On the Dirichlet Problem for the Reaction-Diffusion Equations in Non-smooth Domains

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We study the Dirichlet problem for the parabolic equation

$$u_t = \Delta u^m - bu^\beta,\ m > 0,\ \beta > 0,\ b \in I\!R$$

in a bounded, non-cylindrical and non-smooth domain $\Omega \subset I\!R^{N+1}, N \geq 2$. Existence and boundary regularity results are established. We introduce a notion of parabolic modulus of left-lower (or left-upper) semicontinuity at the points of the lateral boundary manifold and show that the upper (or lower) Hölder condition on it plays a crucial role for the boundary continuity of the constructed solution. The Hölder exponent $\frac{1}{2}$ is critical as in the classical theory of the one-dimensional heat equation $u_t = u_{xx}$.

1 Introduction

Consider the equation

$$u_t = \Delta u^m - bu^\beta, \tag{1.1}$$

where $u = u(x,t), x = (x_1, \dots, x_N) \in I\!R^N, N \geq 2, t \in I\!R_+, \Delta = \sum_{i=1}^{N} \partial^2/\partial x_i^2, m > 0, \beta > 0, b \in I\!R$. We study the Dirichlet problem (DP) for equation (1.1) in a bounded domain $\Omega \subset I\!R^{N+1}$. It can be stated as follows: given any continuous function on the parabolic boundary $\mathcal{P}\Omega$ of Ω, to find a continuous extension of this function to the closure of Ω which satisfies (1.1) in $\overline{\Omega} \backslash \mathcal{P}\Omega$.

Equation (1.1) is usually called a reaction-diffusion equation. It is a simple and widely used model for various physical, chemical and biological problems involving diffusion with a source or with absorption, as for instance in modelling filtration in porous media, transport of thermal energy in a plasma, flow of a chemically reacting fluid from a flat surface, evolution of populations etc.

We make now precise the meaning of solution to DP. Let Ω be bounded open subset of $I\!R^{N+1}, N \geq 2$. Let the boundary $\partial\Omega$ of Ω consists of the closure of a domain $B\Omega$ lying on $t = 0$, a domain $D\Omega$ lying on $t = T \in (0, \infty)$ and a (not necessarily connected) manifold $S\Omega$ lying in the strip $0 < t \leq T$. Denote $\Omega(\tau) = \{(x,t) \in \Omega : t = \tau\}$ and assume that $\Omega(t) \neq \emptyset$ for $t \in (0,T)$. The set $\mathcal{P}\Omega = \overline{B\Omega} \cup S\Omega$ is called a parabolic boundary of Ω. Furthermore, the class of domains with described structure will be denoted by $\mathcal{D}_{0,T}$. Let $\Omega \in \mathcal{D}_{0,T}$ is given and ψ is an arbitrary continous nonnegative function defined on $\mathcal{P}\Omega$. DP consists in finding a solution to equation (1.1) in $\Omega \cup D\Omega$ satisfying initial-boundary condition

$$u = \psi \quad \text{on } \mathcal{P}\Omega \tag{1.2}$$

Obviously, in general the equation (1.1) degenerates at points (x,t), where $u = 0$ and we cannot expect the considered problem to have a classical solution. We shall follow the following notion of weak solution:

Definition 1.1 We shall say that the function $u(x,t)$ is a solution of DP (1.1), (1.2), if

(a) u is nonnegative and continuous in $\overline{\Omega}$, satisfying (1.2).

(b) for any t_0, t_1 such that $0 < t_0 < t_1 \leq T$ and for any domain $\Omega_1 \in \mathcal{D}_{t_0,t_1}$ such that $\overline{\Omega}_1 \subseteq \Omega \cup D\Omega$ and $\partial B\Omega_1, \partial D\Omega_1, S\Omega_1$ being sufficiently smooth manifolds, the following integral identity holds

$$\int\limits_{D\Omega_1} uf dx = \int\limits_{B\Omega_1} uf dx + \int\limits_{\Omega_1} (uf_t + u^m \Delta f - bu^\beta f) dx dt - \int\limits_{S\Omega_1} u^m \frac{\partial f}{\partial \nu} dx dt, \tag{1.3}$$

where $f \in C^{2,1}_{x,t}(\overline{\Omega}_1)$ is an arbitrary function that equals to zero on $S\Omega_1$ and ν is the outward-directed normal vector to $\Omega_1(t)$ at $(x,t) \in S\Omega_1$.

