The Dialectical Tier of Mathematical Proof

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Human mathematics consists in fact in talking about formal proofs, and not actually performing them. One argues quite convincingly that certain formal texts exist, and it would in fact not be impossible to write them down. But it is not done: it would be hard work, and useless because the human brain is not good at checking that a formal text is error-free. Human mathematics is a sort of dance around an unwritten formal text, which if written would be unreadable. This may not seem very promising, but human mathematics has in fact been prodigiously successful.

David Ruelle, ‘Conversations on mathematics with a visitor from outer space’.
Johnson’s Two Tier Model of Argument

The illative core comprises ‘a thesis, \( T \), supported by a set of reasons, \( R \)’, whereas the ‘dialectical tier must be a set of ordered pairs, with each pair consisting of an objection and one or more responses to the objection: thus:

\[
\{\langle O_1, \{A_{1a}, \ldots, A_{1n}\} \rangle, \langle O_2, \{A_{2a}, \ldots, A_{2n}\} \rangle, \ldots, \langle O_N, \{A_{Na}, \ldots, A_{Nn}\} \rangle \}\}
\]

Now, in advancing a Johnson-argument, a proponent has to do two things: (i) he must assert \( T \) because \( R \), and (ii) for every objection, \( O_i \), to \( R-T \), he is obligated to respond with one or more answers, \( A_{i1} - A_{ij} \)’

Hans Hansen, ‘An exploration of Johnson’s sense of “argument”’. 
Johnson: Proofs Are Not Arguments

**P1** Proofs require axioms; arguments do not have axioms.

**P2** Proofs must be deductive; arguments need not be.

**P3** Proofs have necessarily true conclusions; almost all arguments have contingent conclusions.

**P4** ‘[A]n argument requires a dialectical tier, whereas no mathematical proof has or needs to have such’.

Ralph Johnson, *Manifest Rationality*. 
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Proofs and Conclusive Arguments

C1 ‘Its premises would have to be unimpeachable or uncriticizable.’

C2 ‘The connection between the premises and the conclusion would have to be unimpeachable—the strongest possible.’

C3 ‘A conclusive argument is one that can successfully (and rationally) resist every attempt at legitimate criticism.’

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## Some Mathematical Dialogue Types

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<tr>
<th>Type of Dialogue</th>
<th>Initial Situation</th>
<th>Main Goal</th>
<th>Goal of Protagonist</th>
<th>Goal of Interlocutor</th>
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<tr>
<td>Inquiry</td>
<td>Open-mindedness</td>
<td>Prove or disprove conjecture</td>
<td>Contribute to outcome</td>
<td>Obtain knowledge</td>
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<tr>
<td>Deliberation</td>
<td>Open-mindedness</td>
<td>Reach a provisional conclusion</td>
<td>Contribute to outcome</td>
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<td>Persuasion</td>
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<td>Resolve difference of opinion</td>
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<td>Persuade protagonist</td>
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<td>Difference of opinion</td>
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<td>Contribute to outcome</td>
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<tr>
<td></td>
<td></td>
<td>provisional conclusion</td>
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<tr>
<td>Debate (Eristic)</td>
<td>Irreconcilable difference of opinion</td>
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<td>Clarify position</td>
<td>Clarify position</td>
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<tr>
<td>Information-Seeking (Pedagogical)</td>
<td>Interlocutor lacks information</td>
<td>Transfer of knowledge</td>
<td>Disseminate knowledge of results and methods</td>
<td>Obtain knowledge</td>
</tr>
</tbody>
</table>
Tiers of Mathematical Reasoning

If we were to push it to its extreme we should be led to a rather paradoxical conclusion; that we can, in the last analysis, do nothing but point; that proofs are what Littlewood and I call gas, rhetorical flourishes designed to affect psychology, pictures on the board in the lecture, devices to stimulate the imagination of pupils. . . . On the other hand it is not disputed that mathematics is full of proofs, of undeniable interest and importance, whose purpose is not in the least to secure conviction. Our interest in these proofs depends on their formal and aesthetic properties. Our object is both to exhibit the pattern and to obtain assent.

G. H. Hardy, ‘Mathematical proof’.
Epstein’s Picture of Mathematical Proof

A Mathematical Proof
Assumptions about how to reason and communicate.

A Mathematical Inference
Premises

necessity

Conclusion

The mathematical inference is valid.

R. L. Epstein, *Logic as the Art of Reasoning Well*. 
The Parallel Structure of Mathematical Proof

**Argumentational Structure:**
Mathematical Proof, $P_n$
Endoxa: Data accepted by mathematical community

\[ \text{argument} \]
Claim: $I_n$ is sound

**Inferential Structure:**
Mathematical Inference, $I_n$
Premisses: Axioms or statements formally derived from axioms

\[ \text{derivation} \]
Conclusion: An additional formally expressed statement