Learning from our mistakes—
but especially from our fallacies and howlers

Andrew Aberdein

Humanities and Communication,
Florida Institute of Technology,
150 West University Blvd,
Melbourne, Florida 32901-6975, U.S.A.
aberdein@fit.edu

Perspectives on Mathematical Practices 2007

Observations on *Sick Mathematics?*

And you claim to have discovered this ‘common foundation’ of mental life, which has been overlooked by every psychologist, from observations on *sick people?*

Sigmund Freud, 1926, ‘The Question of Lay-Analysis’.

MISTAKE 'a momentary aberration, a slip in writing, or the misreading of earlier work'

HOWLER 'an error which leads *innocently* to a *correct* result'

FALLACY 'leads by *guile* to a *wrong* but plausible conclusion'

E. A. Maxwell, 1959, *Fallacies in Mathematics*, p. 9.

A Preliminary Typology

	True Result	False Result
Sound Method	Correct	Fallacy
Unsound Method	Howler	Mistake

Maxwell's *Fallacies in Mathematics*

MISTAKE 'a momentary aberration, a slip in writing, or the misreading of earlier work'

HOWLER 'an error which leads *innocently* to a *correct* result'

FALLACY 'leads by *guile* to a *wrong* but plausible conclusion'

E. A. Maxwell, 1959, *Fallacies in Mathematics*, p. 9.

A Preliminary Typology

	True Result	False Result
Sound Method	Correct	Fallacy
Unsound Method	Howler	Mistake

Juggling Feats

That some reasonings are genuine, while others seem to be so but are not, is evident. This happens with arguments as also elsewhere, through a certain likeness between the genuine and the sham.

Aristotle, *De Sophisticis Elenchis*, 164a.

For although in the more gross sort of fallacies it happeneth, as Seneca maketh the comparison well, as in juggling feats, which, though we know not how they are done, yet we know well it is not as it seemeth to be; yet the more subtle sort of them doth not only put a man beside his answer, but doth many times abuse his judgment.

Bacon, 1605, *Advancement of Learning*, p. 131.

Juggling Feats

That some reasonings are genuine, while others seem to be so but are not, is evident. This happens with arguments as also elsewhere, through a certain likeness between the genuine and the sham.

Aristotle, *De Sophisticis Elenchis*, 164a.

For although in the more gross sort of fallacies it happeneth, as Seneca maketh the comparison well, as in juggling feats, which, though we know not how they are done, yet we know well it is not as it seemeth to be; yet the more subtle sort of them doth not only put a man beside his answer, but doth many times abuse his judgment.

Bacon, 1605, *Advancement of Learning*, p. 131.

The Guts of Reality

Physicists like to think they're dealing with reality. Some of them are quite arrogant about it and talk as if they were the only ones with a finger in the belly of the real. They think that mathematicians are just playing games, making up our own rules and playing our own games. But with all their physical theories the possibility still exists that space and time are just Kant's categories of apperception, or that physical objects are nothing but ideas in the mind of God. Who can say for sure? Their physical theories can't rule these possibilities out. But in math things are exactly the way they seem. There's no room, no *logical* rule, for deception. I don't have to consider the possibility that maybe seven isn't really a prime, that my mind conditions seven to appear a prime. One doesn't—can't—make the distinction between mathematical appearance and reality, as one can—must—make the distinction between physical appearance and reality. The mathematician can penetrate the essence of his objects in a way the physicist never could, no matter how powerful his theory. We're the ones with our fists deep in the guts of reality.

Rebecca Goldstein, 1983, *The Mind Body Problem*, p. 95

Argument[ation] Schemes

There seems to be general agreement among argumentation theorists that argumentation schemes are principles or rules underlying arguments that legitimate the step from premises to standpoints. They characterize the way that the acceptability of the premise that is explicit in the argumentation is transferred to the standpoint.

Bart Garssen, 1999, 'The Nature of Symptomatic Argumentation', p. 225.

Example: Argument from Analogy

Argumentation Scheme for Argument from Analogy

Similarity Premise Generally, case C_1 is similar to case C_2.

Base Premise A is true (false) in case C_1.

Conclusion A is true (false) in case C_2.

Critical Questions:

1. Are there differences between C_1 and C_2 that would tend to undermine the force of the similarity cited?
2. Is A true (false) in C_1?
3. Is there some other case C_3 that is also similar to C_1, but in which A is false (true)?

Douglas Walton, 2006, *Fundamentals of Critical Argumentation*, pp. 96 f.

Example: Argument from Analogy

Argumentation Scheme for Argument from Analogy

Similarity Premise: Generally, case C_1 is similar to case C_2.

Base Premise: A is true (false) in case C_1.