Furthermore, we assume that $0 < T < +\infty$ if $b \geq 0$ or $b < 0$ and $0 < \beta \leq 1$, and $T \in (0; T^*)$ if $b < 0$ and $\beta > 1$, where $T^* = M^{1-\beta} \big/ (b(1-\beta))$ and $M > \sup \psi$.

After Wiener published his famous work [15], where he accomplished the long line investigations on the DP for Laplace equation in general domains, the DP for the heat equation was continously under the interest of many mathematicians in this century. The analog of the Wiener's condition, namely the necessary and sufficient condition for a boundary point of an arbitrary bounded open subset of $I\!R^{N+1}$ to be regular for the heat equation has been established in [8] (see also [11]). We refer to [16] for the references on the Wiener's type sufficient conditions for boundary regularity in the case of general parabolic equations. Another sufficient condition, so called exterior tusk condition has been established in [7] for the linear heat equation and later in [12] for the linear uniformly parabolic equations.

However, it should be mentioned that Wiener's criterion does not explicitly clear the natural analytic question, which we impose in this paper (see also [4]) for more general nonlinear equation (1.1). Namely, what about the relation between the solvability of the DP or regulartiy of the boundary points and local modulus of continuity of the boundary manifolds. The importance of this question arises in many applications. Almost complete answer to this question was given by Petrowsky [14] in the case of one-dimensional linear heat equation $u_t = u_{xx}$. Results concerning one-dimensional reaction-diffusion equation $u_t = a(u^m)_{xx} + bu^\beta, a > 0, m > 0, b \in I\!R, \beta > 0$ have been presented in recent papers of the author [1,2]. Primarily applying the results of [1], a full description of the evolution of interfaces and of the local solution near the interface for all relevant values of parameters is presented in [3].

At the moment there is a complete well established theory of the boundary value problems in cylindrical domains for general second order nonlinear degenerate parabolic equations, which includes as a particular case (1.1) (see the review article [10]). It seems that the current papers of the author ([4,5]) are the first ones which addressed the DP for the high dimensional nonlinear degenerate or singular parabolic equations in non-cylindrical domains with non-smooth boundaries. In [4] the DP (1.1),(1.2) with $b = 0$ is investigated. A notion of parabolic modulus of left-lower (or left-upper) semicontinuity of the lateral boundary manifold at the given point (see Definition 2.1, Section 2) was introduced. Our main assumption (Assumption $\mathcal{A}$ and (2.1), Section 2) for existence and boundary regularity consisted in upper (or lower)

Hölder condition on the parabolic modulus of left-lower (or left-upper) semincontinuity at each point of the lateral boundary manifold. Moreover, as in the classical theory of one-dimensional heat equation, the critical Hölder exponent is equal to $\frac{1}{2}$. The well-posedness of the DP (1.1),(1.2) with $b = 0$ is completed in another current paper [5], where uniquenes and comparison results, as well as L_1-contraction property and continuous dependence of solution on the initial-boundary data are established. The purpose of this paper is to extend the results of the paper [4] to the case of equation (1.1).

2 Statement of the Main Result

We shall use the usual notation:

$z = (x,t) = (x_1, \dots x_N, t) \in I\!R^{N+1}, N \geq 2, x = (x_1, \overline{x}) = (x_1, x_2, \dots, x_N) \in I\!R^N, \overline{x} = (x_2, \dots, x_N) \in I\!R^{N-1}, |x|^2 = \sum\limits_{i=1}^{N} |x_i|^2, |\overline{x}|^2 = \sum\limits_{i=2}^{N} |x_i|^2$. For a point $z = (x,t) \in I\!R^{N+1}$ we denote by $Q(z;\delta)$ an open ball in $I\!R^{N+1}$ of radius $\delta > 0$ and with center in z.

