f(x)dx = -\int_\mathbb{R} \varphi(x)f'(x)dx, \quad \forall \varphi \in C^\infty_0(\mathbb{R}),$$

where $C^\infty_0(\mathbb{R})$ is the set of all infinitely differentiable function on $\mathbb{R}$ with compact support and $\varphi' = d\varphi/dx$ for $\varphi \in C^\infty_0(\mathbb{R})$. This implies that for any $(a,b)$ open interval and any $\phi \in C^\infty_0(a,b)$, the set of infinitely differentiable functions on $(a,b)$ with compact support, we can take the zero extension of $\phi$, i.e.,

$$\phi_e(x) := \begin{cases} \phi(x) & \text{if } x \in (a,b) \\ 0 & \text{else.} \end{cases}$$

so that $\phi_e \in C^\infty_0(\mathbb{R})$. And hence

$$\int_a^b \varphi'(x)f(x)dx = \int_\mathbb{R} \phi'_e(x)f(x)dx = -\int_\mathbb{R} \varphi_e(x)f'(x)dx = -\int_a^b \varphi(x)f'(x)dx, \quad \forall \varphi \in C^\infty_0(a,b).$$

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This implies by the definition of the weak derivative that $f|_{(a,b)} \in W^{1,1}(a,b)$ and $f'|_{(a,b)} = f'|_{(a,b)}$. This proves the statement for $m = n = 1$. For the case $m, n \geq 1$, it follows now trivially from the notation 4.5.(xxix) and the fact $[a_{ij}]_{i=1,j=1}^{m,n}[a_{ij}|_{(a,b)}]_{i=1,j=1}^{m,n}$ for $[a_{ij}]_{i=1,j=1}^{m,n} \in M_{m,n}(L^1(a,b)) \cup M_{m,n}(L^1_{loc}(\mathbb{R}))$. This completes the proof. ■

**Definition 26** For any compact interval $[a, b] \subseteq \mathbb{R}$, denote by $AC[a, b]$ the space of absolutely continuous functions on $[a, b]$, i.e.,

$$AC[a, b] := \{ f : [a, b] \to \mathbb{C} \mid f \text{ is differentiable a.e., } df/dx \in L^1(a, b), \text{ and } f(u) = f(v) + \int_v^u \frac{df}{dx} \bigg|_{x=t} \, dt, \; \forall u, v \in [a, b] \}.$$  

We denote by $AC_{loc}(\mathbb{R})$ the space of locally absolutely continuous functions on $\mathbb{R}$, i.e.,

$$AC_{loc}(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{C} \mid f \text{ is differentiable a.e., } df/dx \in L^1_{loc}(\mathbb{R}), \text{ and } f(u) = f(v) + \int_v^u \frac{df}{dx} \bigg|_{x=t} \, dt, \; \forall u, v \in \mathbb{R} \}.$$  

For $A := [a_{ij}]_{i=1,j=1}^{m,n} \in M_{m,n}(AC[a, b])$ we define $\frac{dA}{dx} := [\frac{da_{ij}}{dx}]_{i=1,j=1}^{m,n} \in M_{m,n}(L^1(a,b))$. Similarly, for $A := [a_{ij}]_{i=1,j=1}^{m,n} \in M_{m,n}(AC_{loc}(\mathbb{R}))$ we define the matrix $\frac{dA}{dx} := [\frac{da_{ij}}{dx}]_{i=1,j=1}^{m,n} \in M_{m,n}(L^1_{loc}(\mathbb{R}))$.

**Lemma 70** $A \in M_{m,n}(W^{1,1}(a,b))$ if and only if $A(\cdot) \in M_{m,n}(AC[a, b])$ for some representative function $A(\cdot)$ of $A$. Furthermore, this representative function is unique. Moreover, for the absolutely continuous representative $A(\cdot)$ of $A$ we have $A' = \frac{dA(\cdot)}{dx}$ and

$$A(x) = A(u) + \int_u^x A'(t)dt, \; \forall u, x \in [a, b].$$

Similarly, $A \in M_{m,n}(W^{1,1}_{loc}(\mathbb{R}))$ if and only if $A(\cdot) \in M_{m,n}(AC_{loc}(\mathbb{R}))$ for some representative function $A(\cdot)$ of $A$. Furthermore, this representative function is unique. Moreover, for the
locally absolutely continuous representative $A(\cdot)$ of $A$ we have $A' = \frac{dA(\cdot)}{dx}$ and

$$A(x) = A(u) + \int_u^x A'(t)dt, \quad \forall u, x \in \mathbb{R}.$$  

**Proof.** We first prove the statement for $m = n = 1$. By [45, p. 56, §II.2.1, Proposition 2.1.5], if $f \in W^{1,1}(a, b)$ then $f$ has a representative function $f(\cdot) : [a, b] \to \mathbb{C}$ such that

$$f(x) = f(u) + \int_u^x f'(t)dt, \quad \forall u, x \in [a, b].$$

This implies $f(\cdot) \in AC[a, b]$ with $df(\cdot)/dx = f'$. To prove uniqueness we note that if $g \in AC[a, b]$ is another representative function of $f$ then since $f(\cdot)$ is as well we have $g(x) = f(x)$ for a.e. $x \in (a, b)$. But then by the fact absolutely continuous functions on $[a, b]$ are continuous functions on $[a, b]$ this implies $g(x) = f(x)$ for every $x \in [a, b]$. This proves uniqueness. On the other hand, if $f(\cdot) \in AC[a, b]$ then by [45, p. 55, §II.2.1, Proposition 2.1.3] we have $f \in W^{1,1}(a, b)$.

Now suppose $f \in W^{1,1}_{\text{loc}}(\mathbb{R})$. Then by Lemma 69 we have $f|_{(a,b)} \in W^{1,1}(a, b)$ for any bounded open interval $(a, b)$ and $f'|_{(a,b)} = f'|_{(a,b)}$. Hence from what we just proved $f|_{(a,b)}$ has an absolutely continuous representative function $f|_{(a,b)}(\cdot) : [a, b] \to \mathbb{C}$ such that

$$f|_{(a,b)}(x) = f|_{(a,b)}(u) + \int_u^x f|_{(a,b)}'(t)dt = f|_{(a,b)}(u) + \int_u^x f'(t)dt$$

$$= f|_{(a,b)}(u) + \int_u^x f'(t)dt, \quad \forall u, x \in [a, b].$$

Define a function $g(\cdot) : \mathbb{R} \to \mathbb{C}$ by

$$g(x) := f|_{(a,b)}(x), \quad \text{for any bounded interval } (a, b) \text{ with } x \in (a, b).$$

We first show this function is well-defined. Let $(a, b)$ and $(u, v)$ be any two bounded open
interval such that \( I := (a, b) \cap (u, v) \neq \emptyset \) and let \( f|_{(a, b)}(\cdot) \in AC[a, b] \) be an absolutely continuous representative function of \( f|_{(a, b)} \in W^{1,1}(a, b) \) and \( f|_{(u, v)}(\cdot) \in AC[u, v] \) be an absolutely continuous representative function of \( f|_{(u, v)} \in W^{1,1}(u, v) \). Now fix any representative function \( \tilde{f} : \mathbb{R} \to \mathbb{C} \) of \( f \). Then \( f|_{(a, b)}(x) = \tilde{f}(x) \) for a.e. \( x \in [a, b] \) and \( f|_{(u, v)}(x) = \tilde{f}(x) \) for a.e. \( x \in [u, v] \) implying \( f|_{(a, b)}(x) = \tilde{f}(x) = f|_{(u, v)}(x) \) for a.e. \( x \in I \). Hence since \( f|_{(a, b)}(\cdot) \) and \( f|_{(u, v)}(\cdot) \) are continuous on \( I \) this implies \( f|_{(a, b)}(x) = f|_{(u, v)}(x) \) for all \( x \in I \). This proves \( g(\cdot) : \mathbb{R} \to \mathbb{C} \) is a well-defined function. From the definition of \( g \) it also proves \( g|_{(a, b)} = f|_{(a, b)} \in W^{1,1}(a, b) \) for any bounded open interval \((a, b)\) and hence \( g = f \in W^{1,1}_{loc}(\mathbb{R}) \).

