Mathematical Fallacies and Informal Logic

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Maxwell's *Fallacies in Mathematics*

MISTAKE 'a momentary aberration, a slip in writing, or the misreading of earlier work'

HOWLER 'an error which leads *innocently* to a *correct* result'

FALLACY 'leads by *guile* to a *wrong* but plausible conclusion'

E. A. Maxwell, 1959, *Fallacies in Mathematics*, p. 9.

A Preliminary Typology

	True Result	False Result
Sound Method	Correct	Fallacy
Unsound Method	Howler	Mistake

Aristotle's Fallacies

That some reasonings are genuine, while others seem to be so but are not, is evident. This happens with arguments as also elsewhere, through a certain likeness between the genuine and the sham.

Aristotle, *De Sophisticis Elenchis*, 164a.

Bacon's Juggling Feats

For although in the more gross sort of fallacies it happeneth, as Seneca maketh the comparison well, as in juggling feats, which, though we know not how they are done, yet we know well it is not as it seemeth to be; yet the more subtle sort of them doth not only put a man beside his answer, but doth many times abuse his judgment.

Bacon, 1605, *Advancement of Learning*, p. 131.

Threefold Distinction:

Bacon-Gross Something seems wrong (and is).

Bacon-Subtle Everything seems OK (but is not).

Bacon-Surprise Something seems wrong (but is not).

The Guts of Reality

Physicists like to think they're dealing with reality. Some of them are quite arrogant about it and talk as if they were the only ones with a finger in the belly of the real. They think that mathematicians are just playing games, making up our own rules and playing our own games. But with all their physical theories the possibility still exists that space and time are just Kant's categories of apperception, or that physical objects are nothing but ideas in the mind of God. Who can say for sure? Their physical theories can't rule these possibilities out. But in math things are exactly the way they seem. There's no room, no *logical* room, for deception. I don't have to consider the possibility that maybe seven isn't really a prime, that my mind conditions seven to appear a prime. One doesn't—can't—make the distinction between mathematical appearance and reality, as one can—must—make the distinction between physical appearance and reality. The mathematician can penetrate the essence of his objects in a way the physicist never could, no matter how powerful his theory. We're the ones with our fists deep in the guts of reality.

Rebecca Goldstein, 1983, *The Mind Body Problem*, p. 95

Argument[ation] Schemes

There seems to be general agreement among argumentation theorists that argumentation schemes are principles or rules underlying arguments that legitimate the step from premises to standpoints. They characterize the way that the acceptability of the premise that is explicit in the argumentation is transferred to the standpoint.

Bart Garssen, 1999, 'The Nature of Symptomatic Argumentation', p. 225.

From Argument Schemes to Fallacies

Two ways in which an argument scheme may be fallacious:

1. If it is invariably bad (for example, quantifier shift, question begging);
2. If it is used inappropriately.

Hence "seems good" may be analysed as "is an instance of an argument scheme".

Applicability to Mathematics

1. Many mathematical fallacies of this type;
2. Are there any of this type?

Example: Argument from Verbal Classification

Argument Scheme for Argument from Verbal Classification

Individual Premise a has property F.

Classification Premise For all x, if x has property F, then x can be classified as having property G.

Conclusion a has property G.

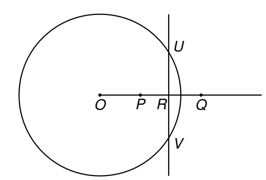
CRITICAL QUESTIONS:

1. What evidence is there that a definitely has property F, as opposed to evidence indicating room for doubt on whether it should be so classified?
2. Is the verbal classification in the classification premise based merely on a stipulative or biased definition that is subject to doubt?

Douglas Walton, 2006, *Fundamentals of Critical Argumentation*, p. 129.

The Fallacy of the Empty Circle 1

To prove that every point inside a circle lies on its circumference.
GIVEN: A circle of centre O and radius r, and an arbitrary point P inside it.



REQUIRED: To prove that P lies on the circumference.
CONSTRUCTION: Let Q be the point on OP produced beyond P such that $OP.OQ = r^2$ and let the perpendicular bisector of PQ cut the circle at U, V. Denote by R the middle point of PQ.

The Fallacy of the Empty Circle 2

PROOF:

$$
\begin{aligned}
OP &= OR - RP \\
OQ &= OR + RQ \\
&= OR + RP \quad [RQ = RP, \text{construction}] \\
\therefore OP.OQ &= (OR - RP)(OR + RP) \\
&= OR^2 - RP^2 \\
&= (OU^2 - RU^2) - (PU^2 - RU^2) \quad [\text{Pythagoras}] \\
&= OU^2 - PU^2 \\
&= OP.OQ - PU^2 \quad (OP.OQ = r^2 = OU^2) \\
\therefore PU &= 0
\end{aligned}
$$

$\therefore$ *P* is at *U*, on the circumference

E. A. Maxwell, 1959, *Fallacies in Mathematics*, pp. 18 f.

Example: Appeal to Expert Opinion

Argument Scheme for Appeal to Expert Opinion

Major Premise Source *E* is an expert in subject domain *S* containing proposition *A*.