Let $\Omega \in \mathcal{D}_{0,T}$ be a given domain. Assume that for arbitrary point $z_0 = (x^0, t_0) \in S\Omega$ (or $z_0 = (x^0, 0) \in \overline{S\Omega}$) there exists $\delta > 0$ and a conitnuous function ϕ such that, after a suitable rotation of x-axes, we have $\overline{S\Omega} \cap Q(z_0, \delta) = \{z \in Q(z_0, \delta) : x_1 = \phi(\overline{x}, t)\}$. Suppose also that sign $(x_1 - \phi(\overline{x}, t)) =$ const for $z \in Q(z_0, \delta) \cap \Omega$. Furthermore, we denote this constant by $d(z_0)$. Obviously, by introducing a new variable $x_1' = -x_1$, if necessary, we could have supposed that $d(z_0) = 1$. However, we describe the conditions for both cases $d(z_0) = \pm 1$ seperately, in order to distinct "left and right boundary points" as in the one-dimensional case.

Let $z_0 = (x^0, t_0) \in S\Omega$ be a given boundary point. For an arbitrary sufficiently small $\delta > 0$, consider a parabolic domain $P(\delta) = \{(\overline{x}, t) : |\overline{x} - \overline{x}^0| < \varepsilon_0(\delta + t - t_0)^{\frac{1}{2}}, t_0 - \delta < t < t_0\}$, where $\varepsilon_0 > 0$ is an arbitrary fixed number.

Definition 2.1 Let $\quad \omega^-(\delta) = max(\phi(\overline{x}^0, t_0) - \phi(\overline{x}, t) : (\overline{x}, t) \in \overline{P(\delta)}),$

$$\omega^+(\delta) = min(\phi(\overline{x}^0, t_0) - \phi(\overline{x}, t) : (\overline{x}, t) \in \overline{P(\delta)}).$$

The function $\omega^-(\delta)$ (respectively $\omega^+(\delta)$) is called the parabolic modulus of left-lower (respectively left-upper) semicontinuity of the function ϕ at the point $(\overline{x}^0, t_0)$.

For suffuciently small $\delta > 0$ these functions are well-defined and converge to zero as $\delta \downarrow 0$. Our main assumption on the behaviour of the function ϕ near z_0 is as follows:

Assumption $\mathcal{A}$. There exists a function $F(\delta)$ which is defined for all positive sufficiently small δ; F is positive with $F(\delta) \downarrow 0$ as $\delta \downarrow 0$ and if $d(z_0) = 1$ (respectively $d(z_0) = -1$) then

$$\omega^-(\delta) \leq \delta^{\frac{1}{2}} F(\delta) \tag{2.1}$$

(respectively $\omega^+(\delta) \geq -\delta^{\frac{1}{2}} F(\delta)$)

We prove in the next section that assumption $\mathcal{A}$ is sufficient for the regularity of the boundary point z_0. Namely, the constructed limit solution takes the boundary value $\psi(z_0)$ at the point $z = z_0$ continuously in $\overline{\Omega}$. It is well-known that in the case of the linear equation ($m = 1, \beta = 1$ in (1.1)) boundary point $z_0 = (x^0, 0) \in \overline{S\Omega}$ is always regular (see e.g. [11, p. 172]). Hence, in this case the assumption $\mathcal{A}$ imposed at every boundary point $z_0 \in S\Omega$ is sufficient for solvability of the DP. It may easily be proved that the solution in this particular case is a unique classical solution (see also Corollary 2.1 below).

However, in general to provide the regularity of the boundary point $z_0 = (x^0, 0) \in \overline{S\Omega}$ we need another assumption. Denote $x_1 = \overline{\phi}(\overline{x}) \equiv \phi(\overline{x}, 0)$.

Definition 2.2 Let $\qquad \omega_0^-(\delta) = max(\overline{\phi}(\overline{x}^0) - \overline{\phi}(\overline{x}) : |\overline{x} - \overline{x}^0| \leq \delta)$

$$\omega_0^+(\delta) = min(\overline{\phi}(\overline{x}^0) - \overline{\phi}(\overline{x}) : |\overline{x} - \overline{x}^0| \leq \delta)$$

The function $\omega_0^-(\delta)$ (respectively $\omega_0^+(\delta)$) is called the modulus of lower (respectively upper) semicontinuity of the function $x_1 = \overline{\phi}(\overline{x})$ at the point $\overline{x} = \overline{x}^0$.

