Rationale of the Mathematical Joke

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Overview: Two Conjectures

Joke $\sim\rightarrow$ Scheme
Overview: Two Conjectures

Joke $\leadsto$ Scheme
Pappus on analysis

in analysis we assume that which is sought as if it were (already) done, and we inquire what it is from which this results, and again what is the antecedent cause of the latter, and so on, until by so retracing our steps we come upon something already known, or belonging to the class of first principles, and such a method we call analysis as being solution backwards (\(\alphaν\alphaπ\alpha\lambda\nu\ λ\upsilon\sigma\nu\))

Proof by reduction to the wrong problem ‘To see that infinite-dimensional coloured cycle stripping is decidable, we reduce it to the halting problem.’

Dana Angluin, 1983, ‘How to prove it’, *ACM SIGACT News*, 15(1)
Argument from Gradualism

Premise  Proposition $A$ is true (acceptable to the respondent).

Premise  There is an intervening sequence of propositions, $B_1$, $B_2$, \ldots $B_{n-1}$, $B_n$, $C$ such that the following conditionals are true: If $A$ then $B_1$; if $B_1$ then $B_2$; \ldots; if $B_{n-1}$ then $B_n$; if $B_n$ then $C$.

Premise  The conditional ‘If $A$ then $C$’ is not, by itself, acceptable to the respondent, nor are shorter sequences from $A$ to $C$ (than the one specified in the second premise) acceptable to the respondent.

Conclusion  Therefore, the proposition $C$ is true (acceptable to the respondent).

Proof by importance A large body of useful consequences all follow from the proposition in question.

Argument from Positive Consequences

Premise  If $A$ is brought about, then good consequences will plausibly occur.

Conclusion  Therefore, $A$ should be brought about.

Critical Questions:

1. How strong is the likelihood that the cited consequences will (may, must) occur?
2. What evidence supports the claim that the cited consequences will (may, must) occur, and is it sufficient to support the strength of the claim adequately?
3. Are there other opposite consequences (bad as opposed to good, for example) that should be taken into account?

Euler on consequences

Euler does not reexamine the grounds for his conjecture . . . he examines only its consequences. He regards the verification of any such consequence as an argument in favor of his conjecture . . . In scientific research as in ordinary life, we believe, or ought to believe, a conjecture more or less according as its observable consequences agree with the facts. In short, Euler seems to think the same way as reasonable people, scientists or non-scientists, usually think.

Argument from Evidence to a Hypothesis

Premise  If $A$ (a hypothesis) is true, then $B$ (a proposition reporting an event) will be observed to be true.

Premise  $B$ has been observed to be true [false], in a given instance.

Conclusion  Therefore, $A$ is true [false].

Critical Questions:

1. Is it the case that if $A$ is true, then $B$ is true?
2. Has $B$ been observed to be true (false)?
3. Could there be some reason why $B$ is true, other than its being because of $A$ being true?

Retroduction

Proof by accumulated evidence  Long and diligent search has not revealed a counterexample.

Proof by wishful citation  The author cites the negation, converse, or generalization of a theorem from the literature to support his claim.

Proof by ghost reference  Nothing even remotely resembling the cited theorem appears in the reference given.

Proof by eminent authority ‘I saw Karp in the elevator and he said it was probably NP-complete.’

Proof by personal communication ‘Eight-dimensional coloured cycle stripping is NP-complete [Karp, personal communication].’

Dana Angluin, 1983, ‘How to prove it’, *ACM SIGACT News*, 15(1)
Proof by reference to inaccessible literature  The author cites a simple corollary of a theorem to be found in a privately circulated memoir of the Slovenian Philological Society, 1883.

Proof by mutual reference  In reference A, Theorem 5 is said to follow from Theorem 3 in reference B, which is shown to follow from Corollary 6.2 in reference C, which is an easy consequence of Theorem 5 in reference A.

Proof by forward reference  Reference is usually to a forthcoming paper by the author, which is often not as forthcoming as at first.