It also implies that for any \( u, x \in \mathbb{R} \), if we take any bounded open interval \((a, b)\) containing \( u, x \), then we have

\[
g(x) = f|_{(a, b)}(x) = f|_{(a, b)}(u) + \int_{u}^{x} f'(t)dt = g(u) + \int_{u}^{x} f'(t)dt.
\]

Thus implying \( g(\cdot) \in AC_{loc}(\mathbb{R}) \) with \( dg(\cdot)/dx = f' \). Hence we have proven \( f \) has a locally absolutely continuous representative \( f(\cdot) := g(\cdot) \in AC_{loc}(\mathbb{R}) \) with \( df(\cdot)/dx = f' \) and such that

\[
f(x) = f(u) + \int_{u}^{x} f'(t)dt, \quad \forall u, x \in \mathbb{R}.
\]

To prove uniqueness we note that if \( f_1, f_2 \in AC_{loc}(\mathbb{R}) \) are two representative functions of \( f \) then \( f_1 \) and \( f_2 \) are equal a.e. on \( \mathbb{R} \) and so since functions in \( AC_{loc}(\mathbb{R}) \) are continuous this implies \( f_1 \) and \( f_2 \) are equal everywhere on \( \mathbb{R} \). This proves uniqueness.

Suppose now that \( f \in AC_{loc}(\mathbb{R}) \). Then we have \( f \in L^1_{loc}(\mathbb{R}) \) and \( df/dx \in L^1_{loc}(\mathbb{R}) \). We need only show now that \( f \) has \( df/dx \) as its weak derivative, i.e., \( f' = df/dx \). To do this we need to show

\[
\int_{\mathbb{R}} \varphi'(t)f(t)dt = -\int_{\mathbb{R}} \varphi(t)\frac{df}{dx}\bigg|_{x=t}dt, \quad \forall \varphi \in C_0^\infty(\mathbb{R}),
\]

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where $C_0^\infty(\mathbb{R})$ is the set of all infinitely differentiable function on $\mathbb{R}$ with compact support. Let $\varphi \in C_0^\infty(\mathbb{R})$ and let $[a, b]$ be any nonempty compact interval such that supp($\varphi$) $\subseteq [a, b]$. Then by integration by parts we have

\[ \int_{\mathbb{R}} \varphi'(t)f(t)dt = \int_a^b \varphi'(t)f(t)dt = f(b)\varphi(b) - f(a)\varphi(a) - \int_a^b \varphi(t) \left. \frac{df}{dx} \right|_{x=t} dt \]

\[ = -\int_a^b \varphi(t) \frac{df}{dx} \bigg|_{x=t} dt = -\int_{\mathbb{R}} \varphi(t) \frac{df}{dx} \bigg|_{x=t} dt. \]

And therefore we have shown

\[ \int_{\mathbb{R}} \varphi'(t)f(t)dt = -\int_{\mathbb{R}} \varphi(t) \frac{df}{dx} \bigg|_{x=t} dt, \quad \forall \varphi \in C_0^\infty(\mathbb{R}), \]

which proves $f' = df/dx \in L^1_{loc}(\mathbb{R})$ and hence $f \in W^{1,1}_{loc}(\mathbb{R})$.

This completes the prove in the case $m = n = 1$. The case $m, n \geq 1$ follows now trivially from the case $m = n = 1$ by the notation 4.5.(xxiv) and 4.5.(xxix). This completes the proof. ■

The following lemma, along with Lemma 69 and Lemma 70, completes our characterization of spaces $M_{m,n}(W^{1,1}(a, b))$ and $M_{m,n}(W^{1,1}_{loc}(\mathbb{R}))$.

**Lemma 71** Let $A : \mathbb{R} \rightarrow M_{m,n}(\mathbb{C})$ be a function such that there exists an increasing sequence of intervals $\{(a_j, b_j)\}_{j=1}^\infty$ with $\mathbb{R} = \bigcup_{j=1}^\infty (a_j, b_j)$ such that for every $j$, $A|_{(a_j, b_j)} \in M_{m,n}(W^{1,1}(a_j, b_j))$. Then $A \in M_{m,n}(W^{1,1}_{loc}(\mathbb{R}))$.

**Proof.** We first prove the statement for $m = n = 1$. Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a function such that there exists an increasing sequence of intervals $\{(a_j, b_j)\}_{j=1}^\infty$ with $\mathbb{R} = \bigcup_{j=1}^\infty (a_j, b_j)$ such that for every $j$, $f|_{(a_j, b_j)} \in W^{1,1}(a_j, b_j)$. It follows from this that on every interval $(a_j, b_j)$ the function $f$ on this interval is equal a.e. to a function which is measurable and integrable on $(a_j, b_j)$ and this implies the function $f$ restricted to $(a_j, b_j)$ is a measurable and
integrable function. Thus since \( \mathbb{R} = \bigcup_{j=1}^{\infty} (a_j, b_j) \) and the restriction of \( f : \mathbb{R} \to \mathbb{C} \) to \((a_j, b_j)\) is measurable and integrable for every \( j \), this implies \( f \) is a measurable function which is locally integrable and so \( f \in L^1_{\text{loc}}(\mathbb{R}) \).

Now \( f|_{(a_j, b_j)} \in W^{1, 1}(a_j, b_j) \) for every \( j \) and so by Lemma 70 there exists an absolutely continuous function \( g_j \in AC[a_j, b_j] \) such that \( f(x) = g_j(x) \) for a.e. \( x \in (a_j, b_j) \) for each \( j \).

But this implies for any fix \( j_0 \) we must have \( g_j(x) = f(x) = g_{j_0}(x) \) for a.e. \( x \in (a_{j_0}, b_{j_0}) \) for any fixed \( j \geq j_0 \). But the functions \( g_j, g_{j_0} \) are continuous on \([a_{j_0}, b_{j_0}]\) and so this implies \( g_j(x) = g_{j_0}(x) \) for all \( x \in [a_{j_0}, b_{j_0}] \) and every \( j \geq j_0 \). But this implies \( g(x) := \lim_{j \to \infty} g_j(x) \) exists for every \( x \in \mathbb{R} \) and so defines a function \( g : \mathbb{R} \to \mathbb{C} \). Moreover, from what we proved we have \( g(x) = g_j(x) \) for all \( x \in [a_{j_0}, b_{j_0}] \) and any \( j \). Thus for every \( j \) we have \( g|_{[a_j, b_j]} = g_j \in AC[a_j, b_j] \) implying \( g \) is differentiable a.e. on \([a_j, b_j]\) with \( dg/dx|_{[a_j, b_j]} = dg_j/dx = g_j' \in L^1(a_j, b_j) \). But since \( \mathbb{R} = \bigcup_{j=1}^{\infty} (a_j, b_j) \) this implies \( g \) is differentiable a.e. on \( \mathbb{R} \) with \( dg/dx \in L^1_{\text{loc}}(\mathbb{R}) \). Hence if \( \varphi \in C_0^\infty(\mathbb{R}) \) then we can find \( j \) sufficiently large so that \( \text{supp}(\varphi) \subseteq [a_j, b_j] \) and so

\[
\int_{\mathbb{R}} \varphi'(t) g(t) dt = \int_{a_j}^{b_j} \varphi'(t) g(t) dt = \int_{a_j}^{b_j} \varphi'(t) g_j(t) dt = \int_{a_j}^{b_j} \varphi(t) g_j'(t) dt
\]

\[= \int_{a_j}^{b_j} \varphi(t) \left. \frac{dg}{dx} \right|_{x=t} dt = \int_{\mathbb{R}} \varphi(t) \left. \frac{dg}{dx} \right|_{x=t} dt.\]

This implies that \( g' = dg/dx \in L^1_{\text{loc}}(\mathbb{R}) \) and therefore \( g \in W^{1, 1}_{\text{loc}}(\mathbb{R}) \). But \( f = g \) a.e. on \( \mathbb{R} \) and so \( f \in W^{1, 1}_{\text{loc}}(\mathbb{R}) \). This proves the statement for \( n = m = 1 \). The case where \( m, n \geq 1 \) follows trivially now from the case \( n = m = 1 \). This completes the proof.  

