Can logical formalization analyze mathematical structure?
What is argumentation theory?
Mavericks need argumentation theory
Foundationalists need argumentation theory

Arguing Over Mathematics

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Two Streams in the Philosophy of Mathematics:
Rival Conceptions of Mathematical Proof
University of Hertfordshire, July 2009
I shall defend four theses:

1. Close analysis of the argumentational structure of mathematics transcends the capabilities of formal logic.
2. Argumentation theory is our best hope for a close analysis of the argumentational structure of mathematics.
3. The ‘maverick stream’ of philosophy of mathematics needs a close analysis of the argumentational structure of mathematics.
4. The ‘foundational stream’ of philosophy of mathematics also needs a close analysis of the argumentational structure of mathematics.

Hence, argumentation theory is pivotal to both streams of philosophy of mathematics.
Can logical formalization analyze mathematical structure?

- Resource intensive;
- Formalization can obscure original structure, even to unintelligibility;
- Some proof steps resist formalization;
- Formalization hopeless at analyzing the informal.
On 5 July 2005, Prof. Dr. Jan Bergstra proposed a list of ten challenging research problems for computer science. The first problem of the list reads as follows:

Formalize and verify by computer a proof of Fermat’s Last Theorem, as proved by A. Wiles in 1995. This problem is in the spirit of a long and strong Dutch tradition.

This is indeed a challenge. I expect the problem to be solved in around fifty years, with a very wide margin.

Proof that $1 + 1 = 2$, final step

Russell & Whitehead, 1910, *Principia Mathematica*
Some proofs resist formalization

In the left-hand knot all of the segments are oriented in a clockwise direction about the point $X$, except for that between points $A$ and $B$. That segment can be “thrown over” the rest of the knot, so this knot is equivalent to the right-hand knot, all segments of which are in a clockwise orientation about $X$.

Hopeless at analyzing the informal

It is obvious on reflection that a mathematician must use non-deductive logic in the first stages of his work on a problem. Mathematics cannot consist just of conjectures, refutations and proofs. Anyone can generate conjectures, but which ones are worth investigating? Which ones are relevant to the problem at hand? Which can be confirmed or refuted in some easy cases, so that there will be some indication of their truth in a reasonable time? Which might be capable of proof by a method in the mathematician’s repertoire? Which might follow from someone else’s theorem? Which are unlikely to yield an answer until after the next review of tenure? The mathematician must answer these questions to allocate his time and effort. But not all answers to these questions are equally good. To stay employed as a mathematician, he must answer a proportion of them well. But to say that some answers are better than others is to admit that some are, on the evidence he has, more reasonable than others, that is, are rationally better supported by the evidence. This is to accept a role for non-deductive logic.

James Franklin, 1987, Non-deductive logic in mathematics, *BJPS*, 38
What is argumentation theory?

- Desirable qualities:
  - respects original structure
  - scalability
  - defeasibility

- A motley of different approaches—not necessarily a shortcoming

- Individual steps can be analyzed in various ways:
  - Toulmin layouts
  - Argumentation schemes

- So can networks of steps

- Argument interchange format
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What is a Toulmin Layout?

(a) Basic Layout

(b) Enhanced Layout

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Toulmin layouts
Argumentation schemes
More complex structure
Argument interchange format

Argumentation schemes

**Argumentation Scheme for Defeasible Modus Ponens**

As a rule, if \( P \), then \( Q \).

\( P \).

It is not the case that there is an exception to the rule that if \( P \), then \( Q \).

Therefore, \( Q \). 

Argument from Analogy

Argumentation Scheme for Argument from Analogy

**Similarity Premise**  Generally, case $C_1$ is similar to case $C_2$.

**Base Premise**  $A$ is true (false) in case $C_1$.

**Conclusion**  $A$ is true (false) in case $C_2$.

Critical Questions:

1. Are there differences between $C_1$ and $C_2$ that would tend to undermine the force of the similarity cited?
2. Is $A$ true (false) in $C_1$?
3. Is there some other case $C_3$ that is also similar to $C_1$, but in which $A$ is false (true)?

