The Role of Diagrams in Mathematical Arguments

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Abstract. Recent accounts of the role of diagrams in mathematical reasoning take a Platonic line, according to which the proof depends on the similarity between the perceived shape of the diagram and the shape of the abstract object. This approach is unable to explain proofs which share the *same* diagram in spite of drawing conclusions about *different* figures. Saccheri's use of the bi-rectangular isosceles quadrilateral in ***Euclides Vindicatus*** provides three such proofs. By forsaking abstract objects it is possible to give a natural explanation of Saccheri's proofs as well as standard geometric proofs and even number-theoretic proofs.

Keywords: Diagram, proof, anti-platonism, mathematical reasoning

§0. Introduction. Plato's doctrine that mathematical investigation by means of images falls short of reality has survived two millennia (*cf*. ***Rep***. 510-b). Tennant 1986 is typical.

> [The diagram] is only an heuristic to prompt certain trains of inference; ... it is dispensable as a proof-theoretic device; ... indeed, ... it has no proper place in the proof as such. For the proof is a syntactic object consisting only of sentences arranged in a finite, inspectable array. (304)

But lately the formalist creed has been challenged by more charitable appraisals of diagrammatic reasoning (e.g., Giaquinto 1992, 1994; Brown 1997, 1999; Lomas 2002). This challenge is warranted by the historical importance of diagrams. Diagrams were essential to mathematical reasoning at least until the flourishing of algebra in the 17th and

18th centuries. To deny that they play an essential role in proof is to disparage the methods and results upon which present day mathematics rests. Moreover, the formalist conception of proof arose only *after* mathematicians developed the notion of abstract structure in the 19th century (cf. Mueller 1969, 298-9; Nagel 1979); and only at *that* time were axioms (such as Pasch's) articulated in order that results which depended previously upon a diagram could be proved by a finite, inspectable array of sentences. Diagrams are, as well, still important in mathematics education. Plainly the movement to accord diagrams a substantial role in mathematics is crucial to a philosophy of *real* mathematics.

Thus far, philosophical accounts of diagrams have proceeded along Platonist lines: Diagrams are a more or less reliable means of learning about objects outside the causal nexus. Brown is unabashedly Platonist. He asserts that in a wide variety of cases a good diagram works by being isomorphic *to the situation it represents* (1999, 38). And in those cases for which there is no isomorphism – cases in which pictures are not really pictures – diagrams still function as "windows to Plato's heaven" (39). Thus, for Brown, diagrams provide access to abstract objects. According to Lomas,

> [T]he perception of shape properties of concrete diagrams is a *surrogate* for conscious awareness of shape properties of abstract geometric objects depicted in the diagrams. (210, his italics)

Here too diagrams are, in virtue of their perceptual properties, assumed to point beyond themselves to abstract objects. Giaquinto's discussion of the diagram in Plato's ***Meno*** (82bff.) makes a similar point: It requires that the inexact diagram be *seen as* an exact one (1993, 90). The subsequent analysis of seeing as proposes that

> [The diagram] must look similar to something which appears exactly square. Seeing the diagram as a square involves both seeing it and observing this similarity of appearance. (ibid.)

This comment needn't imply Platonism, as an empirically admissible account of "appearing exactly square" is not out of the question. Giaquinto doesn't provide one, however, and he's apparently not inclined in that direction because he writes that mathematics is for "getting information about things that [are] not before one's eyes" (95).

It's no accident that Plato used a mathematical diagram as his paradigm case of worldly objects that enable one to recollect the realities which they resemble; the analysis of diagrammatic reasoning along Platonistic lines is perfectly natural. But the appeal to abstract objects presents special problems for an account of diagrammatic reasoning, problems in addition to the usual complaints about epistemic access to abstract objects. For the perceptual properties of diagrams are themselves insufficient to account for a certain variety in the results of diagrammatic reasoning, viz., proofs which share the same diagram in spite of the fact that their conclusions concern figures which are incompatible with one another.

§1 presents such a case, Saccheri's use of the bi-rectangular, isosceles quadrilateral (cf. appendix; Saccheri 1920; Sherry 1999). Platonic accounts of diagrammatic reasoning, which depend on a similarity between the perceived shape of the diagram and the shape of the abstract object, cannot explain Saccheri's use of the *same* diagram in proving theorems about radically incompatible figures. Fortunately, there is no need to appeal to abstract objects in order to give an account of the role of diagrams in mathematical argument. In explaining how such reasoning is possible §2 presents an account of diagrammatic reasoning that makes no use of abstract objects. Instead the account is built from an analysis of the manner in which mathematics is applied to concrete objects. Both the usual problems with abstract objects and the special one just mentioned are avoided in the alternative I'm proposing. Finally, §3 shows that this analysis can be extended to diagrammatic reasoning outside of geometry, in particular to the number theoretic proofs which lead Brown to propose that *some* diagrams are best understood as windows to Plato's heaven.

§1. The Lesson of Saccheri's Quadrilateral. In the 18th century Saccheri tried to demonstrate that Euclid's parallel postulate is a logical consequence of his first four postulates. His proofs employed a figure (fig. 1) known as "a Saccheri quadrilateral," a bi-rectangular, isosceles quadrilateral.

[[INSERT IMAGE SherryFig1.eps]]

fig. 1

Saccheri began from the insight that the parallel postulate and its contraries are equivalent to hypotheses about the summit angles (∠C and ∠D) of a Saccheri quadrilateral.

> Hypothesis of the Right Angle (HRA): The summit angles of a Saccheri quadrilateral are right angles. (HRA is equivalent to Euclid's fifth postulate.)
>
> Hypothesis of the Obtuse Angle (HOA): The summit angles of a Saccheri quadrilateral are obtuse angles.
>
> Hypothesis of the Acute Angle (HAA): The summit angles of a Saccheri quadrilateral are acute angles.