Conclusion: A is true (false) in case C_2.

Critical Questions:

1. Are there differences between C_1 and C_2 that would tend to undermine the force of the similarity cited?
2. Is A true (false) in C_1?
3. Is there some other case C_3 that is also similar to C_1, but in which A is false (true)?

Douglas Walton, 2006, *Fundamentals of Critical Argumentation*, pp. 96 f.

From Argument Schemes to Fallacies

Two ways in which an argument scheme may be fallacious:

1. If it is invariably bad (for example, quantifier shift, question begging);
2. If it is used inappropriately.

Inference Packages

Inference packages . . . are psychologically-bundled ways of phenomenologically exploring the effect of several assumptions at once without explicit recognition of what those assumptions are.

Jody Azzouni, 2005, 'Is there still a Sense in which Mathematics can have Foundations?', p. 9.

A Fallacy from Newton

Suppose the product or rectangle AB increased by continual motion: and that the momentaneous increments of the sides A and B are a and b. When the sides A and B were deficient, or lesser by one half of their moments, the rectangle was $\overline{A - \frac{1}{2}a} \times \overline{B - \frac{1}{2}b}$ *i.e.* $AB - \frac{1}{2}aB - \frac{1}{2}bA + \frac{1}{4}ab$. And as soon as the sides A and B are increased by the other two halves of their moments, the rectangle becomes $\overline{A + \frac{1}{2}a} \times \overline{B + \frac{1}{2}b}$ or $AB + \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$. From the latter rectangle subduct the former, and the remaining difference will be $aB + bA$. Therefore the increment of the rectangle generated by the initial increments a and b is $aB + bA$. *Q.E.D.* But it is plain that the direct and true method to obtain the moment or increment of the rectangle AB, is to take the sides as increased by their whole increments, and so multiply them together $A + a$ by $B + b$, the product whereof $AB + aB + bA + ab$ is the augmented rectangle; whence, if we subduct AB the remainder $aB + bA + ab$ will be the true increment of the rectangle, exceeding that which was obtained by the former illegitimate and indirect method by the quantity ab.

George Berkeley, 1734, *The Analyst*, ¶9.

Shopping List Howler

To make out a bill:

$\frac{1}{4}$ lb. butter	@ $2s.\ 10d.$ per lb.
$2\frac{1}{2}$ lb. lard	@ $10d.$ per lb.
3 lb. sugar	@ $3\frac{1}{4}d.$ per lb.
6 boxes matches	@ $7d.$ per dozen.
4 packets soap-flakes	@ $2\frac{1}{2}d.$ per packet.

The solution is

$$8\tfrac{1}{2}d. + 2s.\ 1d. + 9\tfrac{3}{4}d. + 3\tfrac{1}{2}d. + 10d. = 4s.\ 8\tfrac{3}{4}d.$$

One boy, however, avoided the detailed calculations and simply added all the prices on the right:

$$2s.\ 10d. + 10d. + 3\tfrac{1}{4}d. + 7d. + 2\tfrac{1}{2}d. = 4s.\ 8\tfrac{3}{4}d.$$

E. A. Maxwell, 1959, *Fallacies in Mathematics*, p. 9.

Shopping List Howler

To make out a bill:

$\frac{1}{4}$ lb. butter	@ 2s. 10d. per lb.
$2\frac{1}{2}$ lb. lard	@ 10d. per lb.
3 lb. sugar	@ $3\frac{1}{4}d$. per lb.
6 boxes matches	@ 7d. per dozen.
4 packets soap-flakes	@ $2\frac{1}{2}d$. per packet.

The solution is

$$8\tfrac{1}{2}d. + 2s.\,1d. + 9\tfrac{3}{4}d. + 3\tfrac{1}{2}d. + 10d. = 4s.\,8\tfrac{3}{4}d.$$

One boy, however, avoided the detailed calculations and simply added all the prices on the right:

$$2s.\,10d. + 10d. + 3\tfrac{1}{4}d. + 7d. + 2\tfrac{1}{2}d. = 4s.\,8\tfrac{3}{4}d.$$

E. A. Maxwell, 1959, *Fallacies in Mathematics*, p. 9.

Howler, or Bullshit?

To solve the equation

$$
\begin{aligned}
x^2 + (x+4)^2 &= (x+36)^2. \\
x^2 + x^2 + 4^2 &= x^2 + 36^2 \\
\therefore x^2 + x^2 + 16 &= x^2 + 336 \\
\therefore x^2 + x^2 - x^2 &= 336 - 16 \\
\therefore x^4 &= 320 \\
\therefore x &= 80.
\end{aligned}
$$

Correct.