Minor Premise *E* asserts that proposition *A* (in domain *S*) is true (false).

Conclusion *A* may plausibly be taken to be true (false).

CRITICAL QUESTIONS:

1. Expertise Question: How credible is *E* as an expert source?
2. Field Question: Is *E* an expert in the field that *A* is in?
3. Opinion Question: What did *E* assert that implies *A*?
4. Trustworthiness Question: Is *E* personally reliable as a source?
5. Consistency Question: Is *A* consistent with what other experts assert?

Douglas Walton, 1997, *Appeal to Expert Opinion*, pp. 210, 223.

An Expert Opinion

All the evidence is that there is nothing very systematic about the sequence of digits of π. Indeed, they seem to behave much as they would if you just chose a sequence of random digits between 0 to 9. This hunch sounds vague, but it can be made precise as follows: there are various tests that statisticians perform on sequences to see whether they are likely to have been generated randomly, and it looks very much as though the sequences of digits of π would pass these tests. Certainly the first few million do. One obvious test is to see whether any short sequence of digits, such as 137, occurs with about the right frequency in the long term. In the case of the string 137 one would expect it to crop up about 1/1000th of the time in the decimal expansion of π.
Experience strongly suggests that short sequences in the decimal expansion of the irrational numbers that crop up in nature, such as π, e or $\sqrt{2}$, do occur with the correct frequencies. And if that is so, then we would expect a million sevens in the decimal expansion of π about $10^{-1000000}$ of the time — and it is of course, no surprise, that we will not actually be able to check that directly. And yet, the argument that it does eventually occur, while not a proof, is pretty convincing.

W. T. Gowers, 2006, 'Does mathematics need a philosophy?' in R. Hersh, ed., *18 Unconventional Essays on the Nature of Mathematics*, p. 194.

An Experiment

Here is an open conjecture:

Conjecture. Somewhere in the decimal expansion of π there are one million sevens in a row.

Here is a heuristic argument about the claim (taken from a talk by Prof. Timothy Gowers, University of Cambridge):

[Argument Stated Here]

After having read this argument please say to what extent you are persuaded by it:

not persuaded 1 2 3 4 5 totally persuaded

Matthew Inglis & Juan Pablo Mejia-Ramos, 2006, 'Is it ever appropriate to judge an argument by its author?', *Proceedings of the British Society for Research into Learning Mathematics* **26(2)**, p. 44.

Example: Argument from Popular Opinion

Argument Scheme for Argument from Popular Opinion

General Acceptance Premise *A* is generally accepted as true.

Presumption Premise If *A* is generally accepted as true, that gives a reason in favor of *A*.

Conclusion There is a reason in favor of *A*.

CRITICAL QUESTIONS:

1. What evidence, such as a poll or an appeal to common knowledge, supports the claim that *A* is generally accepted as true?
2. Even if *A* is generally accepted as true, are there any good reasons for doubting it is true?

Douglas Walton, 2006, *Fundamentals of Critical Argumentation*, pp. 91 f.

Example: Argument from Popular Practice

Argument Scheme for Argument from Popular Practice

Premise *A* is a popular practice among those who are familiar with what is acceptable or not with regard to *A*.

Premise If *A* is a popular practice among those familiar with what is acceptable or not with regard to *A*, that gives a reason to think that *A* is acceptable.

Conclusion Therefore, *A* is acceptable in this case.

CRITICAL QUESTIONS:

1. What actions or other indications show that a large majority accepts *A*?
2. Even if a large majority accepts *A* as true, what grounds might there be for thinking they are justified in accepting *A*?

Douglas Walton, 2006, *Fundamentals of Critical Argumentation*, pp. 93 f.

A Richer Typology of Mathematical Error

METHOD *seems*	METHOD *is*	RESULT *seems*	RESULT *is*	
G	G	T	T	Proof
G	G	T	F	∅
G	G	F	T	Surprise
G	G	F	F	∅
G	B	T	T	Howler
G	B	T	F	Fallacy
G	B	F	T	Howler
G	B	F	F	Fallacy
B	G	T	T	Surprise
B	G	T	F	∅
B	G	F	T	Surprise
B	G	F	F	∅
B	B	T	T	Howler
B	B	T	F	(Tempting) Mistake
B	B	F	T	Howler
B	B	F	F	Mistake

A Richer Typology of Mathematical Error

Mathematical Error Where Result is False

	Result *seems* True	Result *seems* False
Method *seems* Sound	Subtle Fallacy	Gross Fallacy
Method *seems* Unsound	(Tempting) Mistake	Mistake

Summary

- Mathematical reasoning can exhibit fallacies—of a variety of types.
- Mathematical fallacies may be characterized in terms of argument schemes.
- Sensitive treatment of fallacies brings to light a richer typology of mathematical error.