Assumption $\mathcal{B}$. There exists a function $F_1(\delta)$ which is defined for all positive sufficiently small $\delta; F_1$ is positive with $F_1(\delta) \downarrow 0$ as $\delta \downarrow 0$ and if $d(z_0) = 1$ (respectively $d(z_0) = -1$) then

$$\omega_0^-(\delta) \leq \delta F_1(\delta) \tag{2.2}$$

(respectively $\omega_0^+(\delta) \geq -\delta F_1(\delta)$)

It may easily be verified that if we redefine $\overline{\phi}$ as $x_1 = \overline{\phi}(\overline{x}) \equiv \phi(\overline{x}, t_0)$ then assumption $\mathcal{B}$ is a consequence of the assumption $\mathcal{A}$ at the boundary point $z_0 = (x^0, t_0) \in S\Omega$. However, assumption $\mathcal{B}$ has a sense for the boundary points $z_0 = (x^0, 0) \in \overline{S\Omega}$ on the bottom of the lateral boundary manifold. We prove in the next section that assumption $\mathcal{B}$ is sufficient for the regularity of the boundary point $z_0 = (x^0, 0) \in \overline{S\Omega}$. Namely, the constructed limit solution takes the boundary value $\psi(z_0)$ at the point $z = z_0$ continuously in $\overline{\Omega}$. Thus our main theorem reads:

Theorem 2.1 *DP* (1.1), (1.2) is solvable in a domain Ω which satisfies the assumption $\mathcal{A}$ at every point $z_0 \in S\Omega$ and assumption $\mathcal{B}$ at every point $z_0 = (x^0, 0) \in \overline{S\Omega}$.

Corollary 2.1 If the constructed solution to *DP* (1.1), (1.2) is positive in Ω, then under the conditions of the Theorem 2.1, it is a unique classical solution.

3 Proof of the Main Result

Step 1. Construction of the limit solution.

Consider a sequence of domains $\Omega_n \in \mathcal{D}_{0,T}, n = 1, 2, \dots$ with $S\Omega_n, \partial B\Omega_n$ and $\partial D\Omega_n$ being sufficiently smooth manifolds. Assume that $\{S\Omega_n\}, \{\partial B\Omega_n\}$ and $\{\partial D\Omega_n\}$ approximate $S\Omega, \partial B\Omega$ and $\partial D\Omega$ respectively. Moreover, let $\overline{S\Omega}_n$ at some neighbourhood of its every point after suitable rotation of $x-$axes has a representation via the sufficiently smooth function $x_1 = \phi_n(\overline{x}, t)$. More precisely, assume that $S\Omega$ in some neighbourhood of its point z_0 is represented by the function $x_1 = \phi(\overline{x}, t), (\overline{x}, t) \in P(\mu_0^{-2})$ with some $\mu_0 > 0$, where ϕ satisfies assumption $\mathcal{A}$ from Section 2. Then we also assume that $S\Omega_n$ in some neighbourhood of its point $z_n = (x_1^{(n)}, \overline{x}^0, T)$ is represented by the function $x_1 = \phi_n(\overline{x}, t), (\overline{x}, t) \in P(\mu_0^{-2})$, where $\{\phi_n\}$ is a sequence of sufficiently smooth functions and $\phi_n \to \phi$ as $n \to +\infty$, uniformly in $P(\mu_0^{-2})$. We can also asssume that ϕ_n satisfies assumption $\mathcal{A}$ from Section 2 uniformly with respect to n. As in [4], we make a similar assumption also regarding the points $z_n = (x^{(n)}, 0) \in \overline{S\Omega}_n$ on the bottom of the lateral boundary manifold. Assume also that for arbitrary compact subset $\Omega^{(0)}$ of $\Omega \cup D\Omega$, there exists a number n_0 which depends on the distance between $\Omega^{(0)}$ and $\mathcal{P}\Omega$, such that $\Omega^{(0)} \subseteq \Omega_n \cup D\Omega_n$ for $n \geq n_0$. Let Ψ be a nonnegative and continuous function in $\mathbb{R}^{N+1}$ which coincides with ψ on $\mathcal{P}\Omega$ and let M is an upper bound for $\psi_n = \Psi + \epsilon_n, n = 1, 2, \dots$ in some compact which contains $\overline{\Omega}$ and $\overline{\Omega}_n, n = 1, 2, \dots$, where $\{\epsilon_n\}$ be a real monotonic sequence such that $\epsilon_n < M$ and $\epsilon_n \downarrow +0$. Consider the following regularized equation