E. A. Maxwell, 1959, *Fallacies in Mathematics*, p. 93.

Howler, or Bullshit?

To solve the equation

$$\begin{aligned} x^2 + (x+4)^2 &= (x+36)^2. \\ x^2 + x^2 + 4^2 &= x^2 + 36^2 \\ \therefore x^2 + x^2 + 16 &= x^2 + 336 \\ \therefore x^2 + x^2 - x^2 &= 336 - 16 \\ \therefore x^4 &= 320 \\ \therefore x &= 80. \end{aligned}$$

Correct.

E. A. Maxwell, 1959, *Fallacies in Mathematics*, p. 93.

Howler, or Bullshit?

To solve the equation

$$\begin{aligned} x^2 + (x+4)^2 &= (x+36)^2. \\ x^2 + x^2 + 4^2 &= x^2 + 36^2 \\ \therefore x^2 + x^2 + 16 &= x^2 + 336 \\ \therefore x^2 + x^2 - x^2 &= 336 - 16 \\ \therefore x^4 &= 320 \\ \therefore x &= 80. \end{aligned}$$

Correct.

E. A. Maxwell, 1959, *Fallacies in Mathematics*, p. 93.

Howler, or Bullshit?

To solve the equation

$$
\begin{aligned}
x^2 + (x+4)^2 &= (x+36)^2. \\
x^2 + x^2 + 4^2 &= x^2 + 36^2 \\
\therefore x^2 + x^2 + 16 &= x^2 + 336 \\
\therefore x^2 + x^2 - x^2 &= 336 - 16 \\
\therefore x^4 &= 320 \\
\therefore x &= 80.
\end{aligned}
$$

Correct.

E. A. Maxwell, 1959, *Fallacies in Mathematics*, p. 93.

Howler, or Bullshit?

To solve the equation

$$\begin{aligned} x^2 + (x+4)^2 &= (x+36)^2. \\ x^2 + x^2 + 4^2 &= x^2 + 36^2 \\ \therefore x^2 + x^2 + 16 &= x^2 + 336 \\ \therefore x^2 + x^2 - x^2 &= 336 - 16 \\ \therefore x^4 &= 320 \\ \therefore x &= 80. \end{aligned}$$

Correct.

E. A. Maxwell, 1959, *Fallacies in Mathematics*, p. 93.

Howler, or Bullshit?

To solve the equation

$$\begin{aligned} x^2 + (x+4)^2 &= (x+36)^2. \\ x^2 + x^2 + 4^2 &= x^2 + 36^2 \\ \therefore x^2 + x^2 + 16 &= x^2 + 336 \\ \therefore x^2 + x^2 - x^2 &= 336 - 16 \\ \therefore x^4 &= 320 \\ \therefore x &= 80. \end{aligned}$$

Correct.

E. A. Maxwell, 1959, *Fallacies in Mathematics*, p. 93.

The fact about himself that the bullshitter hides. . . is that the truth-values of his statements are of no central interest to him; what we are not to understand is that his intention is neither to report the truth nor to conceal it. This does not mean that his speech is anarchically impulsive, but that the motive guiding and controlling it is unconcerned with how the things about which he speaks truly are.

Harry Frankfurt, 1988, *The Importance of What We Care About*, p. 130.

Epistemic Luck

Veritic Epistemic Luck

It is a matter of luck that the agent's belief is true.

Safety

For all agents, if an agent knows a contingent proposition φ, then, in most nearby possible worlds in which she forms her belief about φ in the same way as she forms her belief in the actual world, that agent only believes that φ when φ is true.

Duncan Pritchard, 2005, *Epistemic Luck*, pp. 146; 156.

A Richer Typology of Mathematical Error

METHOD		RESULT		
seems	*is*	*seems*	*is*	
G	G	T	T	Proof
G	G	T	F	$\emptyset$
G	G	F	T	Surprise
G	G	F	F	$\emptyset$
G	B	T	T	Howler
G	B	T	F	Fallacy
G	B	F	T	Howler
G	B	F	F	Fallacy
B	G	T	T	Surprise
B	G	T	F	$\emptyset$
B	G	F	T	Surprise
B	G	F	F	$\emptyset$
B	B	T	T	Howler (Bullshit)
B	B	T	F	(Tempting) Mistake
B	B	F	T	Howler (Bullshit)
B	B	F	F	Mistake

Summary

- Mathematical fallacies may be characterized in terms of argument schemes. This generalizes independent accounts of mathematical reasoning, such as inference packages.
- The howler, as an instance of epistemic luck, raises hard questions: is the history of mathematics a sequence of howlers?
- Sensitive treatment of fallacies and howlers brings to light a richer typology of mathematical error.