The Personality of Mathematical Proofs

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Plan

The plan:

1. Mathematicians like adjectives;
2. ‘Beautiful’ proofs;
3. ‘Explanatory’ proofs;
4. Human personalities;
5. Mathematical personalities?
6. Implications for: (i) beauty; (ii) explanatoriness; (iii) mathematical practice research methods;
7. Next steps.
Some (almost) randomly chosen quotes from MathOverflow:

“[Spectral sequences] have a reputation for being abstruse and difficult. It has been suggested that the name ‘spectral’ was given because, like spectres, spectral sequences are terrifying, evil, and dangerous.”

(Ravi Vakil).
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Mathematicians Like Adjectives

One nice example (from topology) is Tychonoff's Theorem (that a product of compact spaces is compact). No matter how many times I see it, I find the classic proof based on the (Alexandre Subbase Lemma) difficult and opaque. On the other hand if one first develops the theory of nets (aka Moore-Smith Convergence), not only is that a powerful tool for all sorts of other purposes, but its development is a natural and intuitive generalization of sequences, and the place where Zorn's Lemma enters (the proof that any net has a universal subnet) is much clearer than in the proof of the subbase lemma. And of course once one has universal nets, the proof of Tychonoff is the obvious generalization of the trivial proof that a finite product of sequentially compact spaces is sequentially compact.

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Are Proofs Beautiful?

Like some of the speakers in the *Mathematical Ethnographies* project, Poincare also characterised mathematical beauty as simplicity:

“Now, what are the mathematic entities to which we attribute this character of beauty and elegance, and which are capable of developing in us a sort of esthetic emotion? They are those whose elements are harmoniously disposed so that the mind without effort can embrace their totality while realizing the details.”

Henri Poincare
Are Proofs Beautiful?

Michael Atiyah agreed:
“elegance is more or less synonymous with simplicity”
(cited in Wells, 1990)
Beauty

Notice the way these mathematicians characterised mathematical beauty:

- simple
- elegant
- minimal
- concise
- unexpected
- immediate
- short
- wow!
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Lots of Adjectives!

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Tao (2007) claimed “the concept of mathematical quality is a high-dimensional one”

how high?
Key Question

What do mathematicians mean when they use adjectives such as “deep”, “elegant”, “explanatory” etc?
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Our thesis is that this approach is incomplete.
Three main accounts of mathematical beauty:

1. Beauty is simplicity (e.g. Poincare, Atiyah etc)
2. Beauty is enlightenment (Rota)
3. Beauty is related to usefulness (McAllister)
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If you can understand a proof with minimal effort, then it’s beautiful

Obvious Criticism: lots of trivial proofs are easy to understand with minimal effort, but aren’t beautiful
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Accounts of mathematical beauty:

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3. Beauty is related to usefulness (McAllister)

Rota claimed that beauty is just a word mathematicians give to things that they find enlightening: “Mathematicians may say that a theorem is beautiful when they really mean to say that the theorem is enlightening.”
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Montano’s Criticism: if ugliness is the opposite of beauty, then we’d expect that all non-enlightening pieces of mathematics would be ugly, but that’s not the case (e.g. Least Squares Regression is neither ugly nor enlightening).
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Aesthetic induction: if a property (or deductive step) has been shown to be useful very many times in the past, then scientists (or mathematicians) come to regard it as beautiful. So a proof with many useful deductive steps will be beautiful (cf. diagonal argument: Cantor, Turing, Gödel etc.)
Mathematical Beauty

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Criticism: Mathematicians who are ignorant of the history of mathematics should be incapable of appreciating mathematical beauty outside of their own research area (they don’t know whether a particular deductive step has been useful before).

3. Beauty is related to usefulness (McAllister)

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Explanatoriness

• When is a mathematical proof said to be explanatory?

• Two major theories of mathematical explanation.
  1. Mark Steiner
  2. Philip Kitcher
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Proofs are explanatory if they turn on a “characterising property”, a “property unique to a given entity or structure within a family or domain of such entities or structures”.

Explanatoriness

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Proofs are explanatory if they turn on a "characterising property", a "property unique to a given entity or structure within a family or domain of such entities or structures".

Criticism: characterising properties are very hard to identify (e.g. Resnik & Kushner, 1987; Hafner & Mancosu, 2005). What’s the characterising property in Pythagoras’s Theorem?
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Explanatory proofs unify, i.e. they derive a lot of material from a little. This happens when the number of ‘argument patterns’ is minimised.
Explanatoriness

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  Criticism: “Nuclear flyswatter” proofs, where one disproportionately powerful technique is repeatedly applied has to be called explanatory by Kitcher. Seems implausible (Hafner & Mancosu, 2008).
Explanatoriness

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How to Approach this Question?