Using standard Euclidean methodology – setting out a diagram, adding auxiliary constructions and then reasoning about elements of the augmented diagram – Saccheri hoped to show that the contrary hypotheses, in conjunction with Euclid's first four postulates, each entail HRA. He was successful with HOA (Prop. 14) but able to derive only strange results from HAA, results that anticipate fundamental theorems of hyperbolic geometry.[1]

Remarkably, Saccheri used the *same* diagram in proving theorems about each of three figures: a Saccheri quadrilateral whose summit angles are right, a Saccheri quadrilateral whose summit angles are obtuse and a Saccheri quadrilateral whose summit angles are acute. Using in each case the *same* diagram of a bi-rectangular isosceles quadrilateral, Saccheri proved that

> The summit line CD of a Saccheri quadrilateral whose summit angles are right is equal to the base AB.
>
> The summit line CD of a Saccheri quadrilateral whose summit angles are obtuse is less than the base AB.
>
> The summit line CD of a Saccheri quadrilateral whose summit angles are acute is greater than the base AB.

The cases employ different auxiliary constructions, but each of the auxiliaries is possible no matter what the magnitude of the summit angles. This strategy turns out not to be a special case but only an especially powerful illustration of the standard use of diagrams in geometrical reasoning.

How, then, can the diagram do triple duty? The answer to this question depends on both the perceptual ambiguity of the diagram and the manner in which it is actually used in the argument. The *ambiguity* of Saccheri's diagram follows from its being an empirical object. As long as the diagram is constructed by ruler and compass with reasonable care, any result of measuring the summit angles could count equally as an instance of a right, an obtuse, or an acute angle. Empirical

[1] HOA is consistent and yields an elliptic geometry *provided* one drops

predicates, by their nature, admit borderline cases, and an angle that measures 90±n degrees (context determines n) is a borderline case of an angle that is right, but also a borderline case of an angle that is slightly obtuse and a borderline case of an angle that is slightly acute. This is not to suggest that measurement or careful construction, are important to the use Saccheri makes of his diagram. We shall see that, on the contrary, more sophisticated empirical observations, such as measurement, play no role in diagrammatic reasoning – even if they can be useful for suggesting theorems.

The ambiguity in the magnitude of the summit angles of a Saccheri quadrilateral presented no impediment to his reasoning by means of the diagram. But the ease with which Saccheri put his diagram to radically different uses is an anomaly for any attempt to understand diagrammatic reasoning in terms of our perception of the diagram and the capacity of that perception to provide access to abstract objects. For an ambiguous empirical object can't, on its own, explain how in one case it provides access to, say, a Saccheri quadrilateral with right summit angles, while in another case it provides access to a Saccheri quadrilateral with acute summit angles. The perceptual characteristics of an ambiguous diagram may provide *topological* information (e.g., information about incidence or betweeness), but not the *metric* information that is necessary for Saccheri to complete his different demonstrations (e.g., information about the magnitude of angles and lines).[2] Some thing or activity *besides* the diagram seems necessary to connect Saccheri's diagram with its appropriate Platonic counterpart.

Euclid's second postulate that a line may be extended indefinitely.

[2] Not all topological features need be exhibited by the diagram. For instance, gaps in the lines or corners of the diagram will not affect the proof even though continuity is a topological characteristic of the figure.

Once the extra element has been made explicit, I believe, the Platonic counterpart will be seen to be unnecessary.

Before discussing what else is necessary for the diagram in Saccheri's proof to perform its function, it's worth considering an objection to the claim that perception does not contain the resources by which a single diagram provides access to multiple objects. Our experience includes diagrams that can be perceived in more than one way. For example, fig. 2

[[INSERT IMAGE SherryFig2.tiff]]

fig. 2

can be perceived as both a duck and a rabbit. By analogy, shouldn't it be possible to perceive Saccheri's diagram in some circumstances as a figure with right summit angles and other times as a figure with acute or obtuse summit angles? Such ambiguity in perception is undeniable, but it's not, in my experience, what happens as people work their way through Saccheri's proofs. I've presented these proofs several times and frequently people will say they have trouble with the proofs because they can see the summit angles *only* as right angles. But never has anyone reported the aspect change characteristic of the duck-rabbit. More generally, aspect change involves a shift between radically different perceptions, rather than a shift between properties that share borderline cases. We shall see at the close of §2 that perception is involved in Saccheri's proof, but not the perception of the magnitude of the summit angles. The summit angles are perceived either as having no particular magnitude, or they are perceived as right angles and the reasoner simply puts this perceptual information aside while proceeding with the proof. That is, Saccheri's proofs succeed in spite of the rectangular appearance of the diagram.

§2. The Possibility of Saccheri's Proofs. Perfectly ordinary means suffice to put the same quadrilateral to use in proofs about quadrilaterals with different summit angles. It takes only the standard practice of letting a diagram be a Φ, where Φ is some appropriate mathematical concept. Thus, Saccheri's ploy is no more remarkable than the pedestrian practice of, say, letting ABC (fig. 3) be an isosceles triangle – something we can do even if its sides appear unequal.

[[INSERT IMAGE SherryFig3.eps]]

fig. 3

A poor diagram does not generally affect proofs in which it occurs, even if it may affect our success at guessing theorems. For example, a diagram of a triangle with sides of distinctly different lengths and a ray that makes a distinctly oblique angle with the base (fig. 4) does not affect a proof[3] that the bisector of the summit angle of an isosceles triangle is the perpendicular bisector of the base.

[[INSERT IMAGE SherryFig4.eps]]

fig. 4

Let ABC be an isosceles triangle with AC=BC and let CD bisect ∠ACB. Since CD bisects ∠ACB, ∠ACD=∠DCB. Further, since AC=BC and CD=CD, ΔACD≅ΔBCD (side-angle-side). But then AD=DB and

[3] I'm using "proof" in the sense in which mathematicians use it in their day-to-day work.

∠ADC=∠BDC; and since ∠ADC and ∠BDC are equal and adjacent, they are right. Therefore, CD is the perpendicular bisector of AB.