\[\boxed{\text{Lemma 72}}\]

Let \((a, b) \subseteq \mathbb{R}\) be a bounded interval. Then the matrix multiplication \( \cdot : M_{m,n}(W^{1, 1}(a, b)) \times M_{n,r}(W^{1, 1}(a, b)) \to M_{m,r}(W^{1, 1}(a, b)) \), where \( A \cdot B := AB \) for \( A \in M_{m,n}(W^{1, 1}(a, b)) \), \( B \in M_{n,r}(W^{1, 1}(a, b)) \), is a continuous bilinear map and

\[(AB)' = A'B + AB'.\]
Proof. We first prove the statement for $m = n = r = 1$. To begin, let $(a, b) \subseteq \mathbb{R}$ be a bounded interval. Then by [45, p. 65, §II.2.3, Proposition 2.3.1], multiplication $\cdot : W^{1,1}(a, b) \times W^{1,1}(a, b) \rightarrow W^{1,1}(a, b)$, where $f \cdot g := fg$ for $f, g \in W^{1,1}(a, b)$, is a continuous bilinear map and

$$(fg)' = f'g + fg'.$$

This proves the statement for $m = n = r = 1$. The statement for $m, n, r \geq 1$ now follows from this, Lemma 77, and notation 4.5.(xxix). This completes the proof. □

Lemma 73 Matrix multiplication $\cdot : M_{m,n}(W^{1,1}_{\text{loc}}(\mathbb{R})) \times M_{n,r}(W^{1,1}_{\text{loc}}(\mathbb{R})) \rightarrow M_{m,r}(W^{1,1}_{\text{loc}}(\mathbb{R}))$, where $A \cdot B := AB$ for $A \in M_{m,n}(W^{1,1}_{\text{loc}}(\mathbb{R}))$, $B \in M_{n,r}(W^{1,1}_{\text{loc}}(\mathbb{R}))$, is a well-defined map and

$$(AB)' = A'B + AB'.$$

Proof. We first prove the statement for $m = n = r = 1$. To begin, let $(a, b) \subseteq \mathbb{R}$ be any bounded interval. It follows from Lemma 69 and Lemma 72 that if $f, g \in W^{1,1}_{\text{loc}}(\mathbb{R})$ then we have $(fg)|_{(a,b)} = f|_{(a,b)}g|_{(a,b)} \in W^{1,1}(a, b)$. Now since this is true for every bounded interval $(a, b) \subseteq \mathbb{R}$, this implies by Lemma 71 that $fg \in W^{1,1}_{\text{loc}}(\mathbb{R})$. Moreover, by Lemma 69 and Lemma 72 we have for every bounded interval $(a, b) \subseteq \mathbb{R}$,

$$(fg)'|_{(a,b)} = (f|_{(a,b)}g|_{(a,b)})' = (f|_{(a,b)})'g|_{(a,b)} + f|_{(a,b)}(g|_{(a,b)})'$$

$$= f'|_{(a,b)}g|_{(a,b)} + f|_{(a,b)}g'|_{(a,b)}.$$

It follows from this that $f'g + fg' \in L^1_{\text{loc}}(\mathbb{R})$ and for any $\varphi \in C_0^\infty(\mathbb{R})$ we can find a bounded interval $(a, b) \subseteq \mathbb{R}$ such that $\text{supp}(\varphi) \subseteq (a, b)$ implying

$$\int_{\mathbb{R}} \varphi'(x)(fg)(x)dx = \int_{a}^{b} \varphi'(x)(fg)|_{(a,b)}(x)dx.$$
\[
= - \int_a^b \varphi(x)(f'|_{(a,b)}(x)g|_{(a,b)}(x) + f|_{(a,b)}(x)g'|_{(a,b)}(x))\,dx
\]

\[
= - \int_a^b \varphi(x)(f'(x)g(x) + f(x)g'(x))\,dx
\]

\[
= - \int_a^b \varphi(x)(f'g + fg')(x)\,dx.
\]

And because this is true for every \( \varphi \in C_0^\infty(\mathbb{R}) \) this means the weak derivative of \( fg \) is \( f'g + fg' \), i.e.,

\[
(fg)' = f'g + fg'.
\]

This proves the statement for \( m = n = r = 1 \). The statement for \( m, n, r \geq 1 \) now follows trivially from this one by notation 4.5.(xxix). This completes the proof. \( \square \)

**Lemma 74** Let \( E = M_{m,n}(L^p_{\text{loc}}(\mathbb{R})) \) or \( M_{m,n}(W^{1,1}_{\text{loc}}(\mathbb{R})) \), where \( 1 \leq p \leq \infty \). The translation operator \( \mathcal{L}_d : E \to E \), where \( (\mathcal{L}_d A)(\cdot) = A(\cdot + d) \) for \( A \in E \), is a well-defined map. Moreover, for any \( A \in M_{m,n}(W^{1,1}_{\text{loc}}(\mathbb{R})) \) we have

\[
(\mathcal{L}_d A)' = \mathcal{L}_d A'.
\]

**Proof.** We first prove the statement for \( n = m = 1 \). Let \( f \in L^p_{\text{loc}}(\mathbb{R}) \). Then \( (\mathcal{L}_d f)(\cdot) : \mathbb{R} \to \mathbb{C} \) given by \( (\mathcal{L}_d f)(x) := f(x + d), x \in \mathbb{R} \) is a measurable function. And for any bounded interval \( (a, b) \subseteq \mathbb{R} \), since \( ||(\mathcal{L}_d f)|_{(a,b)}||_p = ||f|_{(a+d,b+d)}||_p \), we have \( (\mathcal{L}_d f)|_{(a,b)} \in L^p(a,b) \). This proves \( \mathcal{L}_d f \in L^p_{\text{loc}}(\mathbb{R}) \). Now if \( f \in W^{1,1}_{\text{loc}}(\mathbb{R}) \) then \( f' \in L^1_{\text{loc}}(\mathbb{R}) \) and from what we have just proved, \( \mathcal{L}_d f' \in L^1_{\text{loc}}(\mathbb{R}) \). We will now show \( \mathcal{L}_d f \in W^{1,1}_{\text{loc}}(\mathbb{R}) \) with \( (\mathcal{L}_d f)' = \mathcal{L}_d f' \). To start, we have \( (\varphi(\cdot - d))' = \varphi'(\cdot - d) \in C_0^\infty(\mathbb{R}) \) for all \( \varphi \in C_0^\infty(\mathbb{R}) \) since \( (\cdot)' = d/dx \) is just the derivative in the classical sense on smooth functions. From this and the fact the Lebesgue integral on
\[ \int_{\mathbb{R}} \phi'(x)(L_d f)(x)dx = \int_{\mathbb{R}} \phi'(x-d)(L_d f)(x-d)dx \]
\[ = \int_{\mathbb{R}} \phi'(x-d)f(x)dx = \int_{\mathbb{R}} \phi(x-d)f(x)dx \]
\[ = \int_{\mathbb{R}} (\phi(-d))'(x)f(x)dx = \int_{\mathbb{R}} \phi(x-d)f'(x)dx \]
\[ = \int_{\mathbb{R}} \phi(x)f'(x+d)dx = \int_{\mathbb{R}} \phi(x)(L_d f')(x)dx, \quad \varphi \in C_0^\infty(\mathbb{R}). \]

But this implies \( (L_d f)' = L_d f' \) and, since we have already shown \( L_d f' \in L_{loc}^1(\mathbb{R}) \), this proves \( L_d f \in W^{1,1}_{loc}(\mathbb{R}) \). This proves the statement for \( m = n = 1 \). The statement for \( m, n, r \geq 1 \) now follows trivially from this one by notation 4.5.(xxix). This completes the proof. \[ \blacksquare \]