$$u_t = a\Delta u^m - bu^\beta + b\theta_b \epsilon_n^\beta, \tag{3.1}$$

where $\theta_b = (1,\ if\ b > 0;\ 0,\ if\ b \leq 0)$. We then consider the DP in Ω_n for the equation (3.1) with the initial boundary data ψ_n. This is a nondegenerate parabolic problem and classical theory ([9,13]) implies the existence of a unique classical solution u_n which satisfies $\epsilon_n \leq u_n(x,t) \leq \psi^1(t)$ in $\overline{\Omega}_n$, where

$$\psi^1(t) = \begin{cases} \Big[M^{1-\beta} - b(1-\theta_b)(1-\beta)t\Big]^{1/(1-\beta)}, & \text{if } \beta \neq 1, \\ M\exp\left(-b(1-\theta_b)t\right), & \text{if } \beta = 1. \end{cases}$$

Next we take a sequence of compact subsets $\Omega^{(k)}$ of $\Omega \bigcup D\Omega$ such that

$$\Omega = \bigcup_{k=1}^{\infty} \Omega^{(k)}, \Omega^{(k)} \subseteq \Omega^{(k+1)}, k = 1, 2, \ldots \tag{3.2}$$

By our construction, for each fixed k, there exists a number n_k such that $\Omega^{(k)} \subseteq \Omega_n \bigcup D\Omega_n$ for $n \geq n_k$. It is a well-known result of the modern theory of degenerate parabolic equations (which includes (1.1) as a model example) that the sequence of uniformly bounded solutions $u_n, n \geq n_k$ to equation (1.1) is uniformly equicontinuous in a fixed compact $\Omega^{(k)}$ (see e.g. [6, Theorem 1 & Proposition 1 and Theorem 7.1]). From (3.2), by diagonalization argument and the Arzela-Ascoli theorem, we may find a subsequence $n^{'}$ and a limit function $\tilde{u} \in C(\Omega \cup D\Omega)$ such that $u_{n'} \to \tilde{u}$ as $n' \to +\infty$, pointwise in $\Omega \cup D\Omega$ and the convergence is uniform on compact subsets of $\Omega \cup D\Omega$. Now consider a function $u(x,t)$ such that $u(x,t) = \tilde{u}(x,t)$ for $(x,t) \in \Omega \cup D\Omega, u(x,t) = \psi$ for $(x,t) \in \mathcal{P}\Omega$. Obviously the function u satisfies the integral identity (1.3). It is also continuous in $B\Omega$, since above mentioned result on the equicontinuity of the sequence u_n is true up to some neighbourhood of every point $z \in B\Omega$ [6, Theorem 6.1]. Hence, the constructed function u is a solution of the Dirichlet Problem (1.1), (1.2), if it is continuous in $\mathcal{P}\Omega \backslash B\Omega$.

Step 2. Boundary regularity. Let $z_0 = (x^0, t_0) \in S\Omega$. We shall prove that z_0 is regular, namely that

$$\lim\ u(z) = \psi(z_0) \text{ as } z \to z_0, z \in \Omega \cup D\Omega \tag{3.3}$$

Without loss of generality assume that $d(z_0) = 1$. First, assume that $t_0 = T$. If $\psi(z_0) > 0$, we shall prove that for arbitrary sufficiently small $\varepsilon > 0$ the following two inequalities are valid

$$\liminf u(z) \geq \psi(z_0) - \varepsilon \text{ as } z \to z_0, z \in \Omega \cup D\Omega \tag{3.4}$$

$$\limsup u(z) \leq \psi(z_0) + \varepsilon \text{ as } z \to z_0, z \in \Omega \cup D\Omega \tag{3.5}$$