Proof by deferral  ‘We’ll prove this later in the course.’

Paul Renteln and Alan Dundes, 2005, ‘Foolproof: A sampling of mathematical folk humor’, Notices of the AMS, 52(1)
Appeal to Expert Opinion

**Major Premise**  
Source $E$ is an expert in subject domain $S$ containing proposition $A$.

**Minor Premise**  
$E$ asserts that proposition $A$ is true (false).

**Conclusion**  
$A$ is true (false).

**Critical Questions:**

1. **Expertise Question:** How credible is $E$ as an expert source?
2. **Field Question:** Is $E$ an expert in the field that $A$ is in?
3. **Opinion Question:** What did $E$ assert that implies $A$?
4. **Trustworthiness Question:** Is $E$ personally reliable as a source?
5. **Consistency Question:** Is $A$ consistent with what other experts assert?
6. **Backup Evidence Question:** Is $E$’s assertion based on evidence?

Proof by vehement assertion  It is useful to have some kind of authority relation to the audience.

Proof by vigorous handwaving  Works well in a classroom or seminar setting.

Proof by intimidation  ‘Trivial’

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Argument from Danger

Premise If you (the respondent) bring about $A$, then $B$ will occur.

Premise $B$ is a danger to you.

Conclusion Therefore (on balance) you should not bring about $A$.

Proof by appeal to intuition  Cloud-shaped drawings frequently help here.


Proof by seduction  ‘Convince yourself that this is true!’

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G. H. Hardy, proving by seduction:
The reader will convince himself of the truth of the following assertion . . . given any rational \( r \), and any positive integer \( n \), we can find another rational number lying on either side of \( r \) and differing from \( r \) by less than \( 1/n \).

Intuition

Hardy on intuitive obviousness

When one says ‘such and such a theorem is almost obvious’ one may mean one or other of two things. One may mean ‘it is difficult to doubt the truth of the theorem,’ ‘the theorem is such as common sense instinctively accepts,’ as it accepts, for example, the truth of the propositions ‘$2 + 2 = 4$’ or ‘the base angles of an isosceles triangle are equal.’ That a theorem is ‘obvious’ in this sense does not prove that it is true, since the most confident of the intuitive judgments of common sense are often found to be mistaken; and even if the theorem is true, the fact that it is also ‘obvious’ is no reason for not also proving it, if a proof can be found. The object of mathematics is to prove that certain premises imply certain conclusions; and the fact that the conclusions may be as ‘obvious’ as the premises never detracts from the necessity, and often not even from the interest of the proof.

Intuition

Hardy on intuitive obviousness

... But sometimes (as for example here [If $\phi(n)$ and $\psi(n)$ tend to limits $a$, $b$, then $\phi(n) + \psi(n)$ tends to the limit $a + b$.]) we mean by ‘this is almost obvious’ something quite different from this. We mean ‘a moment’s reflection should not only convince the reader of the truth of what is stated, but should also suggest to him the general lines of a rigorous proof.’ And often, when a statement is ‘obvious’ in this sense, one may well omit the proof, not because the proof is unnecessary, but because it is a waste of time to state in detail what the reader can easily supply for himself.

Argument from Position to Know

Major Premise  Source a is in position to know about things in a certain subject domain S containing proposition A.

Minor Premise  a asserts that A is true (false).

Conclusion  A is true (false).

Critical Questions:

1. Is a in position to know whether A is true (false)?
2. Is a an honest (trustworthy, reliable) source?
3. Did a assert that A is true (false)?

Meta-Argument

**Proof by metaproof** A method is given to construct the desired proof. The correctness of the method is proved by any of these techniques.

**Proof by funding** How could three different government agencies be wrong?

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Ethotic Argument

Argument Scheme for Ethotic Argument

Premise If $x$ is a person of good [bad] moral character, then what $x$ contends ($A$) should be accepted [rejected] (as more [less] plausible).