**Lemma 75** For \( 1 \leq p \leq \infty \) and \( m, n \in \mathbb{N} \), the adjoint extends via \( A^*(\cdot) := A(\cdot)^* \) for \( A \in M_{m,n}(L_p^p(\mathbb{R})) \), to a well-defined map \( ^*: M_{m,n}(L_p^p(\mathbb{R})) \to M_{n,m}(L_p^p(\mathbb{R})) \). Moreover, if \( A \in M_{m,n}(W^{1,1}_{loc}(\mathbb{R})) \) then \( A^* \in M_{n,m}(W^{1,1}_{loc}(\mathbb{R})) \) and
\[ (A^*)' = (A')^*. \]

**Proof.** In the case \( m = n = 1 \), it just follows from the facts that \( f \in L^1_{loc}(\mathbb{R}) \) if and only if \( \text{Re}(f), \text{Im}(f) \in L^1_{loc}(\mathbb{R}) \) and \( f \in W^{1,1}_{loc}(\mathbb{R}) \) if and only if \( \text{Re}(f), \text{Im}(f) \in W^{1,1}_{loc}(\mathbb{R}), \text{Re}(f') = (\text{Re}(f))', \text{Im}(f') = (\text{Im}(f))', \) and \( \overline{f'} = f' \). In the case \( m, n \geq 1 \), if \( A := \{a_{ij}\}_{i=1,j=1}^{m,n} \in M_{m,n}(L^1_{loc}(\mathbb{R})) \) then, by the case \( m = n = 1 \), we have \( A^* := \{a_{ij}^*\}_{j=1,i=1}^{n,m} \in M_{n,m}(L^1_{loc}(\mathbb{R})) \) and if \( A \in M_{m,n}(W^{1,1}_{loc}(\mathbb{R})) \) then \( A^* \in M_{n,m}(W^{1,1}_{loc}(\mathbb{R})) \) with
\[ (A^*)' = \{\overline{a_{ij}}\}_{j=1,i=1}^{n,m} = \{\overline{a_{ij}}\}_{j=1,i=1}^{n,m} = (A')^*. \]

This completes the proof. \[ \blacksquare \]
Lemma 76 If \( A \in M_{m,n}(L^p(\mathbb{T})) \) and \( 1 \leq p \leq \infty \) then \( A^* \in M_{m,n}(L^p(\mathbb{T})) \) with \( ||A^*||_{L^p(\mathbb{T})} = ||A||_{L^p(\mathbb{T})} \).

Proof. By Lemma 75 the map \( ^* : M_{m,n}(L^p_{\text{loc}}(\mathbb{R})) \rightarrow M_{m,n}(L^p_{\text{loc}}(\mathbb{R})) \) extending the adjoint by \( A^*(\cdot) = A(\cdot)^* \) for \( A \in M_{m,n}(L^p_{\text{loc}}(\mathbb{R})) \) is well-defined. By Lemma 74, the translation operator \( \mathcal{L}_d : M_{m,n}(L^p_{\text{loc}}(\mathbb{R})) \rightarrow M_{m,n}(L^p_{\text{loc}}(\mathbb{R})) \) given by \( (\mathcal{L}_dA)(\cdot) = A(\cdot + d) \) for \( A \in M_{m,n}(L^p_{\text{loc}}(\mathbb{R})) \) is well-defined. Thus for any \( A \in M_{m,n}(L^p(\mathbb{T})) \), since \( M_{m,n}(L^p(\mathbb{T})) \subseteq M_{m,n}(L^p_{\text{loc}}(\mathbb{R})) \), then \( A^*, \mathcal{L}_dA^*, (\mathcal{L}_dA)^* \) are all well-defined and belong to \( M_{m,n}(L^p_{\text{loc}}(\mathbb{R})) \). Also, \( B \in M_{m,n}(L^p(\mathbb{T})) \) if and only if \( M_{m,n}(L^p_{\text{loc}}(\mathbb{R})) \) and \( \mathcal{L}_dB = B \). Thus since \( A^* \in M_{m,n}(L^p_{\text{loc}}(\mathbb{R})) \) and

\[
\mathcal{L}_dA^* = A^*(\cdot + d) = A(\cdot + d)^* = (\mathcal{L}_dA)^* = A^*
\]

then \( A^* \in M_{m,n}(L^p(\mathbb{T})) \).

Now we prove \( ||A^*||_{L^p(\mathbb{T})} = ||A||_{L^p(\mathbb{T})} \) for \( A \in M_{m,n}(L^p(\mathbb{T})) \). We start with the case \( m = n = 1 \). If \( f \in L^p(\mathbb{T}) \) then as we just proved \( \overline{f} \in L^p(\mathbb{T}) \) and

\[
||\overline{f}||_{L^p(\mathbb{T})} = \left( \frac{1}{d} \int_0^d |\overline{f}(x)|^p dx \right)^{\frac{1}{p}} = \left( \frac{1}{d} \int_0^d |f(x)|^p dx \right)^{\frac{1}{p}} = ||f||_{L^p(\mathbb{T})} \quad (1 \leq p < \infty),
\]

\[
||\overline{f}||_{L^\infty(\mathbb{T})} = ||\overline{f}||_{\infty} = ||f||_{\infty} = ||f||_{L^\infty(\mathbb{T})} \quad (p = \infty).
\]

This proves the statement for \( m = n = 1 \). Consider now the case \( m, n \geq 1 \). If \( A := \{a_{ij}\}_{i=1,j=1}^{m,n} \in M_{m,n}(L^p(\mathbb{T})) \) then \( A^* = \{\overline{a_{ij}}\}_{i=1,j=1}^{n,m} \in M_{n,m}(L^p_{\text{loc}}(\mathbb{R})) \) and so

\[
||A^*||_{L^p(\mathbb{T})} = \left( \sum_{j=1}^n \sum_{i=1}^m ||\overline{a_{ij}}||_{L^p(\mathbb{T})}^2 \right)^{\frac{1}{2}} = \left( \sum_{i=1}^m \sum_{j=1}^n ||a_{ij}||_{L^p(\mathbb{T})}^2 \right)^{\frac{1}{2}} = ||A||_{L^p(\mathbb{T})}.
\]

And hence the statement is true for \( m, n \geq 1 \). This completes the proof. \( \blacksquare \)
General Statement

In this section we give the statements we need for general Banach spaces. We split this section into statements that do not involve the frequency $\omega$ and those that do.

Parameter Independent

This is the collection of statements which do not involve the frequency parameter.

Definition 27 Let $E_1$, $E_2$, and $E_3$ be Banach spaces. A function $\cdot : E_1 \times E_2 \to E_3$ is called a continuous bilinear map provided it satisfies the following three properties:

(i) For any $a_1 \in E_1$, the map $a_2 \mapsto a_1 \cdot a_2$ is a linear map from $E_2$ to $E_3$.

(ii) For any $a_2 \in E_2$, the map $a_1 \mapsto a_1 \cdot a_2$ is a linear map from $E_1$ to $E_3$.

(iii) There exists a constant $C > 0$ such that $\|a_1 \cdot a_2\|_{E_3} \leq C \|a_1\|_{E_1} \|a_2\|_{E_2}$ for all $a_1 \in E_1$, $a_2 \in E_2$.

Lemma 77 Let $E_1$, $E_2$, and $E_3$ be Banach spaces and $\cdot : E_1 \times E_2 \to E_3$ be a bilinear map. Define the map $\cdot : M_{m,n}(E_1) \times M_{n,r}(E_2) \to M_{m,r}(E_3)$ defined by

$$A \cdot B := \left[ \sum_{k=1}^{n} a_{ik} \cdot b_{kj} \right]_{i=1,j=1}^{m,r}$$

for any $A := [a_{ij}]_{i=1,j=1}^{m,n} \in M_{m,n}(E_1)$, $B := [b_{ij}]_{i=1,j=1}^{n,r} \in M_{n,r}(E_2)$. Then $\cdot : M_{m,n}(E_1) \times M_{n,r}(E_2) \to M_{m,r}(E_3)$ is a continuous bilinear map.