Since $\varepsilon > 0$ is arbitrary, from (3.4) and (3.5), (3.3) follows. If $\psi(z_0) = 0$, however, then it is sufficient to prove (3.5), since (3.4) follows directly from the fact that $u \geq 0$ in $\overline{\Omega}$. Let $\psi(z_0) > 0$. Take an arbitrary $\varepsilon \in (0, \psi(z_0))$ and prove (3.4). For arbitrary $\mu > 0$, consider a function $w_n(x,t) = f(\xi) \equiv M_1(\xi/h(\mu))^\alpha$, where $\xi = h(\mu)+\phi_n(\overline{x}^0,T)-x_1-\mu[T-t+\varepsilon_0^{-2}|\overline{x}-\overline{x}^0|^2]$,

$$M_1 = \psi(z_0) - \varepsilon,\ h(\mu) = M_3\mu^{-1}F(\mu^{-2}), M_3 = [(M_2/M_1)^{\frac{1}{\alpha}} - 1]^{-1}, M_2 = \psi(z_0) - \varepsilon/2,$$

If $b \leq 0$, we take the two cases:

(a) $\alpha > m^{-1}$, if $0 < m \leq 1$ and (b) $m^{-1} < \alpha < (m-1)^{-1}$ if $m > 1$.

If $b > 0$ we take four different cases

I $\quad m^{-1} < \alpha \leq \ min\ \Big((m-1)^{-1}; (1-\beta)^{-1}\Big), \quad if\ m > 1,\ \ 0 < \beta < 1,$

II $\quad m^{-1} < \alpha \leq (m-1)^{-1}, \quad if\ m > 1,\ \ \beta \geq 1,$

III $\quad \alpha > m^{-1}, \quad if\ 0 < m \leq 1,\ \ \beta \geq m,$

IV $\quad m^{-1} < \alpha \leq (m-\beta)^{-1}, \quad if\ 0 < m \leq 1,\ \ 0 < \beta < m.$

Then we set $V_n = \{(x,t) : \phi_n(\overline{x},t) < x_1 < \phi_{1n}(\overline{x},t), (\overline{x},t) \in P(\mu^{-2})\}$,

$$\phi_{1n}(\overline{x},t) = \phi_n(\overline{x},t) + (1+M_3)\mu^{-1}F(\mu^{-2}) - \mu[T-t+\varepsilon_0^{-2}|\overline{x}-\overline{x}^0|^2],$$

Lemma 3.1 If $\mu > 0$ is chosen such that $F(\mu^{-2}) \leq (1+M_3)^{-1}$ then the parabolic boundary of V_n consists of two boundary surfaces $x_1 = \phi_n(\overline{x},t)$ and $x_1 = \phi_{1n}(\overline{x},t)$.

The proof of the Lemma 3.1 is given in [4] (see Lemma 3.1 in [4]). Thus, V_n is a domain in $I\!R^{N+1}$, lying in the strip $T-\delta_* < t < T$, $\delta_* = (1+M_3)\mu^{-2}F(\mu^{-2})$, its boundary consists of a single point lying on $\{t = T-\delta_*\}$, a domain DV_n lying on $\{t = T\}$ and a connected manifold SV_n lying in the strip $\{T-\delta_* < t \leq T\}$. Boundary manifold SV_n consists of two boundary surfaces $x_1 = \phi_n(\overline{x},t)$ and $x_1 = \phi_{1n}(\overline{x},t)$. Our purpose is to estimate u_n in V_n via the barrier function $\tilde{w}_n =\ max\ (w_n; \epsilon_n/2)$. Obviously, $\tilde{w}_n = \epsilon_n/2$ for $x_1 \geq \theta_n(\overline{x},t)$; $\tilde{w}_n = w_n$ for $x_1 < \theta_n(\overline{x},t)$, where

$\theta_n(\overline{x},t) = (1-((2M_1)^{-1}\epsilon_n)^{\frac{1}{\alpha}})h(\mu) + \phi_n(\overline{x}^0,T) - \mu[T-t+\varepsilon_0^{-2}|\overline{x}-\overline{x}^0|^2]$. In the next lemma we estimate u_n via the barrier function $\tilde{w}_n$ on the parabolic boundary of V_n. For that the special structure of V_n plays an important role. Namely, our barrier function takes the value $\epsilon_n/2$, which is less than a minimal value of u_n, on the part of the parabolic boundary of V_n which lies in Ω_n. Hence it is enough to compare u_n and $\tilde{w}_n$ on the part of the lateral boundary of Ω_n, which may easily be done in view of boundary condition for u_n.