Apparently, the theorem does not result from *inspecting* a less than exact diagram, since the desired result is nowhere to be seen. Yet the diagram is not *merely* an heuristic device here. It's necessary, for example, in locating the bisector of the summit angle as intersecting the opposite side (a tidbit of topological information); indeed, existence assertions and relations of incidence are *often* justified by a diagram (cf. Greaves 2002, 20). In suitable circumstances one might reject this proof as too informal, but no one would reject it simply because the diagram fails to satisfy the conditions of the theorem. To see why this is so, it's necessary to look more closely at the activity of letting a diagram be **Φ**.

Letting a diagram be **Φ** is the conscious activity of a subject, not the causal result of a subject's seeing a diagram and observing the similarity of appearance between the diagram and something which appears exactly **Φ**. Since authors concerned with diagrams often discuss them in the context of the geometrical demonstration in Plato's ***Meno***, it will be instructive to use this example to illustrate the subject's conscious activity.

Socrates begins the demonstration by drawing a figure (fig. 5).

[[INSERT IMAGE SherryFig5.eps]]

fig. 5

Tell me, boy, do you know that a figure like this is a square?

The boy replied, I do.

And do you know that a square figure has these four lines [the sides] equal?

Certainly.

And these lines which I have drawn through the middle of the square are also equal?

Yes. (82b-c)

Socrates' questions are not *merely* aimed at finding out how his drawing looks to the slave, for there's nothing in the slave's responses to indicate that he is reporting what he observes. *Otherwise* he might have stopped to scrutinize elements of the figure, perhaps spanning them with thumb and forefinger. If Socrates is typical, the diagram would not have been found square, and were the slave's concern *only* for appearance, he would have responded differently.

Socrates' first four questions are concerned with finding out whether the slave has mastered the skill of *treating* a diagram *as*[4] a square, in other words, the skill of *letting* a diagram be a square. Not every empirical figure can be treated as a square; the figure needs the same number of empirically distinguishable elements as there are elements of a paradigm empirical square (like a tile), 4 sides and 4 angles, and the figure should be roughly convex. The point of Socrates' *first* question is to determine whether the slave understands the characteristics necessary for an empirical object to be treated as a square. But the capacity to treat the diagram as a square also requires being a competent participant in inferential practices involving terms like "square," "line," and "equal." So, once the slave acknowledges that the diagram is the kind of object that can be treated as a square, Socrates proceeds by checking whether, regardless of appearance, the slave deduces correct conclusions given the stipulation that the figure is a square. The answer to the question "are the four

[4] I use the expression "treating as" rather than "seeing as" because the former emphasizes the activity of the subject and so avoids any

sides equal?" reflects logical mastery, *not* a report of a perceptual characteristic of the diagram, and especially not a perceptual characteristic revealed by a sophisticated observation such as measurement. The conversation preceding the demonstration establishes, therefore, that the slave is capable of applying mathematical concepts to an empirical object.

In general, treating a diagram as an instance of the mathematical concept Φ consists in being prepared to deduce various conclusions about the elements of the diagram. It is this capacity of the reasoner which explains how Saccheri is able to use the same diagram to prove theorems about distinct quadrilaterals. The difference between treating Saccheri's diagram as one quadrilateral rather than another consists entirely in the different mathematical concepts (right, obtuse, acute) by means of which he treats the summit angles, in other words, in the different inferences he is prepared to draw about the summit angles. For example, Saccheri treats one of the summit angles as obtuse when he infers that it is not equal to a base angle, which is right (Appendix, Prop. III.ii). In general, diagrams are not used by geometers to represent metric or proportional data; rather, their function is to represent topological properties like incidence relations (Greaves 2002, 29-31). Thus, the summit angles of Saccheri's figure have *no* exact magnitude until *after* one treats them as instances of exact concepts.

One treats the elements of a diagram as instances of a mathematical concept because doing so enables one to draw *further* inferences about elements of the diagram in accordance with rules already at one's disposal. For instance, after treating AC and BC in

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suggestion that the diagram is ambiguous in the way Jastrow's duck-rabbit is.

fig. 4 as equal and the two angles formed by the ray CD as equal, side-angle-side warrants the inference that ΔACD is congruent with ΔBCD. This conclusion, and further rules, warrant further conclusions. In this respect the ***Meno*** demonstration and Saccheri's proofs proceed similarly.

To avoid abstract objects I appeal to inference rules in this context, rather than mathematical truths. Doing so makes more sense of the ***Meno*** demonstration than Plato's own appeal to recollection. In order to make his case that learning is recollection, Socrates purports to be conversing with an individual who has not been taught geometry, and Meno swears this is the case (85e). But Meno *also* certifies that the slave is a competent speaker of Greek (82b), and this, I take it, includes having mastered deductive techniques that enable members of the community to exchange money for goods, construct buildings, etc. Without these skills, the demonstration could never get off the ground, as the slave hasn't been taught pure mathematics, and observation of the diagram is insufficient for the answers he gives. Indeed, Plato lends plausibility to the idea of recollection only by *ignoring* the practical background which makes possible the discussion between Socrates and the slave. Conversely, recollection loses its plausibility to the extent that the slave boy's discovery can be tied to his practical training.

Calling attention to the practical foundation of the ***Meno*** demonstration emphasizes that the diagram's role is precisely that played by any physical object - a farmer's field, a building, a roomful of guests and morsels of food, etc. We learn more about a diagram just as we learn more about physical objects, viz., by applying mathematical concepts and drawing inferences in accordance with mathematical rules. Recognizing that a diagram is just one among other physical objects is the crucial step in understanding the role of diagrams in mathematical argument.