- What do mathematicians mean when they use adjectives such as “deep”, “elegant”, “explanatory” etc?
How to Approach this Question?

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• The philosophical approach to answering these questions involves proposing characterisations of a given property (e.g. beauty), and then critiquing those characterisations by reference to extreme examples.
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• Might there be another way?
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- Might there be another way?

- Our suggestion: convert the philosophical question into a social psychology question.
Human Personalities

• Very many adjectives are used to describe human characteristics.
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- For example, you might hear someone say “David is loud, talkative, excitable, outgoing, shameless...”
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How many basic ways of characterising a person are there?

This sounds like it’s an impossibly complicated question to answer.

Actually it’s not. The answer’s five.
Big Five Personality Traits

- Probably the greatest achievement of 20th century psychology was the discovery that there are only five broad dimensions on which a person’s personality varies.
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Big Five Personality Traits

• Probably the greatest achievement of 20th century psychology was the discovery that there are only five broad dimensions on which a person’s personality varies.

• This was discovered by asking people to think of a person, then rate how well a long list of adjectives described them, then looking at the correlations between these ratings (performing a PCA).

• Those characteristics which almost always go together are in some sense the same characteristic.
The Big Five

Openness to Experience:
inventive/curious versus consistent/cautious
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Conscientiousness:
efficient/organised versus easy-going/careless
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compassionate/friendly versus cold/unkind

Neuroticism:
sensitive/nervous versus secure/confident
The Big Five

APPENDIX
The 40-Item Mini-Marker Set

How Accurately Can You Describe Yourself?

Please use this list of common human traits to describe yourself as accurately as possible. Describe yourself as you see yourself at the present time, not as you wish to be in the future. Describe yourself as you are generally or typically, as compared with other persons you know of the same sex and of roughly your same age.

Before each trait, please write a number indicating how accurately that trait describes you, using the following rating scale:

<table>
<thead>
<tr>
<th>Inaccurate</th>
<th>Moderate</th>
<th>Accurate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Extremely</td>
<td>Very</td>
<td>Slightly</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Bashful</td>
<td>Energetic</td>
<td>Moody</td>
</tr>
<tr>
<td>Bold</td>
<td>Envious</td>
<td>Organized</td>
</tr>
<tr>
<td>Careless</td>
<td>Extraverted</td>
<td>Philosophical</td>
</tr>
<tr>
<td>Cold</td>
<td>Fretful</td>
<td>Practical</td>
</tr>
<tr>
<td>Complex</td>
<td>Harsh</td>
<td>Quiet</td>
</tr>
<tr>
<td>Cooperative</td>
<td>Imaginative</td>
<td>Relaxed</td>
</tr>
<tr>
<td>Creative</td>
<td>Inefficient</td>
<td>Rude</td>
</tr>
<tr>
<td>Deep</td>
<td>Intellectual</td>
<td>Shy</td>
</tr>
<tr>
<td>Disorganized</td>
<td>Jealous</td>
<td>Sloppy</td>
</tr>
<tr>
<td>Efficient</td>
<td>Kind</td>
<td>Sympathetic</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Systematic</td>
</tr>
</tbody>
</table>
The Big Five

Big 5 profiles predict:

- u/g mathematics achievement and behaviour (Alcock, Attridge, Kenny & Inglis, submitted);
- perceived well-being (Hayes & Joseph, 2003);
- job performance (Barrick & Mount, 1991);
- tendency to share money (Paunonen & Ashton, 2001);
- drug consumption (Paunonen & Ashton, 2001);
- use of swear words (Mehl et al., 2006);
- frequency of group conversations in day-to-day life (Mehl et al., 2006);
- etc etc etc.