Proof. The map is obviously linear in both of its arguments and so properties (i) and (ii) are satisfied of Definition 27. Now choose any $C > 0$ such that $\|a_1 \cdot a_2\|_{E_3} \leq C \|a_1\|_{E_1} \|a_2\|_{E_2}$.
for all $a_1 \in E_1$, $a_2 \in E_2$. Then

$$
||A \cdot B||_{E_3} = \left( \sum_{i=1}^{m} \sum_{j=1}^{r} \left( \sum_{k=1}^{n} a_{ik} \cdot b_{kj} \right)^2 \right)^{\frac{1}{2}} \leq \left( \sum_{i=1}^{m} \sum_{j=1}^{r} \sum_{k=1}^{n} ||a_{ik} \cdot b_{kj}||_{E_3}^2 \right)^{\frac{1}{2}}
$$

$$
\leq \left( \sum_{i=1}^{m} \sum_{j=1}^{r} \sum_{k=1}^{n} C^2 ||a_{ik}||_{E_1}^2 ||b_{kj}||_{E_3}^2 \right)^{\frac{1}{2}}
$$

$$
\leq C \left( \left( \sum_{i=1}^{m} \sum_{k=1}^{n} ||a_{ik}||_{E_1}^2 \right) \left( \sum_{k=1}^{n} \sum_{j=1}^{r} ||b_{kj}||_{E_3}^2 \right) \right)^{\frac{1}{2}}
$$

$$
= C ||A||_{M_{m,n}(E_1)} ||B||_{M_{n,r}(E_2)}.
$$

This proves property (iii) of Definition 27 and thus we have proved the lemma. ■

**Parameter Dependent**

This is the collection of statements which involve the frequency parameter.

**Lemma 78** Let $E_1$, $E_2$, and $E_3$ be Banach spaces and $\cdot : E_1 \times E_2 \to E_3$ be a bilinear map. If $u \in \mathcal{O}(\Omega, E_1)$ and $v \in \mathcal{O}(\Omega, E_2)$ then $u \cdot v \in \mathcal{O}(\Omega, E_3)$ and

$$(u \cdot v)_\omega = u_\omega \cdot v + u \cdot v_\omega.$$  

**Proof.** First we choose any $C > 0$ such that $||a_1 \cdot a_2||_{E_3} \leq C ||a_1||_{E_1} ||a_2||_{E_2}$ for all $a_1 \in E_1$, $a_2 \in E_2$. Now let $u \in \mathcal{O}(\Omega, E_1)$ and $v \in \mathcal{O}(\Omega, E_2)$. Let $\omega_0 \in \Omega$. Then

$$
|| (\omega - \omega_0)^{-1}(u(\omega) \cdot v(\omega) - u(\omega_0) \cdot v(\omega_0)) - (u_\omega(\omega_0) \cdot v(\omega_0) + u(\omega_0) \cdot v_\omega(\omega_0)) ||_{E_3}
$$

$$
= ||[(\omega - \omega_0)^{-1}(u(\omega) - u_\omega(\omega_0))] \cdot v(\omega) + u(\omega_0) \cdot [(\omega - \omega_0)^{-1}(v(\omega) - v_\omega(\omega_0))]|_{E_3}
$$

$$
\leq C ||(\omega - \omega_0)^{-1}(u(\omega) - u_\omega(\omega_0))||_{E_1} ||(\omega - \omega_0)^{-1}(v(\omega) - v_\omega(\omega_0))||_{E_2}
$$

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Thus as $\omega \to \omega_0$ the RHS of that inequality goes to zero implying the LHS of that inequality does as well which implies the function $(u \cdot v)(\omega) := u(\omega) \cdot v(\omega)$ is differentiable at $\omega_0$ and

$$(u \cdot v)_\omega(\omega_0) = (u_\omega \cdot v)(\omega_0) + (u \cdot v_\omega)(\omega_0).$$

Therefore $u \cdot v \in \mathcal{O}(\Omega, E)$ and $(u \cdot v)_\omega = u_\omega \cdot v + u \cdot v_\omega$. ■

Lemma 79 Let $E_1$ and $E_2$ be Banach spaces. If $\mathcal{L} \in \mathcal{O}(\Omega, \mathcal{L}(E_1, E_2))$ and $u \in \mathcal{O}(\Omega, E_1)$ then $\mathcal{L}u \in \mathcal{O}(\Omega, E_2)$ and

$$(\mathcal{L}u)_\omega = \mathcal{L}_\omega u + \mathcal{L}u_\omega.$$  

Proof. From the Banach spaces $\mathcal{L}(E_1, E_2)$ and $E_1$, we define the map $\cdot : \mathcal{L}(E_1, E_2) \times E_1 \to E_2$ by $\mathcal{L} \cdot u := \mathcal{L}u$. Then it is a continuous bilinear map with $||\mathcal{L} \cdot u||_{E_2} \leq ||\mathcal{L}||_{\mathcal{L}(E_1, E_2)} ||u||_{E_1}$. Thus it follows from Lemma 78 above that if $\mathcal{L} \in \mathcal{O}(\Omega, \mathcal{L}(E_1, E_2))$ and $u \in \mathcal{O}(\Omega, E_1)$ then $\mathcal{L}u \in \mathcal{O}(\Omega, E_2)$ and $(\mathcal{L}u)_\omega = \mathcal{L}_\omega u + \mathcal{L}u_\omega$. This completes the proof. ■

Lemma 80 Let $E$ be a Banach space. Then $\mathcal{O}(\Omega, M_{m,n}(E)) = M_{m,n}(\mathcal{O}(\Omega, E))$ and for any $A := \{a_{ij}\}_{i=1,j=1}^{m,n} \in \mathcal{O}(\Omega, M_{m,n}(E))$,

$$A_\omega = \{(a_{ij})_\omega\}_{i=1,j=1}^{m,n}.$$  

Proof. Let $(E, || \cdot ||)$ be a Banach space and $(M_{m,n}(E), || \cdot ||_{M_{m,n}(E)})$ the Banach space whose norm is defined in (4.81). To begin, we know that any two norms on a finite dimensional vector space over $\mathbb{C}$ are equivalent so that there exists constants $C_1, C_2 > 0$ such that

$$C_1 ||B||_{M_{m,n}(\mathbb{C}), \infty} \leq ||B||_{M_{m,n}(\mathbb{C}), 2} \leq C_2 ||B||_{M_{m,n}(\mathbb{C}), \infty}, \quad \forall B \in M_{m,n}(\mathbb{C}),$$

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where $\| \cdot \|_{M_{m,n}(\mathbb{C}),\infty}, \| \cdot \|_{M_{m,n}(\mathbb{C}),2} : M_{m,n}(\mathbb{C}) \to [0, \infty)$ are the norms defined by

$$
\|B\|_{M_{m,n}(\mathbb{C}),\infty} := \max_{1 \leq i \leq m, 1 \leq j \leq n} |b_{ij}|, \quad \|B\|_{M_{m,n}(\mathbb{C}),2} := \left( \sum_{i=1}^{m} \sum_{j=1}^{n} |b_{ij}|^2 \right)^{\frac{1}{2}}
$$

for any $B := [b_{ij}]_{i=1}^{m}_{j=1} \in M_{m,n}(\mathbb{C})$.

Now we define a function $| \cdot | : M_{m,n}(E) \to [0, \infty)$ by

$$
|A| := \|[a_{ij}]\|_{i=1}^{m}_{j=1}^{n}, \quad \forall A := [a_{ij}]_{i=1}^{m}_{j=1} \in M_{m,n}(E).
$$

We define two norms $\| \cdot \|_{M_{m,n}(E),\infty}, \| \cdot \|_{M_{m,n}(E),2} : M_{m,n}(E) \to [0, \infty)$ by

$$
\|A\|_{M_{m,n}(E),\infty} := \|[A]\|_{M_{m,n}(\mathbb{C}),\infty}, \quad \|A\|_{M_{m,n}(E),2} := \|[A]\|_{M_{m,n}(\mathbb{C}),2}
$$

for any $A := [a_{ij}]_{i=1}^{m}_{j=1} \in M_{m,n}(E)$. But since $\| \cdot \|_{M_{m,n}(E),2} = \| \cdot \|_{M_{m,n}(E)}$ it follows that

$$
C_1 \|A\|_{M_{m,n}(E),\infty} \leq \|A\|_{M_{m,n}(E)} \leq C_2 \|A\|_{M_{m,n}(E),\infty}, \quad \forall A \in M_{m,n}(E).
$$

From this it follows that the identity map $\iota : (M_{m,n}(E), \| \cdot \|_{M_{m,n}(E)}) \to (M_{m,n}(E), \| \cdot \|_{M_{m,n}(E),\infty})$ belongs to $\mathcal{L}(M_{m,n}(E), \| \cdot \|_{M_{m,n}(E)})$ and has a continuous inverse. These facts and Lemma 79 imply that $\mathcal{O}(\Omega, (M_{m,n}(E), \| \cdot \|_{M_{m,n}(E)})) = \mathcal{O}(\Omega, (M_{m,n}(E), \| \cdot \|_{M_{m,n}(E),\infty}))$ and $(\iota A)_{\omega} = \iota A_{\omega} = A_{\omega}$ for all $A \in \mathcal{O}(\Omega, (M_{m,n}(E), \| \cdot \|_{M_{m,n}(E)}))$.