Simple empirical features of objects - that they can be counted, that they are proper parts of further objects, that they can be placed on top of one another, that one element lies between two others, etc. - play the same role in inferences involving diagrams that they play in inference from one empirical proposition to another. Imagine an individual who, from a point A, observes two distant points B and C, and suppose he sets out on a straight path contained within the angle BAC. We are prepared to infer that he *must* eventually cross the line between B and C if he keeps walking in a straight line. Of course, he may exit the triangle without crossing BC, but then we would say that he *must* have deviated from the straight path. Such inferences are no doubt connected to our experience with triangular objects. But the inferential practice does not arise simply from induction; for no one with the relevant experience imagines he could encounter a triangle that lacks this property. Were a neophyte to question the validity of the intersection of the straight path and the side opposite A, we could do little beyond drawing a diagram for him, or, perhaps, having him try it himself. Here we have a primitive case of diagrammatic reasoning, a case in which an inference is licensed by a diagram rather than some previously articulated rule, like side-angle-side. How shall we understand such cases?

Wittgenstein can help here. Often he describes mathematical propositions as empirical propositions that have hardened into rules (Wittgenstein 1983, VI, 22-3; cf. I, 67; IV, 29; VII, 74). An empirical proposition has hardened into a rule once it has come to be used by a community for inferring one empirical proposition from another.[5] That is, it is no longer brought into question and it serves to *exclude* certain possibilities from our experience. In order for an empirical proposition to harden into a rule, there must be overwhelming agreement among people, not only in their observations, but also in their reactions to them. This agreement reflects, presumably, biological and anthropological facts about human beings. An empirical proposition that has hardened into a rule very likely has practical value, underwriting inferences in commerce, architecture, etc. Many of the inference practices which Socrates and the slave bring to their conversation are undoubtedly traceable to empirical propositions that have hardened into a rule.

Propositions corresponding to empirical regularities are good candidates to become rules. For example, people observe that combining groups of five sheep and seven sheep yields a group of twelve sheep, combining five horses with seven yields a group of twelve horses, etc. A proposition corresponding to this regularity, say, "combining a group of five with a group of seven yields a group of twelve," is, by virtue of its use, hardened into a rule for inferring one empirical proposition from another. That is, certain possibilities are excluded from our experience, and someone who combines groups of five and seven to obtain a group larger or smaller than twelve must have miscounted. Inferences that depend on a diagram are similar. It is surely an empirical regularity that a ray drawn from a vertex through the interior of a triangle intersects the opposite side. But, unlike the arithmetic case, the regularity is so obvious that no one bothers to

[5] Cf. Wittgenstein 1961, 6.211: "Indeed in real life a mathematical proposition is never what we want. Rather, we make use of mathematical propositions *only* in inferences from propositions that do not belong to mathematics to others that likewise do not belong to mathematics." The passage continues: "In philosophy the question, 'what do we actually use this word or this proposition for?' repeatedly leads to valuable

formulate a corresponding proposition. The absence of an explicit proposition is not so important here. A diagram or perhaps a vague amalgam of past experience can perform the same function, viz., serving as a paradigm in accordance with which one proposition is inferred from another. As long as no one calls the result into question and as long as neophytes whose rays fail to intersect the opposite side are corrected, a diagram is as good as a rule.

Eventually, of course, just such a rule - Pasch's axiom - *was* articulated in the context of formal axiomatics, which consciously rejects appeal to experience. Formal axiomatics emerged only in the late 19th century, after geometry embraced objects with no empirical counterparts (e.g., Poncelet's points at infinity; cf. Greaves 2002, 42 ff.). But when geometry is construed as a set of inference rules applicable to objects of experience, the conclusion that the ray emerging from one angle of a triangle intersects the side opposite is as necessary as any, even though its warrant may be a diagram.

Inference rules arising from empirical regularities are important enough to examine in the context of a different example, the principle of superposition. The principle of superposition licenses one to infer the equality of lines, angles and areas from determining that certain of their elements can be made to coincide (cf. Heath 1956, vol. 1, 225). It's plausible that Meno's slave agrees that the diagonal bisects the square (84e) on the basis of superposition. The principle of superposition also arises from empirical regularities, the results of placing one brick on top of another, folding paper or cloth back on itself, etc. Ritual altars were, in fact, constructed using the rule that the parts of a brick broken along the diagonal could be

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insights." The latter remark suggests that this view of mathematics is carried into his later philosophy.

superimposed on one another (cf. Seidenberg 1977, 337). Here again is an empirical regularity elevated to a rule by its use as a *criterion* for having divided a brick along its diagonal; parts that can't be superimposed indicate a poorly constructed brick or diagonal.

Commentators complain about superposition. Greaves, for instance, regards it is a weak point in the system of diagrammatic representation (Greaves 2002, 30-2), though Heath claims that no ancient geometer doubted its legitimacy (ibid.). There *is* something untoward about superposition, but not what Greaves imagines. Superposition lies outside the usual system of diagrammatic representation because it is not part of our experience of drawing figures on paper or on the ground. Instead, the grasp of superposition arises from our experience with placing objects on top of one another. Apparently, traditional geometry comprises more than "different types of graphical construction combined … with sentential deductive arguments" (cf. Greaves 2002, 20). Ancient geometers used superposition sparingly, not because they held it to be less reliable than the inference techniques which arose from experience with drawing figures, but presumably because it was more complicated and so less pleasing aesthetically.[6]

I have emphasized that diagrammatic reasoning recapitulates habits of applied mathematical reasoning. On this view, diagrams are not representations of abstract objects, but simply physical objects, which are sometimes used to represent other physical objects. Contrary to what one may imagine, this account is consistent with the aims of both the ***Meno*** demonstration and Saccheri's proofs, viz., obtaining mathematical knowledge. For obtaining mathematical knowledge, according to the present analysis, consists not in gaining information about abstract objects, but in constructing inference rules. Mathematical proof is the construction of such rules, and oftentimes the construction involves a diagram. The new rule can be used like the prior techniques, viz., as a warrant for inferring something about an empirical object on the basis of treating its elements as instances of certain exact concepts. But the new rule is part of pure mathematics *as long as* the object to which the inference techniques are applied – the diagram - is not subject to the usual caveats of applied mathematics. Since there is *no question* whether Socrates' drawing is better treated as a square than a rhombus and *no question* whether his diagonals are really straight and really bisect their respective squares, the result in ***Meno*** *is* pure mathematics. The same is true of the isosceles triangle theorem and each of the Saccheri quadrilateral theorems.