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More complex structure

Argument interchange format

Appeal to Expert Opinion

Argument Scheme for Appeal to Expert Opinion

**Major Premise** Source $E$ is an expert in subject domain $S$ containing proposition $A$.

**Minor Premise** $E$ asserts that proposition $A$ is true (false).

**Conclusion** $A$ is true (false).

Critical Questions:

1. Expertise Question: How credible is $E$ as an expert source?
2. Field Question: Is $E$ an expert in the field that $A$ is in?
3. Opinion Question: What did $E$ assert that implies $A$?
4. Trustworthiness Question: Is $E$ personally reliable as a source?
5. Consistency Question: Is $A$ consistent with what other experts assert?
6. Backup Evidence Question: Is $E$’s assertion based on evidence?

Ethotic Argument

Argument Scheme for Ethotic Argument

**Premise**  If $x$ is a person of good [bad] moral character, then what $x$ contends ($A$) should be accepted [rejected] (as more [less] plausible).

**Premise**  $a$ is a person of good [bad] moral character.

**Conclusion**  Therefore, what $a$ contends ($A$) should be accepted [rejected] (as more [less] plausible).

Critical Questions:

1. Is $a$ a person of good [bad] moral character?
2. Is the question of $a$’s character relevant, in the context of dialogue in the given case?
3. How strong a weight of presumption in favor of [against] $A$ is claimed, and is that strength warranted by the case?

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A complex argument

An Overview of the Same-Sex Marriage Debate

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Wigmore diagrams

J.H. Wigmore, 1931, *The Principles of Judicial Proof*
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Carneades diagrams


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**Argument Interchange Format**

**Presumptive Inference Scheme Description**

A presumptive inference scheme description is a tuple \(\langle PD, \alpha, cd, \Psi, \Gamma, \text{entails} \rangle\) where:

- \(PD \subseteq \mathcal{N}_F^{\text{Prem}}\) is a set of premise descriptors;
- \(\alpha \in \mathcal{S}^R\) is the scheme;
- \(cd \in \mathcal{N}_F^{\text{Conc}}\) is a conclusion descriptor;
- \(\Psi \subseteq \mathcal{N}_F^{\text{Pres}}\) is a set of presumption descriptors;
- \(\Gamma \subseteq \mathcal{S}^C\) is a set of exceptions; and
- \(\text{entails} \subseteq \mathcal{N}_F^{\text{Prem}} \times \mathcal{N}_F^{\text{Pres}}\)

such that

- \(\alpha \xrightarrow{\text{hasConcDesc}} cd\);
- \(\forall pd \in PD\) we have \(\alpha \xrightarrow{\text{hasPremiseDesc}} pd\);
- \(\forall \psi \in \Psi\) we have \(\alpha \xrightarrow{\text{hasPresumption}} \psi\);
- \(\forall \gamma \in \Gamma\) we have \(\alpha \xrightarrow{\text{hasException}} \gamma\).

A \textit{presumptive argument} based on presumptive inference scheme description

$$\langle PD, \alpha, cd, \Psi, \Gamma, \text{entails} \rangle$$

is a tuple $$\langle P, \tau, c \rangle$$ where:

- $$P \subseteq \mathcal{N}_I$$ is a set of nodes denoting premises;
- $$\tau \in \mathcal{N}_S^{RA}$$ is a rule of inference application node;
- $$c \in \mathcal{N}_I$$ is a node denoting the conclusion

such that:

- $$\tau \xrightarrow{\text{edge}} c; \ \forall p \in P \ \text{we have} \ p \xrightarrow{\text{edge}} \tau$$;
- $$\tau \xrightarrow{\text{fulfillsScheme}} \alpha; \ c \xrightarrow{\text{fulfillsConcDesc}} cd; \ \text{and}$$
- $$\xrightarrow{\text{fulfillsPremiseDesc}} \subseteq P \times PD$$ corresponds to a one-to-one correspondence from $$P$$ to $$PD$$.