Now it follows from the definition of the norm $\| \cdot \|_{M_{m,n}(E),\infty}$ that the statement of this lemma is true for $\mathcal{O}(\Omega, (M_{m,n}(E), \| \cdot \|_{M_{m,n}(E),\infty}))$ and hence $\mathcal{O}(\Omega, (M_{m,n}(E), \| \cdot \|_{M_{m,n}(E)})) = \mathcal{O}(\Omega, (M_{m,n}(E), \| \cdot \|_{M_{m,n}(E),\infty})) = M_{m,n}(\mathcal{O}(\Omega, E))$ with

$$
[(a_{ij})_{\omega}]_{i=1}^{m}_{j=1} = (\iota A)_{\omega} = \iota A_{\omega} = A_{\omega}
$$

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for every \( A := [a_{ij}]_{i,j=1}^{m,n} \in \mathcal{O}(\Omega, (M_{m,n}(E), \| \cdot \|_{M_{m,n}(E)})) \). This completes the proof of the lemma.

\[ \square \]

**Lemma 81** If \( f \in \mathcal{O}(\Omega, E) \) then \( f_\omega \in \mathcal{O}(\Omega, E) \).

**Proof.** This statement is proven in [19, pp. 21–22, Theorem 1.8.4 & Theorem 1.8.5].

\[ \square \]

**Specific Statements**

In the second section we give the statements we need for the specific Banach spaces used in this chapter. We split this section into statements that do not involve the frequency \( \omega \) and those that do.

**Parameter Independent**

This is the collection of statements which do not involve the frequency parameter.

**Lemma 82** If \( p, q, s \in \mathbb{R} \) with \( 1 \leq p, q, s \leq \infty \) and \( \frac{1}{p} + \frac{1}{q} = \frac{1}{s} \) then matrix multiplication \( \cdot : M_{m,n}(L^p(a,b)) \times M_{n,r}(L^q(a,b)) \to M_{m,r}(L^s(a,b)) \), where \( A \cdot B := AB \) for \( A \in M_{m,n}(L^p(a,b)) \), \( B \in M_{n,r}(L^q(a,b)) \), is a continuous bilinear map.

**Proof.** Multiplication \( \cdot : L^p(a,b) \times L^q(a,b) \to L^s(a,b) \), where \( f \cdot g := fg \) for \( f \in L^p(a,b) \), \( g \in L^q(a,b) \), is a bilinear map since it follows from Holder’s inequality that

\[ \|fg\|_s \leq \|f\|_p \|g\|_q. \]
Thus by Lemma 77 it follows that matrix multiplication $\cdot : M_{m,n}(L^p(a,b)) \times M_{n,r}(L^q(a,b)) \to M_{m,r}(L^s(a,b))$, where $A \cdot B := AB$ for $A \in M_{m,n}(L^p(a,b))$ and $B \in M_{n,r}(L^q(a,b))$, is also continuous bilinear map. This completes the proof. ■

**Lemma 83** Let $1 \leq p \leq \infty$. Then the map of matrix multiplication $\cdot : M_{m,n}(L^p(a,b)) \times M_{n,r}(W^{1,1}(a,b)) \to M_{m,r}(L^p(a,b))$, where $A \cdot B := AB$ for $A \in M_{m,n}(L^p(a,b))$ and $B \in M_{n,r}(W^{1,1}(a,b))$, is a continuous bilinear map.

**Proof.** Let $(C[a,b], || \cdot ||_\infty)$ denote the Banach space of all continuous functions on $[a,b]$ with norm $||g||_\infty := \sup_{x \in [a,b]} |g(x)|$. By our convention, if $g \in W^{1,1}(a,b)$ then $g$ is absolutely continuous on $[a,b]$ and hence $g \in C[a,b]$. Thus we have $W^{1,1}(a,b) \subseteq C[a,b]$. Now by [45, p. 57, §II.2.1, Proposition 2.1.7] we have for the inclusion map $\iota : W^{1,1}(a,b) \to C[a,b]$ given by $\iota g = g$ that $\iota \in \mathcal{L}(W^{1,1}(a,b), C[a,b])$. This implies there exists a $C > 0$ such that

$$\sup_{x \in [a,b]} |g(x)| = ||g||_\infty \leq C ||g||_{1,1}, \text{ for every } g \in W^{1,1}(a,b).$$

It follows from this that for all $f \in L^p(a,b), g \in W^{1,1}(a,b)$ we have

$$||fg||_p \leq ||f||_p ||g||_\infty \leq C ||f||_p ||g||_{1,1}.$$ 

This implies that multiplication $\cdot : L^p(a,b) \times W^{1,1}(a,b) \to L^p(a,b)$, where $f \cdot g := fg$ for $f \in L^p(a,b), g \in W^{1,1}(a,b)$, is a continuous bilinear map. Therefore it follows by Lemma 77 that the matrix multiplication $\cdot : M_{m,n}(L^p(a,b)) \times M_{n,r}(W^{1,1}(a,b)) \to M_{m,r}(L^p(a,b))$, where $A \cdot B := AB$ for $A \in M_{m,n}(L^p(a,b)), B \in M_{n,r}(W^{1,1}(a,b))$, is also a continuous bilinear map. This completes the proof. ■

**Lemma 84** Let $1 \leq p \leq \infty$. Then the map of matrix multiplication $\cdot : M_{m,n}(L^p_{loc}(\mathbb{R})) \times M_{n,r}(W^{1,1}_{loc}(\mathbb{R})) \to M_{m,r}(L^p_{loc}(\mathbb{R}))$, where $A \cdot B := AB$ for $A \in M_{m,n}(L^p_{loc}(\mathbb{R}))$ and $B \in M_{n,r}(W^{1,1}_{loc}(\mathbb{R}))$, is a well-defined map.
Proof. Let $A \in M_{m,n}(L^p_{\text{loc}}(\mathbb{R}))$ and $B \in M_{n,r}(W^{1,1}_{\text{loc}}(\mathbb{R}))$. Then, for any bounded interval $(a, b) \subseteq \mathbb{R}$ by Lemma 83, $(AB)|_{(a,b)} = A|_{(a,b)}B|_{(a,b)} \in M_{m,r}(L^p(a, b))$. But this implies $AB \in M_{m,r}(L^p_{\text{loc}}(\mathbb{R}))$. This completes the proof. 