Thus far, Euclidean proof has been the paradigm for diagrammatic reasoning. Employment of this paradigm invites the objection that we are skipping an important – and arguably more interesting – case of diagrammatic reasoning: so-called "proof without words" or "picture proofs" (cf. Nelsen 1993). Indeed, the discussions of diagrammatic reasoning which I have classified as Platonistic, are largely concerned with diagrams like (fig. 6)

[[INSERT IMAGE SherryFig6.eps]]

fig. 6

rather than ones that occur in Euclid.

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[6] For a valuable discussion of the significance of superposition in Euclid, see Seidenberg 1975, 267-9.

The pair of diagrams in fig. 5 is an ancient way of bringing people to see that the square constructed on the hypotenuse of a right triangle has the same area as the squares constructed on its sides. How different is this picture-proof from a traditional proof? Picture-proofs don't show their results on their sleeve, as it were; it's necessary to study them for a while, before they reveal their treasure. Brown puts this point nicely:

> Remember, pictures may make a result 'obvious', but obvious and immediate need not be the same thing. Often you will have to work at it for a while. (1999, 43)

The proof above takes some work. Among other things, it's crucial to recognize that the two figures are congruent squares, each containing congruent right triangles that are also congruent with the triangles in the sister figure. These relations will be suggested by a well drawn diagram, while a poorly drawn diagram might prevent someone from recognizing the intended congruences. But none of this means that the diagram does its work by resembling an abstract object. For the congruence relations could also be suggested by labeling or hash-marks – even if the diagram were poorly drawn. The virtue of a well drawn diagram is the virtue of the hash-marks – to indicate with which stipulations the reasoning should commence. Only by treating various lengths and angles as *exactly* equal is the reasoner able to reach the desired result. As we've observed repeatedly, mere perception is impotent for establishing exact metrical relations. No matter how carefully the diagrams are drawn the result is not simply read off the diagram. Some individual may fail to see the result implicit in our picture-proof because he's not convinced that, say, the sister figures really are congruent. But that failure will be due *not* to a perceived inequality, but rather to an inability to grasp a conceptual relation.

Picture-proofs, as well as proofs in Euclid, presuppose that the reasoner has mastered the skill of treating an empirical object by means of exact, mathematical concepts. They differ from Euclidean proofs only in their informality, i.e., in the way in which they leave in the background the conceptual maneuvers necessary for the proof to succeed.

At this point, a general characterization of the role of a diagram in geometrical argumentation is appropriate. There is, in fact, a dual role. In the first place, a diagram serves as the ground for synthesizing a mathematical rule from existing concepts and inference rules. Without the diagram there is no object to which the rules may be applied, and so no construction is possible. The formalist will not brook this claim, of course, but even he has to recognize within it a kernel of truth. Formal proofs proceed by first assuming some object a possesses a property Φ and then drawing further inferences about a by instantiating the axioms to a and applying them. Without the individual a, reasoning cannot get off the ground. But although the diagram is eliminable in favor of an orthographic symbol, in practice geometrical reasoning is carried on in light of a diagram. Even Hilbert's ***Foundations of Geometry*** employs diagrams to spare its readers the pain of purely formal proofs (e.g., Hilbert 1921, 17). Perhaps, though, it is better to qualify our claim: In light of our cognitive capacities the diagram provides an efficient ground for synthesizing new inference rules from existing concepts and rules.

The second role of a geometrical diagram is to warrant further inferences in virtue of its simple empirical characteristics. One example has already been presented: It is an empirical characteristic of a triangular diagram that a line which enters the triangle through a vertex will intersect the side opposite that vertex, and in a mathematical context this characteristic is understood to be a necessary characteristic of any triangle. A different sort of example occurs in the proof of Saccheri's third proposition (appendix). At one point in the proof Saccheri seeks to show the absurdity of a proposition he's just derived, ∠MAL=∠HLA (fig. 7).

[[INSERT IMAGE SherryFig7.eps]]

fig. 7

The absurdity arises from two circumstances. On the one hand the stipulation that ∠HDA is acute[7] entails that ∠HLA, which is internal and opposite to ∠HDA, is also less than ∠HDA and so itself acute. On the other hand, the stipulation that ∠MAD is right, in conjunction with the *diagrammatically evident fact* that ∠MAL contains ∠MAD, entails that ∠HLA (which equals ∠MAL) is obtuse. This would be evident even if the diagram were so poorly drawn that ∠MAL and ∠MAD both appeared to be acute. Here too, although the diagrammatically evident fact is an empirical feature of the diagram, we treat that feature as a rule. Thus, some but not all perceptual characteristics of a diagram function in a geometric proof; they are, I reiterate, topological rather than metric characteristics. They correspond to the simplest and most general aspects of our experience with physical objects, aspects which are so pervasive that they have been elevated to the position of rules.

§3. Beyond Geometrical Diagrams. My account of diagrammatic reasoning is modeled upon the use of diagrams in traditional geometry. But perhaps not all diagrammatic reasoning fits this mold. Brown suggests this when he distinguishes a diagram which functions by being "isomorphic, or at least homomorphic, to the situation it represents" from a diagram which exhibits no such relation (fig. 8).

[[INSERT IMAGE SherryFig8.eps]]

fig. 8

Brown illustrates the case with fig. 8, which proves that $\sum_{1}^{n} n = \frac{n^2}{2} + \frac{n}{2}$ (1999,35).[8] Brown is worried about the diagram in fig. 8 because it applies only to a particular number but establishes the result for a whole class of numbers. Such diagrams are isomorphic to number structures with that particular cardinality, but they are not isomorphic to all the numbers. And even though the diagram is homomorphic to the whole number structure, Brown argues that there is no reason to suppose that the homomorphism sheds light on the whole structure (ibid.). This worry reflects too great a concern for the perceptual characteristics of the diagram and insufficient concern for the conceptual resources – the exact metric concepts - which the reasoner brings to bear upon the diagram.