Mavericks need argumentation theory

Descriptive  Structural analysis of informal mathematics . . .
Evaluative  . . . leading to resolution of substantive issues
Empirical  Potential for research on how mathematicians reason
Mathematical  May even be of use in mathematics itself
I used to dislike intensely, *but have come to appreciate and even search for* ... the situation where one has *two*, watertight well-designed arguments that lead inexorably to opposite conclusions. ... Remember that research in mathematics involves a foray into the unknown. We may not know which of the two conclusions is correct or even have any feeling or guess. Proof at this point is our only arbiter. And it seems to have let us down. I have known myself to be in this situation for months on end. It induces obsessive and anti-social behaviour. Perhaps we have found an inconsistency in mathematics. But no, *eventually a crack is found in one of the arguments* and it begins to look more and more shaky. Eventually we kick ourselves for being so utterly stupid and life goes on. But it was *no tool of logic that saved us*. The search for a chink in the armour often involved *many tricks including elaborate thought experiments and perhaps computer calculations*. Much *structural understanding is created*, which is why I now so value this process. One’s feeling of having obtained truth at the end is approaching the absolute. Though I should add that I have been forced to reverse the conclusion on occasions

## Proof*: Some mathematical dialogue types

<table>
<thead>
<tr>
<th>Type of Dialogue</th>
<th>Initial Situation</th>
<th>Main Goal</th>
<th>Goal of Protagonist</th>
<th>Goal of Interlocutor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inquiry</td>
<td>Open-mindedness</td>
<td>Proof or disproof</td>
<td>Contribute to outcome</td>
<td>Obtain knowledge</td>
</tr>
<tr>
<td>Deliberation</td>
<td>Open-mindedness</td>
<td>Reach provisional conclusion</td>
<td>Contribute to outcome</td>
<td>Obtain warranted belief</td>
</tr>
<tr>
<td>Persuasion</td>
<td>Difference of opinion</td>
<td>Resolve difference of opinion with rigour</td>
<td>Persuade interlocutor</td>
<td>Persuade protagonist</td>
</tr>
<tr>
<td>Negotiation</td>
<td>Difference of opinion</td>
<td>Resource-bounded conclusion</td>
<td>Contribute to outcome</td>
<td>Maximize value of exchange</td>
</tr>
<tr>
<td>Debate (Eristic)</td>
<td>Irreconcilable differences</td>
<td>Reveal deeper conflict</td>
<td>Clarify position</td>
<td>Clarify position</td>
</tr>
<tr>
<td>Information-Seeking (Pedagogical)</td>
<td>Interlocutor lacks information</td>
<td>Transfer of knowledge</td>
<td>Disseminate results and methods</td>
<td>Obtain knowledge</td>
</tr>
<tr>
<td>Information-Seeking (Oracular)</td>
<td>Protagonist lacks information</td>
<td>Transfer of ‘knowledge’</td>
<td>Obtain information</td>
<td>Presumably inscrutable</td>
</tr>
</tbody>
</table>
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### Toulmin layout as classifier of persuasiveness

<table>
<thead>
<tr>
<th>Type</th>
<th>Foci of attention (in <strong>bold</strong>) in the given argument</th>
<th>Question answered in terms of the focus of attention</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.</td>
<td><strong>Data</strong></td>
<td>How strongly do you trust the <strong>Data</strong>?</td>
</tr>
<tr>
<td>1.</td>
<td><strong>Conclusion</strong></td>
<td>How strongly do you believe the <strong>Conclusion</strong>?</td>
</tr>
<tr>
<td>2.</td>
<td><strong>Data</strong>—<strong>Warrant</strong>—<strong>Conclusion</strong>&lt;br&gt;<strong>Backing</strong></td>
<td>How strongly does the given <strong>Warrant</strong> (and its associated backing) support the conclusion? (i.e. what is the appropriate qualifier for the argument?)</td>
</tr>
<tr>
<td>3.</td>
<td><strong>Data</strong>—<strong>Warrant</strong>—<strong>Qualifier</strong>—<strong>Conclusion</strong>&lt;br&gt;<strong>Backing</strong></td>
<td>How appropriate is the <strong>Qualifier</strong> (and its associated rebuttal) given the rest of the argument?</td>
</tr>
<tr>
<td>4.</td>
<td><strong>Context</strong>&lt;br&gt;<strong>Rebuttal</strong>&lt;br&gt;<strong>Data</strong>—<strong>Warrant</strong>—<strong>Qualifier</strong>—<strong>Conclusion</strong>&lt;br&gt;<strong>Backing</strong></td>
<td>How admissible is the argument in a given <strong>Context</strong>?</td>
</tr>
</tbody>
</table>