1.47 million Google scholar hits for “Big 5 personality”
Our Idea

• Prior to the Big 5 the field was stuck with the Myers-Briggs scale.
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• Hypothesis: The state of the literature on proof characteristics is analogous to the state of the personality literature prior to the Big Five.
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Our Idea

- Prior to the Big 5 the field was stuck with the Myers-Briggs scale.
- Hypothesis: The state of the literature on proof characteristics is analogous to the state of the personality literature prior to the Big Five.
- Terrence Tao has claimed that mathematical quality has many dimensions. But how many?
- Let’s adopt a similar strategy to the social psychologists and see what happens.
Empirical Work

- We created a list of eighty adjectives which have often been used to describe mathematical proofs.
- Each had more than 250 hits on google for “XXX proof” and “mathematics”.
- For example: 21,000 webpages contain the phrase “conceptual proof” and “mathematics”; 1290 contain the phrase “obscure proof” and “mathematics”.
<table>
<thead>
<tr>
<th>Definitive</th>
<th>Precise</th>
<th>Innovative</th>
<th>Subtle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clear</td>
<td>Useful</td>
<td>Cute</td>
<td>Clumsy</td>
</tr>
<tr>
<td>Simple</td>
<td>Beautiful</td>
<td>Worthless</td>
<td>Flimsy</td>
</tr>
<tr>
<td>Rigorous</td>
<td>Minimal</td>
<td>Explanatory</td>
<td>Informative</td>
</tr>
<tr>
<td>Strong</td>
<td>Unambiguous</td>
<td>Plausible</td>
<td>Crudely</td>
</tr>
<tr>
<td>Striking</td>
<td>Accurate</td>
<td>Illustrative</td>
<td>Appealing</td>
</tr>
<tr>
<td>General</td>
<td>Tedious</td>
<td>Creative</td>
<td>Careless</td>
</tr>
<tr>
<td>Non-Trivial</td>
<td>Ambitious</td>
<td>Insightful</td>
<td>Enlightening</td>
</tr>
<tr>
<td>Elegant</td>
<td>Elaborate</td>
<td>Deep</td>
<td>Inspired</td>
</tr>
<tr>
<td>Obvious</td>
<td>Weak</td>
<td>Profound</td>
<td>Bold</td>
</tr>
<tr>
<td>Practical</td>
<td>Ingenious</td>
<td>Deep</td>
<td>Polished</td>
</tr>
<tr>
<td>Trivial</td>
<td>Clever</td>
<td>Awful</td>
<td>Charming</td>
</tr>
<tr>
<td>Intuitive</td>
<td>Applicable</td>
<td>Ugly</td>
<td>Unpleasant</td>
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<tr>
<td>Natural</td>
<td>Robust</td>
<td>Speculative</td>
<td>Unsublime</td>
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<tr>
<td>Conceptual</td>
<td>Sharp</td>
<td>Confusing</td>
<td>Awkward</td>
</tr>
<tr>
<td>Abstract</td>
<td>Intricate</td>
<td>Dense</td>
<td>Exploratory</td>
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<tr>
<td>Efficient</td>
<td>Loose</td>
<td>Expository</td>
<td>Inefficient</td>
</tr>
<tr>
<td>Careful</td>
<td>Pleasing</td>
<td>Lucid</td>
<td>Sublime</td>
</tr>
<tr>
<td>Effective</td>
<td>Sketchy</td>
<td>Obscure</td>
<td>Shallow</td>
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<tr>
<td>Incomplete</td>
<td>Dull</td>
<td>Delicate</td>
<td>Fruitful</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Meticulous</td>
<td>Disgusting</td>
</tr>
</tbody>
</table>
Empirical Work

Participants asked to pick a proof they’d recently read or refereed and to state how accurately each of our 80 adjectives described it.

Participants were 255 research mathematicians based in US universities (follow-up studies with British, Irish and Australian participants give similar results).

Asked to participate by email via their department secretaries.
Empirical Work

<table>
<thead>
<tr>
<th>Word</th>
<th>Very Inaccurate</th>
<th>Moderately Inaccurate</th>
<th>Neither Inaccurate nor Accurate</th>
<th>Moderately Accurate</th>
<th>Very Accurate</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. shallow</td>
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<td>b. informative</td>
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<td>c. efficient</td>
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<tr>
<td>d. careless</td>
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<td>e. dense</td>
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<td>f. inspired</td>
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<td>g. profound</td>
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<td>h. rigorous</td>
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<tr>
<td>i. intricate</td>
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<td>j. meticulous</td>
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<td>k. explanatory</td>
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<td>l. precise</td>
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<td>m. striking</td>
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<td>n. applicable</td>
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<td>o. flimsy</td>
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<td>p. crude</td>
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<tr>
<td>q. unpleasant</td>
<td></td>
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<td></td>
<td></td>
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<tr>
<td>r. careful</td>
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<tr>
<td>s. ingenious</td>
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<tr>
<td>t. difficult</td>
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<td></td>
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<tr>
<td>u. practical</td>
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</table>
Analysis

- Performed a Principal Components Analysis (Euclidean Squared Metric, Ward’s Method, Varimax Rotation).
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• Both Horn’s Parallel Analysis and Cattell’s Scree Test suggested extracting five components.
Analysis

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• Both Horn’s Parallel Analysis and Cattell’s Scree Test suggested extracting five components.