The diagram in fig. 8 contains seven squares along its left side and its base. This is learned by counting, though one probably needs to count just the squares in the side or the base, as the symmetry of the figure leads one to conclude that the number is the same in both cases. As well, the figure contains 21 white squares and 7 half-black and half-white squares. Do any of these specific numbers function in

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[7] I challenge any reader who still suspects that the summit angles of a Saccheri quadrilateral are analogous to the duck-rabbit to try to see ∠HDA as acute.

[8] This result is often expressed in the form $\frac{n(n+1)}{2}$. Brown's alternative form is more natural for this particular proof.

the proof, once we make explicit the steps in the reasoning? As long as one recognizes that the left side and base of the figure contain the same number of squares, the specific number of squares along the side or in the whole is irrelevant.

The result contains two terms, $\frac{n^2}{2}$ and $\frac{n}{2}$. The former corresponds to the white portion of the figure and the latter to the black portion. The magnitude of the white portion depends on the geometrical-arithmetic rule that the area of a square is the product of the length of its sides and the purely geometric rule that the diagonal of a square cuts it in half. Those rules are applicable only if one understands that fig. 8 is part of a square with n smaller squares along its side. The magnitude of the black portion depends on the geometric rule that a square's diagonal divides it in two, as well as the diagrammatically evident feature that there are the same number of half black squares as there are squares along the left side (or base), which, again, does *not* depend on counting the particular number of squares. Perhaps the general quantitative rule that a sum of a half of each of a number of things is half the sum of each of those things is also involved. Whether or not one connects these results to the simple arithmetic rule that 1+2+…+7=28 is immaterial to the success of this marvelous little proof.

What is necessary for the proof, though, is that one treats the diagram in accordance with exact metrical concepts. The smaller squares must be treated as exactly square and congruent with one another if the entire figure is to be understood as a portion of a large square whose sides are exactly the same length as the sides of the diagram. Likewise the lines which divide each of the smaller black and white squares must be treated as straight lines that divide those squares exactly in two rather than as lines which merely divide them into two portions. Hence, this number-theoretic result depends upon just the kind of geometric reasoning from which our account of diagrams was developed. The same is true of other diagrams which Brown presents as proofs of number-theoretic results (e.g., 1999, 36-7; 43-4).

The issue that concerns Brown most, the generality of a diagram employed in a number-theoretic proof, turns out to be no greater problem than that posed by using a particular diagram in proving a result about all triangles or all squares. In order to generalize properly from a diagram, no features of the diagram (including features that result from letting a particular element in the diagram be Φ) can be employed unless they are shared by any figure which the result is alleged to cover. Fig. 8 applies unproblematically to any whole number because the proof appeals to no features of the particular numbers that occur in the diagram (such as the fact that 7 is prime and 28 is perfect) in deriving the result. Brown's worry that the elements of the diagram are not isomorphic to number structures with different cardinalities is not an issue; I suspect he has this worry because he assigns to important a role to the perceptual characteristics of the diagram. But the success of the proof does not depend on our perception of the number of elements in one or another part of the diagram. It depends only on our being able to see that certain parts of the diagram have the same number of elements, a fact which we can perceive without knowing the specific numbers, i.e., without counting.

The account of diagrammatic reasoning gleaned from traditional geometric practice extends, apparently, *beyond* that practice to proofs in number theory. Moreover, the fact that the example discussed employs both geometric and arithmetic rules fits well with the earlier contention that diagrammatic reasoning is just a special case of applied mathematics. A craftsman who applies mathematical rules to objects in the world pays little attention to the circumstance that some of these rules are geometric and others arithmetic. It is not surprising that the same should be true in the context of a diagrammatic proof, if such proofs are just a special case of applied mathematics.

Conclusion. I won't claim to have given a complete picture of diagrammatic reasoning. But I believe I have shown that an account of diagrammatic reasoning that avoids abstract objects competes well with existing Platonistic accounts, especially because the latter fail to capture the range of practices comprised in diagrammatic reasoning.[9]

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[9] Mike Malone, Ian Dove and an anonymous referee provided useful criticism of earlier versions of this essay.

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Appendix

***Euclides vindicatus*, Prop. I-III**

Proposition I. If ∠A and ∠B are right, then if AD=BC, ∠C=∠D.

[[INSERT IMAGE SherryFigA1.eps]]

Proof: Draw DB and CA. ΔDAB≅ΔCBA because ∠A=∠B, AD=BC, and they share a common side, AB. Hence, DB=CA. ΔDCB≅ΔCDA because AD=BC, DB=CA and DC is common. It follows that ∠C=∠D. (Saccheri uses Prop. I in contrapositive form, If ∠A and ∠B are right, then if ∠C≠∠D, then AD≠BC.)

Proposition II. If ∠A and ∠B are right and AD=BC, then if E and F are midpoints of AB and DC, respectively, then FE is perpendicular to AB and DC.

[[INSERT IMAGE SherryFigA2.eps]]

Proof: Draw DE and CE. ΔDEA≅ΔCEB (SAS). Therefore, DE=CE. Thus, ΔDFE≅ΔCFE (SSS). Hence ∠DFE=∠CFE, and by Euclid's 10th

definition they are right. Next, draw FA and FB. By Prop. I, ∠D=∠C, so ΔDAF≅ΔCBF (SAS). Thus AF=BF. Hence, ΔFAE≅ΔFBE (SSS). Thus, ∠AEF=∠FEB and so both are right (def. 10).

Proposition III. If ABCD is a Saccheri quadrilateral, then (i) if its summit angles are right, then CD=AB; (ii) if its summit angles are obtuse, then CD<AB; and (iii) if its summit angles are acute, then CD>AB.

[[INSERT IMAGE SherryFigA3.eps]]

Proof (i): Suppose ∠C and ∠D are right and that DC>AB. Take CK=AB and join AK. Then ABCK is a Saccheri quadrilateral and ∠BAK=∠CKA (Prop. I). But by construction, ∠BAK< right ∠BAD. Moreover, ∠CKA is external to ΔDAK, so ∠CKA> right ∠ADC (Elements I.16). Hence ∠BAK≠∠CKA, contrary to hypothesis. Therefore, ~(DC>AB). A similar proof shows ~(AB>DC). Thus DC=AB.