Premise $a$ is a person of good [bad] moral character.

Conclusion Therefore, what $a$ contends ($A$) should be accepted [rejected] (as more [less] plausible).

Critical Questions:

1. Is $a$ a person of good [bad] moral character?
2. Is the question of $a$’s character relevant, in the context of dialogue in the given case?
3. How strong a weight of presumption in favor of [against] $A$ is claimed, and is that strength warranted by the case?

Argument from Ignorance

Major Premise  If \( A \) were true, then \( A \) would be known to be true.

Minor Premise  It is not the case that \( A \) is known to be true.

Conclusion  Therefore \( A \) is not true.

Critical Questions:

1. How far along has the search for evidence progressed?

2. Which side has the burden of proof in the dialogue as a whole? In other words, what is the ultimate *probandum* and who is supposed to prove it?

3. How strong does the proof need to be in order for this party to be successful in fulfilling the burden?

Epistemic Closure

Proof by cosmology  The negation of the proposition is unimaginable or meaningless. Popular for proofs of the existence of God.

Proof by cumbersome notation  Best done with access to at least four alphabets and special symbols.

Proof by exhaustion  An issue or two of a journal devoted to your proof is useful.

Proof by obfuscation  A long plotless sequence of true and/or meaningless syntactically related statements.

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Generalization

Proof by example  The author gives only the case $n = 2$ and suggests that it contains most of the ideas of the general proof.

Proof by omission  ‘The reader may easily supply the details.’

‘The other 253 cases are analogous.’

‘…’

Gábor Fejes Tóth on Wu-Yi Hsiang

I think there is hope that Hsiang’s strategy will work: at least the main inequalities seem to hold. As far as details are concerned, my opinion is that many of the key statements have no acceptable proofs. Typically, we are given arguments such as ‘the most critical case is...’ followed by a statement that ‘the same method will imply the general case.’ The problem with arguments of this kind is not only that they require the reader to redo some pages of calculations, but, notoriously, that they occur at places where we expect difficulties and most frequently it is impossible to see how the same method works in the general case.

Argument from Example

Premise  In this particular case, the individual $a$ has property $F$ and also property $G$.

Conclusion  Therefore, generally, if $x$ has property $F$, then it also has property $G$.

Critical Questions:

1. Is the proposition claimed in the premise in fact true?
2. Does the example cited support the generalization it is supposed to be an instance of?
3. Is the example typical of the kinds of cases the generalization covers?
4. How strong is the generalization?
5. Do special circumstances of the example impair its generalizability?

Proof by picture  A more convincing form of proof by example. Combines well with proof by omission.

Dana Angluin, 1983, ‘How to prove it’, *ACM SIGACT News*, 15(1)
Definition

Proof by semantic shift Some of the standard but inconvenient definitions are changed for the statement of the result.

Argument from Definition to Verbal Classification

Individual Premise  $a$ fits definition $D$.

Classification Premise  For all $x$, if $x$ has property $D$, then $x$ can be classified as having property $G$.

Conclusion  $a$ has property $G$.

Critical Questions:

1. What evidence is there that $D$ is an adequate definition, in light of other possible alternative definitions that might exclude $a$’s having $G$?

2. Is the verbal classification in the classification premise based merely on a stipulative or biased definition that is subject to doubt?

Summary of Schemes

Retroduction
- Argument from Gradualism
- Argument from Positive Consequences
- Argument from Evidence to a Hypothesis

Citation
- Appeal to Expert Opinion
- Argument from Danger

Intuition
- Argument from Position to Know

Meta-Argument
- Ethotic Argument

Epistemic Closure
- Argument from Ignorance

Generalization
- Argument from Example

Definition
- Argument from Definition to Verbal Classification
Conclusions

- Mathematical folk humour draws attention to risky inferences.
- Argumentation schemes can analyse these risky inferences.
- Informal mathematical reasoning reflects informal non-mathematical reasoning.
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