• However, Component 2 seemed to be different to the others: crude, careless, shallow, flimsy, inefficient, weak, sketchy, loose, worthless, incomplete, ugly, clumsy, awful, awkward.
Analysis

Component 2 seemed to be those words with low ratings.
Analysis

We correlated each word’s loadings on Component 2 with its mean rating on the five-point scale.

$r = -.94$
• Suggests that Component 2 was just a measure of “non-use”.

• In other words, mathematicians tend not think that mathematical proofs are “crude”, “careless”, “shallow” or “flimsy”.

• So, I’m ignoring Component 2 for the remainder of the discussion (you get essentially the same factor structure if you delete all the C2 words and re-run the PCA).
The Four Components

Adjectives listed if absolute component loadings > .5
The Four Components

striking
ingenious
inspired
profound
creative
deep
sublime
innovative
beautiful
elegant
charming
clever
bold
appealing
pleasing
enlightening
ambitious
delicate
insightful
strong

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The Four Components

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- ingenious
- inspired
- profound
- creative
- deep
- sublime
- innovative
- beautiful
- elegant
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- clever
- bold
- appealing
- pleasing
- enlightening
- ambitious
- delicate
- insightful
- strong

Adjectives listed if absolute component loadings > .5
The Four Components

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ingenious
inspired
profound
creative
deep
sublime
innovative
beautiful
elegant
charming
clever
bold
appealing
pleasing
enlightening
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dense
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confusing
tedious
not simple

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Intricacy

Precision

Adjectives listed if absolute component loadings > .5
• Mathematical proofs have ‘personalities’ which vary on four dimensions: aesthetics, intricacy, precision and utility.

• What does this mean for mathematical beauty? For explanatoriness?

• Implications for our understanding of mathematical practice?
The Poincare Conjecture

• Poincare and others have conjectured that the defining characteristic of beautiful proofs is their simplicity.
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- “But I think there is beauty in the sense, particularly if it is simple proof where you can immediately follow it the first time round, that is obviously a very beautiful proof.” (Maria Zaturska, Mathematical Ethnographies project)
The Poincare Conjecture

- Poincare and others have conjectured that the defining characteristic of beautiful proofs is their simplicity.
- “But I think there is beauty in the sense, particularly if it is simple proof where you can immediately follow it the first time round, that is obviously a very beautiful proof.” (Maria Zaturska, Mathematical Ethnographies project)
- This is a traditional view (see Atiyah).
In fact this is not the case.

“Beautiful” loaded onto the aesthetics component, whereas “simple” (negatively) loaded onto the intricacy component.

The correlation between “beautiful” and “simple” was essentially zero $r_s = .078$, $p = .217$.

In other words, it is not the case that proofs described as beautiful are also described as simple.
Beauty as Simplicity

Why might mathematicians claim that beautiful proofs are simple?
Beauty as Simplicity

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SOME CORRELATES OF RATED MEMORABILITY OF SENTENCES

By MICHAEL M. GRUNEBERG, JOSEPH MONKS, ROBERT N. SYKES AND DAVID J. OBORNE

Department of Psychology, University College of Swansea

This study examines the effect of instruction to generate memorable as opposed to grammatical sentences from letter strings of varying lengths. Results indicate that the effect of instruction is limited, but where it proves effective higher rated Memorability is associated with higher rated Sentence Simplicity and Positive Affect. Correlations between the characteristics examined in the study and Memorability fail to reveal any relationship between rated Bizarreness and Memorability, but correlations are found between rated Memorability and rated Imageability, Meaningfulness and Sentence Simplicity, characteristics found by independent research to be related to Memorability. Other interesting correlations between characteristics examined in the study are also discussed.
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Beauty as Enlightenment

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- As Montano predicted, ‘ugly’ is a poor reverse correlate of ‘enlightening’, $r_S = -.271$, but a good(ish) reverse correlate of ‘beautiful’, $r_S = -.514$. 
Beauty as Usefulness

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- ‘Beautiful’ - ‘useful’ correlation, $r_s = .319$.
- Seems unlikely that usefulness is driving beauty judgements.
- Might be unfair comparison: McAllister is talking about steps in the proof being useful rather than the proof itself. Maybe proofs with lots of useful steps aren’t necessarily useful?
Explanatoriness

- Both standard accounts of explanatoriness in mathematics offer single characterisations/definitions of what counts as an explanatory proof (Steiner/Kitcher).
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Our data suggest that this is misguided, and that explanatoriness is a multi-dimensional concept.