[[INSERT IMAGE SherryFigA4.eps]]

Proof (ii): Suppose ∠C and ∠D are obtuse. Bisect AB and CD at M and H respectively, and join MH. ∠DHM and ∠AMH are both right (Prop. II). Since ∠DAM is right and ∠ADH is obtuse, AM≠DH (Prop. I, contrapositive). But also ~(DH>AM). For suppose DH>AM. Then as before cut off KH=AM. By Prop. I, ∠AKH=∠MAK. But this is absurd because ∠MAK is less than a right angle while ∠AKH is greater than an obtuse angle, ∠ADK (I.16). Hence DH<AM and so CD<AB.

[[INSERT IMAGE SherryFigA5.eps]]

Proof (iii): Suppose ∠C and ∠D are acute. Construct MH as in (ii). As before, since ∠HDA≠∠DAM, DH≠AM (Prop. I). But neither is DH<AM. For suppose DH<AM. Produce HD to HL so that HL=AM. By Prop. I, ∠MAL=∠HLA. But this is absurd because by construction ∠MAL> right ∠MAD and ∠HLA is internal and opposite, and so less than acute ∠HDA. Hence, DH>AM and so CD>AB.

Fig. 1

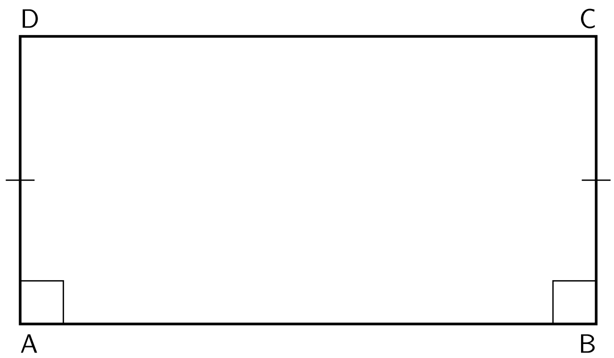


Fig. 2

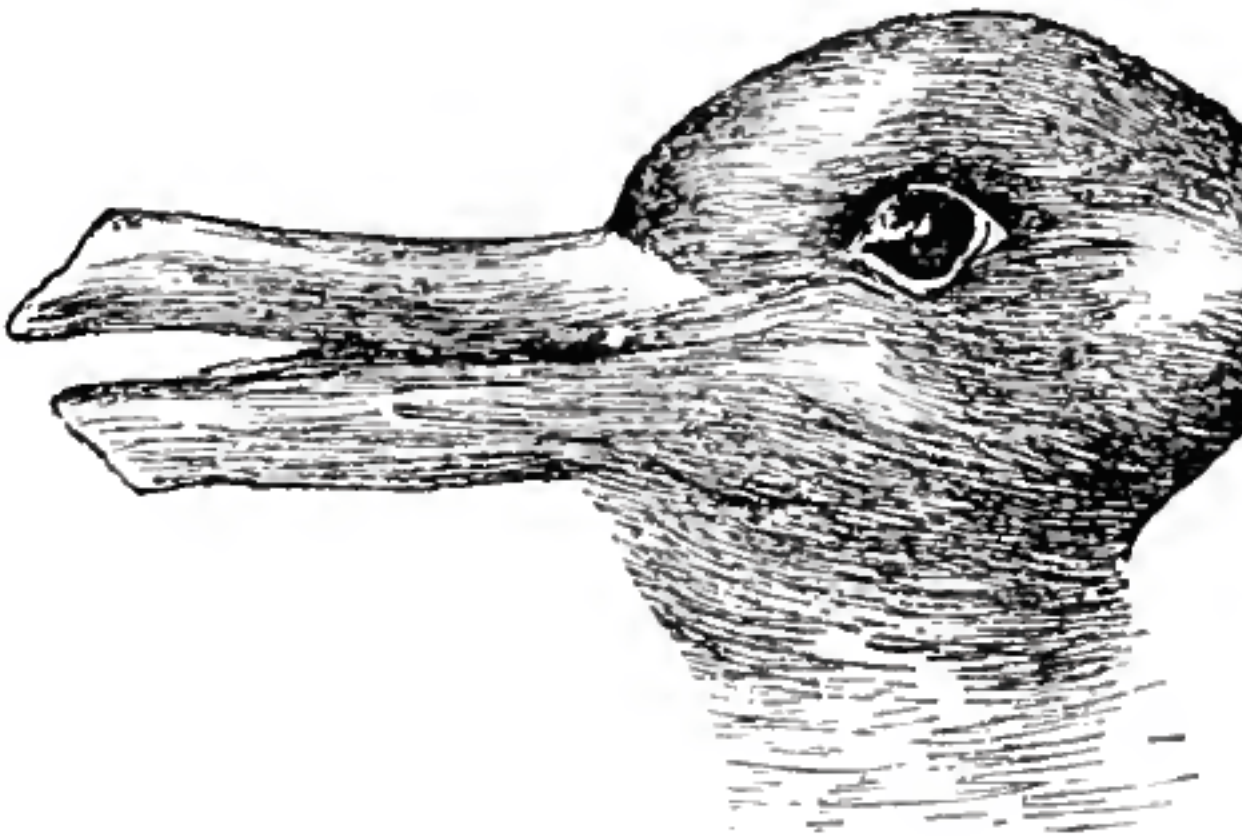


Fig. 3

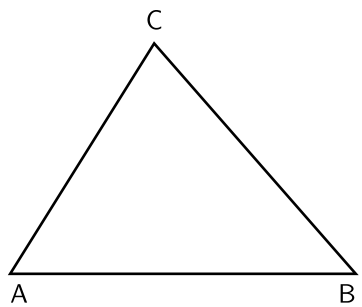
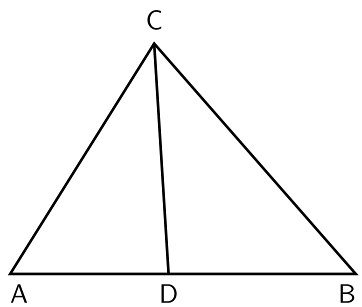