Lemma 85 For each $x \in [a, b]$, the evaluation map at $x$ defined by $\delta_x : M_{m,n}(W^{1,1}(a, b)) \rightarrow M_{m,n}(\mathbb{C})$ where $\delta_x A := A(x)$ for $A \in M_{m,n}(W^{1,1}(a, b))$. Then

$$\delta_x \in \mathcal{L}(M_{m,n}(W^{1,1}(a, b)), M_{m,n}(\mathbb{C})). \quad (4.82)$$

Proof. It follows from the proof of Lemma 83 that there exists a constant $C > 0$ such that

$$|\delta_x g| \leq C||g||_{1,1}, \quad \text{for every } g \in W^{1,1}(a, b).$$

Now the evaluation map is linear so we need only prove its bounded. If $A := [a_{ij}]_{i=1,j=1}^{m,n} \in M_{m,n}(W^{1,1}(a, b))$ we have

$$||\delta_x A||_{\mathbb{C}} = ||A(x)||_{\mathbb{C}} = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}(x)|^2\right)^{\frac{1}{2}} \leq m^{\frac{1}{2}}n^{\frac{1}{2}} \max_{1 \leq i \leq m, 1 \leq j \leq n} |a_{ij}(x)|$$

$$\leq Cm^{\frac{1}{2}}n^{\frac{1}{2}} \max_{1 \leq i \leq m, 1 \leq j \leq n} ||a_{ij}||_{1,1} \leq Cm^{\frac{1}{2}}n^{\frac{1}{2}} \left(\sum_{i=1}^{m} \sum_{j=1}^{n} ||a_{ij}||_{1,1}^2\right)^{\frac{1}{2}} = Cm^{\frac{1}{2}}n^{\frac{1}{2}}||A||_{1,1}.$$ 

This implies that $||\delta_x||_{\mathcal{L}(M_{m,n}(W^{1,1}(a, b)), M_{m,n}(\mathbb{C}))} \leq Cm^{\frac{1}{2}}n^{\frac{1}{2}}$ and hence it follows that $\delta_x \in \mathcal{L}(M_{m,n}(W^{1,1}(a, b)), \mathbb{C})$. This completes the proof. 

Lemma 86 Let $a, b \in \mathbb{R}$ with $a < b$ and let $u \in [a, b]$. We define the integral map by

$$(I A)(x) := \int_{u}^{x} A(t)dt, \quad A \in M_{m,n}(L^1(a, b)), \quad x \in (a, b).$$
Then \( \mathcal{I} \in \mathcal{L}(M_{m,n}(L^1(a, b)), M_{m,n}(W^{1,1}(a, b))) \) and

\[(\mathcal{I}A)' = A,\]

for any \( A \in M_{m,n}(L^1(a, b)) \).

**Proof.** By [45, p. 56, §II.2.1, Proposition 2.1.5] it follows that the map \( \mathcal{I} : M_{m,n}(L^1(a, b)) \to M_{m,n}(W^{1,1}(a, b)) \) is a well-defined map which is linear and \( (\mathcal{I}A)' = A \). Thus we need only prove that it is bounded to complete the proof. Now if \( f \in L^1(a, b) \) then it follows from the definition of the norm \( || \cdot ||_{1,1} \) on \( W^{1,1}(a, b) \) that

\[||\mathcal{I}f||_{1,1} = ||(\mathcal{I}f)'||_1 + ||If||_1 = ||f||_1 + ||If||_1 \leq (b - a + 1)||f||_1.\]

Hence it follows that if \( A := [a_{ij}]_{i=1,j=1}^{m,n} \in M_{m,n}(L^1(a, b)) \) then

\[||\mathcal{I}A||_{1,1} = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} ||\mathcal{I}a_{ij}||^2_{1,1}\right)^{\frac{1}{2}} \leq m^{\frac{1}{2}}n^{\frac{1}{2}} \max_{1\leq i \leq m} ||\mathcal{I}a_{ij}||_{1,1}\]

\[\leq (b - a + 1)m^{\frac{1}{2}}n^{\frac{1}{2}} \max_{1\leq i \leq m} ||a_{ij}||_1 \leq (b - a + 1)m^{\frac{1}{2}}n^{\frac{1}{2}} \left(\sum_{i=1}^{m} \sum_{j=1}^{n} ||a_{ij}||^2_1\right)^{\frac{1}{2}}\]

\[= (b - a + 1)m^{\frac{1}{2}}n^{\frac{1}{2}} ||A||_1.\]

This implies \( ||\mathcal{I}||_{\mathcal{L}(M_{m,n}(L^1(a, b)), M_{m,n}(W^{1,1}(a, b)))} \leq (b - a + 1)m^{\frac{1}{2}}n^{\frac{1}{2}} \) and hence it follows that \( \mathcal{I} \in \mathcal{L}(M_{m,n}(L^1(a, b)), M_{m,n}(W^{1,1}(a, b))) \). This completes the proof. 

**Lemma 87** The integral map \( \mathcal{I} : M_{m,n}(L^1_{loc}(\mathbb{R})) \to M_{m,n}(W^{1,1}_{loc}(\mathbb{R})) \) defined by

\[(\mathcal{I}A)(x) := \int_0^x A(t)dt, \quad A \in M_{m,n}(L^1_{loc}(\mathbb{R})), \quad x \in \mathbb{R},\]
is well-defined and satisfies

\[(\mathcal{I}A)' = A,\]

for any \(A \in M_{m,n}(L^1_{\text{loc}}(\mathbb{R})).\)

**Proof.** Let \(A \in M_{m,n}(L^1_{\text{loc}}(\mathbb{R})).\) Then it follows that the function \(B_A(\cdot) : \mathbb{R} \to M_{m,n}(\mathbb{C}),\) defined by

\[B_A(x) := \int_0^x A(t)dt, \quad x \in \mathbb{R},\]

is well-defined. Moreover, by Lemma 86 we have that \(B_A|_{(a,b)} \in M_{m,n}(W^{1,1}_{\text{loc}}(a,b))\) for any bounded interval \((a, b)\) in \(\mathbb{R}\) containing 0 and \(B_A'|_{(a,b)} = A|_{(a,b)}\). By Lemma 71 it follows that \(B_A \in M_{m,n}(W^{1,1}_{\text{loc}}(\mathbb{R}))\) and \(B_A'|_{(a,b)} = A|_{(a,b)}\) for any bounded interval \((a, b)\) in \(\mathbb{R}\) containing 0. This proves \(B_A' = A\). But by the definition of the integral map we have \(\mathcal{I}A = B_A\) for any \(A \in M_{m,n}(L^1_{\text{loc}}(\mathbb{R})).\) Therefore \(\mathcal{I} : M_{m,n}(L^1_{\text{loc}}(\mathbb{R})) \to M_{m,n}(W^{1,1}_{\text{loc}}(\mathbb{R}))\) is well-defined and \((\mathcal{I}A)' = A\) for any \(A \in M_{m,n}(L^1_{\text{loc}}(\mathbb{R})).\) This completes the proof. \(\blacksquare\)

**Lemma 88** Let \(1 \leq p \leq \infty\) and \((a, b) \subseteq \mathbb{R}\) be a bounded open interval. Define the restriction map by \(A \mapsto A|_{(a,b)}\) where \(A \in M_{m,n}(L^p(\mathbb{T})).\) Then \(|_{(a,b)} \in \mathcal{L}(M_{m,n}(L^p(\mathbb{T})), M_{m,n}(L^p(a, b))).\)

**Proof.** Fix \(p \in [1, \infty]\) and a bounded open interval \((a, b).\) Pick \(q \in \mathbb{N}\) such that \((a, b) \subseteq [-qd,qd].\) Let \(f \in L^p(\mathbb{T}).\) Then \(f \in L^p_{\text{loc}}(\mathbb{R})\) and \(f(x + d) = f(x)\) for a.e. \(x \in \mathbb{R}.\) If \(p = \infty\) then

\[\|f|_{(a,b)}\|_{\infty} \leq \|f|_{(-qd,qd)}\|_{\infty} = \|f|_{(0,d)}\|_{\infty} = \|f\|_{L^\infty(\mathbb{T})}.\]
If $p \in [0, \infty)$ then by periodicity we have $\int_{(j-1)d}^{jd} |f(x)|^p dx = \int_0^d |f(x)|^p dx$ for any $j \in \mathbb{Z}$. Hence implying

$$\|f\|_{(a,b)}^p = \int_a^b |f(x)|^p dx \leq \int_{-qd}^{qd} |f(x)|^p dx$$

$$= \sum_{j=1}^q \int_{(j-1)d}^{jd} |f(x)|^p dx + \sum_{j=1}^q \int_{-jd}^{-(j-1)d} |f(x)|^p dx$$