M. Inglis and J.P. Mejía-Ramos, 2008, How persuaded are you? A typology of responses, *Research in Mathematics Education* 10
Effect of named authority on persuasiveness

Some twenty years ago, I decided to write a proof of the Schroeder-Bernstein theorem for an introductory mathematics class. The simplest proof I could find was in Kelley’s classic general topology text . . . I realized that Kelley’s proof was wrong. Recently, I wanted to illustrate a lecture on my proof style with a convincing incorrect proof, so I turned to Kelley. I could find nothing wrong with his proof; it seemed obviously correct! Reading and rereading the proof convinced me that either my memory had failed, or else I was very stupid twenty years ago. Still, Kelley’s proof was short and would serve as a nice example, so I started rewriting it as a structured proof. Within minutes, I rediscovered the error.

Leslie Lamport, 1995, How to write a proof, *AMM* 102
Any formalization project requires an account of the pragmatics of the pre-formal language.

Not everything can be justified formally, e.g., axioms of set theory.

Applicability of mathematics.
Abstraction by analogy

We lay down a fundamental principle of generalization by abstraction: The existence of analogies between central features of various theories implies the existence of a general theory which underlies the particular theories and unifies with respect to those central features.

E.H. Moore, 1906, Introduction to a form of general analysis, *American Mathematical Society Colloquium Lectures*

[T]he process of abstraction is actually the process of recognising and then exploring patterns and analogies.

‘Believing the axioms’

Not only would a clear account of the structure and rationality of non-demonstrative set theoretic arguments provide solace for the practitioners and philosophers of the subject, but it might even help with the very real problem of locating new rules of thumb and new axiom candidates for the solution of the continuum problem. I should emphasize that this is not a project of importance only to those with a Platonistic bent. It is central to any philosophical position for which the size of the continuum is a real issue: all realistic philosophies of set theory, even those that eschew mathematical objects; modalist accounts that depend on full second-order models; and even some versions of Field’s nominalism.

Penelope Maddy, 1988, Believing the axioms. II, *Journal of Symbolic Logic* 53
Applicability of mathematics

The situation with differential equations is very different. It is true that there are theorems about differential equations which have been rigorously proved, but these tend not to answer the questions which users of differential equations actually ask. The typical situation is as follows: Consider some interesting real-world system. It is in principle possible to write down the differential equations which describe how the system varies with time, according to the laws of nature as currently understood (and ignoring the effects of the Uncertainty Principle); but these equations will be far too complicated to use as they stand. One therefore needs to make radical simplifications, hoping that the solutions of the simplified model will still describe to a good approximation the behaviour of the original system. Currently, this process seems to be a matter of pure faith; but for some systems there may be scope for a rigorous treatment.

Applicability of mathematics

For example, ... Nosé managed to reduce the thermodynamics of the universe to three first order ordinary differential equations. To this simplified system one applies whatever tools seem appropriate. For the Nosé equations ... these are of three kinds:

1. Genuinely rigorous arguments.
2. Arguments whose conclusion takes the form ‘Such-and-such happens unless there is a rather unlikely-looking coincidence.’
3. Information about particular trajectories, obtained numerically.

These will not be enough to determine the behaviour of the system even qualitatively; but among the possible qualitative descriptions compatible with the information obtained there will usually be a simplest one—and an appeal to Ockham’s Razor should lead us to adopt that description. This process must be regarded as a justification of the conclusion rather than a proof of it; but for differential equations there seems little prospect of ever being able to do better.

Conclusions

- Mavericks need argumentation theory
- Foundationalists need argumentation theory
- A common project?
- A rapprochement?