Fig. 4



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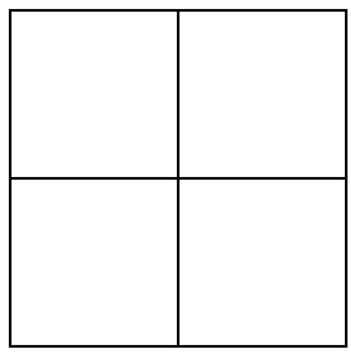

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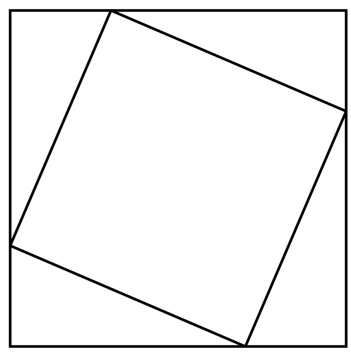


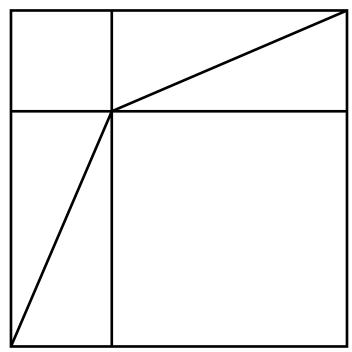


Fig. 7

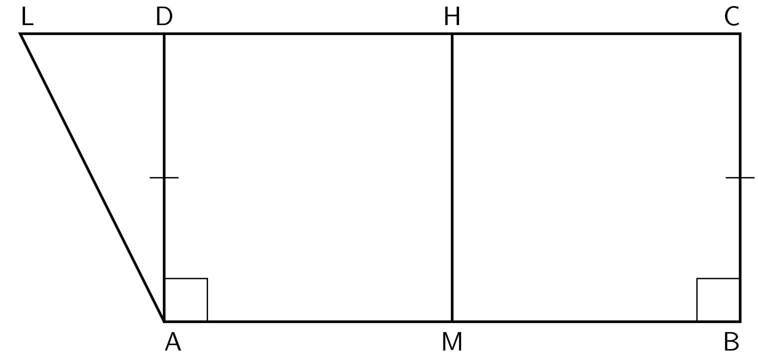
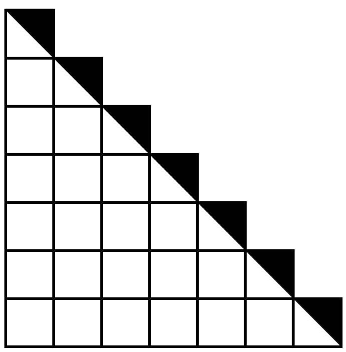
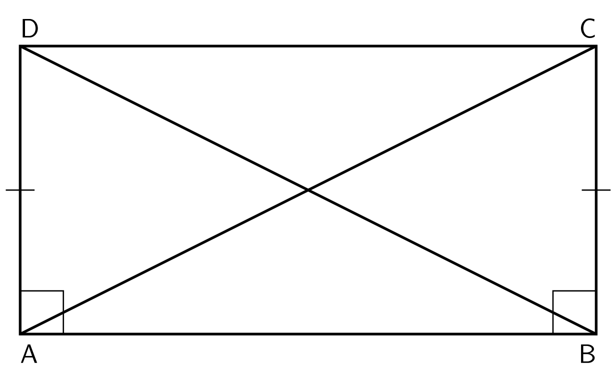


Fig. 8



Appx Fig. 1



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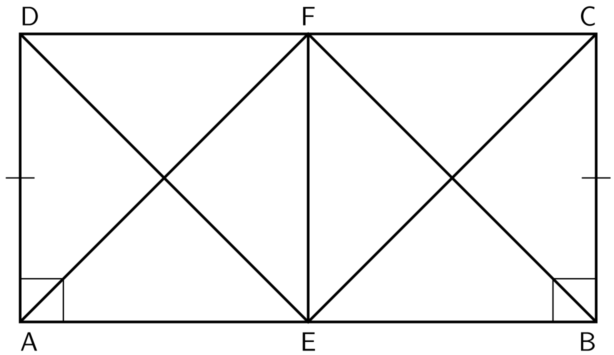

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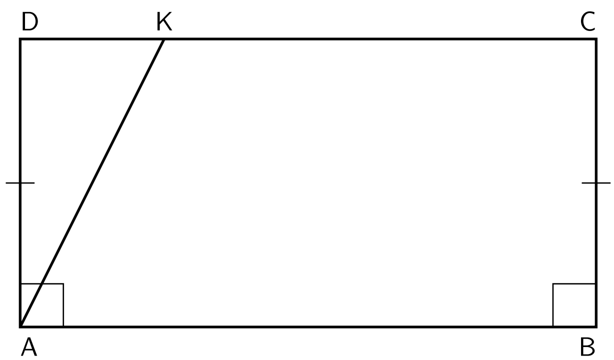

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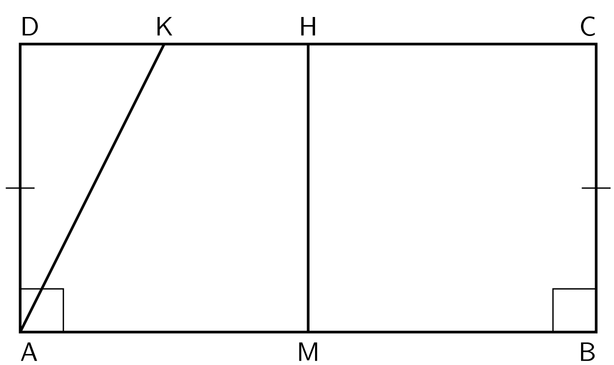

Appx Fig. 5