$$= \sum_{j=1}^q \int_0^d |f(x)|^p dx + \sum_{j=1}^q \int_0^d |f(x)|^p dx$$

$$= 2q \int_0^d |f(x)|^p dx = 2qd \|f\|_{L^p(\mathbb{T})}^p.$$  

Thus for any $p \in [1, \infty]$ we have $\|f\|_{(a,b)}^p \leq (2qd)^{\frac{1}{p}} \|f\|_{L^p(\mathbb{T})}$. But then for any $A := [a_{ij}]_{i=1,j=1}^{m,n} \in M_{m,n}(L^p(\mathbb{T}))$ we have

$$\|A\|_{(a,b)}^p = \left(\sum_{i=1}^m \sum_{j=1}^n \|a_{ij}\|_{(a,b)}^2\right)^{\frac{1}{2}} \leq \max_{1 \leq i \leq m} \max_{1 \leq j \leq n} \|a_{ij}\|_{(a,b)}^2$$

$$\leq (2qd)^{\frac{1}{p}} m^{\frac{1}{2}} n^{\frac{1}{2}} \max_{1 \leq i \leq m} \max_{1 \leq j \leq n} \|a_{ij}\|_{L^p(\mathbb{T})}$$

$$\leq (2qd)^{\frac{1}{p}} m^{\frac{1}{2}} n^{\frac{1}{2}} \left(\sum_{i=1}^m \sum_{j=1}^n \|a_{ij}\|_{L^p(\mathbb{T})}^2\right)^{\frac{1}{2}} = (2qd)^{\frac{1}{p}} m^{\frac{1}{2}} n^{\frac{1}{2}} \|A\|_{L^p(\mathbb{T})}.$$  

And hence we conclude the restriction map $|_{(a,b)} : M_{m,n}(L^p(\mathbb{T})) \rightarrow M_{m,n}(L^p(a,b))$ is a bounded linear operator with norm bounded above by $(2qd)^{\frac{1}{p}} m^{\frac{1}{2}} n^{\frac{1}{2}}$. This proves the statement for $p \in [1, \infty]$.  

Parameter Dependent

This is the collection of statements which involve the frequency parameter.

Lemma 89 Let $p, q, s \in \mathbb{R}$ with $1 \leq p, q, s \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$. If $A \in \mathcal{O}(\Omega, M_{m,n}(L^p(a,b)))$
and $B \in \mathcal{O}(\Omega, M_{n,r}(L^s(a,b)))$ then $AB \in \mathcal{O}(\Omega, M_{m,r}(L^s(a,b)))$ and

$$(AB)_\omega = A_\omega B + AB_\omega.$$ 

**Proof.** By Lemma 82 we know that the map of matrix multiplication $\cdot : M_{m,n}(L^p(a,b)) \times M_{n,r}(L^s(a,b)) \to M_{m,r}(L^s(a,b))$, where $A \cdot B := AB$ for $A \in M_{m,n}(L^p(a,b))$ and $B \in M_{n,r}(L^s(a,b))$, is a continuous bilinear map. Thus by Lemma 78 we must have $AB \in \mathcal{O}(\Omega, M_{m,r}(L^s(a,b)))$ and

$$(AB)_\omega = A_\omega B + AB_\omega.$$ 

This completes the proof. $lacksquare$

**Lemma 90** Let $1 \leq p \leq \infty$. If $A \in \mathcal{O}(\Omega, M_{m,n}(L^p(a,b)))$ and $B \in \mathcal{O}(\Omega, M_{n,r}(W^{1,1}(a,b)))$ then $AB \in \mathcal{O}(\Omega, M_{m,r}(L^p(a,b)))$ and

$$(AB)_\omega = A_\omega B + AB_\omega.$$ 

**Proof.** By Lemma 83 we know that the map of matrix multiplication $\cdot : M_{m,n}(L^p(a,b)) \times M_{n,r}(W^{1,1}(a,b)) \to M_{m,r}(L^p(a,b))$, where $A \cdot B := AB$ for $A \in M_{m,n}(L^p(a,b))$, $B \in M_{n,r}(W^{1,1}(a,b))$, is a continuous bilinear map. Thus by Lemma 78 we must have $AB \in \mathcal{O}(\Omega, M_{m,r}(L^p(a,b)))$ and

$$(AB)_\omega = A_\omega B + AB_\omega.$$ 

This completes the proof. $lacksquare$

**Lemma 91** Let $x \in [a,b]$ and $\delta_x : M_{n,n}(W^{1,1}(a,b)) \to M_{n,n}(\mathbb{C})$ be the evaluation map
at \( x \), i.e., \( \delta_x A = A(x) \) for all \( A \in M_{m,n}(W^{1,1}(a,b)) \). If \( A \in \mathcal{O}(\Omega, M_{m,n}(W^{1,1}(a,b))) \) then \( \delta_x A \in \mathcal{O}(\Omega, M_{m,n}(\mathbb{C})) \) and

\[
(\delta_x A)_\omega = \delta_x A_\omega.
\]

**Proof.** By Lemma 85 we know \( \delta_x \in \mathcal{L}(M_{m,n}(W^{1,1}(a,b)), M_{m,n}(\mathbb{C})). \) Thus by Lemma 79, if \( A \in \mathcal{O}(\Omega, M_{m,n}(W^{1,1}(a,b))) \) then \( \delta_x A \in \mathcal{O}(\Omega, M_{m,n}(\mathbb{C})) \) and

\[
(\delta_x A)_\omega = \delta_x A_\omega.
\]

This completes the proof. \( \blacksquare \)

**Lemma 92** Let \( a, b \in \mathbb{R}, \ a < b, \) and \( u \in [a, b] \). Let \( \mathcal{I} : M_{m,n}(L^1(a,b)) \to M_{m,n}(W^{1,1}(a,b)) \) be the integral map, i.e.,

\[
(\mathcal{I}A)(x) := \int_u^x A(t)dt, \quad A \in M_{m,n}(L^1(a,b)), \quad x \in (a,b).
\]

If \( A \in \mathcal{O}(\Omega, M_{m,n}(L^1(a,b))) \) then \( \mathcal{I}A \in \mathcal{O}(\Omega, M_{m,n}(W^{1,1}(a,b))) \) and

\[
(\mathcal{I}A)_\omega = \mathcal{I}A_\omega.
\]

**Proof.** By Lemma 86 we know that \( \mathcal{I} \in \mathcal{L}(M_{m,n}(L^1(a,b)), M_{m,n}(W^{1,1}(a,b))). \) Thus by Lemma 79, if \( A \in \mathcal{O}(\Omega, M_{m,n}(L^1(a,b))) \) then \( \mathcal{I}A \in \mathcal{O}(\Omega, M_{m,n}(W^{1,1}(a,b))) \) and

\[
(\mathcal{I}A)_\omega = \mathcal{I}A_\omega.
\]

This completes the proof. \( \blacksquare \)
Lemma 93 Let $p \in [1, \infty)$ and $(a, b)$ be any bounded open interval in $\mathbb{R}$. Let $|_{(a, b)} : M_{m,n}(L^p(\mathbb{T})) \to M_{m,n}(L^p(a, b))$ be the restriction map, i.e., the domain restriction map $A \mapsto A|_{(a, b)}$ for $A \in M_{m,n}(L^p(\mathbb{T}))$. If $A \in \mathcal{O}(\Omega, M_{m,n}(L^p(\mathbb{T})))$ then $A|_{(a, b)} \in \mathcal{O}(\Omega, M_{m,n}(L^p(a, b)))$ and

$$(A|_{(a, b)})_\omega = A_\omega|_{(a, b)}.$$  

Proof. By Lemma 88 we know that $|_{(a, b)} \in \mathcal{L}(M_{m,n}(L^p(\mathbb{T})), M_{m,n}(L^p(a, b)))$. Thus by Lemma 79, if $A \in \mathcal{O}(\Omega, M_{m,n}(L^p(\mathbb{T})))$ then we have $A|_{(a, b)} \in \mathcal{O}(\Omega, M_{m,n}(L^p(a, b)))$ and

$$(A|_{(a, b)})_\omega = A_\omega|_{(a, b)}.$$  

This completes the proof. ■