Specifically, if it’s anything, it seems to be located at the conjunction of three dimensions.
Explanatoriness

“Explanatory” loaded negatively onto intricacy, positively onto precision and positively (albeit weakly) onto utility.

<table>
<thead>
<tr>
<th>aesthetics</th>
<th>intricacy</th>
<th>precision</th>
<th>utility</th>
</tr>
</thead>
<tbody>
<tr>
<td>explanatory</td>
<td>0.096</td>
<td>-0.314</td>
<td>0.472</td>
</tr>
</tbody>
</table>

Suggests that proofs which are (i) not intricate, (ii) precise and (iii) maybe useful, have a chance of being called “explanatory” by mathematicians.

Doesn’t seem to support either the Steiner ‘defining characteristic’ account, or the Kitcher ‘unification’ account.
Explanatoriness

- Carnap (1950) distinguished between explanation and explication.
Explanatoriness

- Carnap (1950) distinguished between *explanation* and *explication*.
- Explanation is a relationship between a fact or natural phenomenon and its ostensible explanation.
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• A good explication should meet four criteria: “The explicatum is to be similar to the explicandum ... [characterised] in an exact form, ... a fruitful concept, ... [and] as simple as possible” (Carnap, 1950).
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So a good explication should be precise, useful and not intricate.

Maybe mathematical explanations are just Carnapian explications? (Aberdein & Inglis, in progress).
Implications for Mathematical Practice Research

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We should not trust introspection (e.g., Evans & Wason, 1976; Johansson, Hall & Chater, 2012; Nisbett & Wilson, 1977; White, 1988; Wilson & Bar-Anan, 2008; etc etc etc).
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Many mathematicians, including Poincare, think/thought that beauty is about simplicity, but it isn’t.
Implications for Mathematical Practice Research

This is consistent picture:
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- Mathematicians think that they ignore sources when evaluating arguments, but they don’t (Inglis & Mejia-Ramos, 2009);
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If you want to understand mathematical practice, you have to study how mathematicians behave in mathematical situations.
Next Steps

- Just finished testing a very short (20 item) scale to measure the personality of mathematical proofs.

- With four items per dimension (plus the one ‘non-use’ distractor dimension) we have internal reliabilities of $\alpha = .864$ (aesthetics), $\alpha = .770$ (intricacy), $\alpha = .778$ (precision) and $\alpha = .838$ (utility).
The Personality of Mathematical Proofs

Please think of a particular proof in a paper or book which you have recently refereed or read. Keeping this specific proof in mind, please use the rating scale below to describe how accurately each word in the table below describes the proof. Describe the proof as it was written, not how it could be written if improved or adapted.

So that you can describe the proof in an honest manner, you will not be asked to identify it or its author, and your responses will be kept in absolute confidence.

Please read each word carefully, and then select the option that corresponds to how well you think it describes the proof.

<table>
<thead>
<tr>
<th>Word</th>
<th>Very Inaccurate</th>
<th>Moderately Inaccurate</th>
<th>Neither Inaccurate nor Accurate</th>
<th>Moderately Accurate</th>
<th>Very Accurate</th>
</tr>
</thead>
<tbody>
<tr>
<td>shallow</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
</tr>
<tr>
<td>flimsy</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
</tr>
<tr>
<td>precise</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
</tr>
<tr>
<td>useful</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
</tr>
<tr>
<td>rigorous</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
<td>○</td>
</tr>
<tr>
<td>careless</td>
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Next Steps

- What is this scale useful for? Example questions:
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2. To what extent are ‘personality’ judgements properties of the proof, to what extent are they properties of the mathematician doing the judging?

3. Are there systematic differences in judgements between different research areas? Is it possible that a proof will be judged to be aesthetic by topologists, but ugly by algebraists?
Conclusion

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Conclusion

- Mathematical proofs have ‘personalities’.
- They can be characterised, roughly speaking, with four dimensions.
- Poincare and others were wrong to suggest that beauty is related to simplicity. Maybe simple proofs are easier to remember, so when asked to think of beautiful proofs they’re the ones that come to mind.
- There’s reason to doubt both Rota’s and McAllister’s characterisations of beauty.