Lemma 3.2 If $\mu > 0$ is chosen large enough, then

$$u_n > \tilde{w}_n \text{ on } \overline{SV}_n, \text{ for } n \geq n_1(\epsilon) \tag{3.6}$$

The proof of the Lemma 3.2 is given in [4] (see Lemma 3.2 in [4] and replace n^{-1} with ϵ_n).

Lemma 3.3 If $\mu > 0$ is chosen large enough, then at the points of V_n with $x_1 < \theta_n(\overline{x},t)$, we have

$$Lw_n \equiv w_{n_t} - \Delta w_n^m + bw_n^\beta - b\theta_b\epsilon_n^\beta < 0 \tag{3.7}$$

Proof At the points of V_n with $x_1 < \theta_n(\overline{x},t)$, we have

$$Lw_n = \mu h^{-1}(\mu)\alpha M_1^{\frac{1}{\alpha}} f^{\frac{\alpha-1}{\alpha}} - h^{-2}(\mu)\alpha m(\alpha m-1)M_1^{\frac{2}{\alpha}} f^{\frac{\alpha m-2}{\alpha}} \times$$

$$(1+4\mu^2\varepsilon_0^{-4}|\overline{x}-\overline{x}^0|^2) + 2\mu h^{-1}(\mu)\alpha m\varepsilon_0^{-2}(N-1)M_1^{\frac{1}{\alpha}} f^{\frac{\alpha m-1}{\alpha}} + bf^\beta - b\theta_b\epsilon_n^\beta. \tag{3.8}$$

In view of our construction of V_n, we have $w_n \leq M_2$ in $\overline{V}_n$ (see (3.8) in [6]). Hence, if $m > 1$ and either $b \leq 0$ or $b > 0$ and m, β belong to one of the regions I, II, then from (3.8) it follows that

$$Lw_n \leq h^{-2}(\mu)\alpha M_1^{\frac{1}{\alpha}} f^{\frac{\alpha-1}{\alpha}}\{M_3F(\mu^{-2}) - m(\alpha m-1)M_1^{\frac{1}{\alpha}}M_2^{m-1-\frac{1}{\alpha}}$$

$$+2M_3m\varepsilon_0^{-2}(N-1)M_2^{m-1}F(\mu^{-2}) + b\theta_bM_2^{\beta-1+\frac{1}{\alpha}}\alpha^{-1}M_1^{-\frac{1}{\alpha}}h^2(\mu)\}. \tag{3.9}$$

Hence, if μ is chosen large enough, from (3.9), (3.7) follows. If $0 < m \leq 1$ and either $b \leq 0$ or $b > 0$ and m, β belong to one of the regions III, IV, then from (3.8) similarly we derive that

$$Lw_n \leq h^{-2}(\mu)\alpha M_1^{\frac{1}{\alpha}} f^{\frac{\alpha m-1}{\alpha}}\{M_3F(\mu^{-2})M_2^{1-m} - m(\alpha m-1)M_1^{\frac{1}{\alpha}}M_2^{-\frac{1}{\alpha}}$$

$$+2M_3m\varepsilon_0^{-2}(N-1)F(\mu^{-2}) + b\theta_bM_2^{\beta-m+\frac{1}{\alpha}}\alpha^{-1}M_1^{-\frac{1}{\alpha}}h^2(\mu)\} \tag{3.10}$$

If μ is chosen large enough, from (3.10), (3.7) again follows. Lemma is proved.

Thus $\tilde{w}_n$ is the maximum of two smooth subsolutions of the equation (3.1) in V_n. By the standard maximum principle, from Lemma 3.1, (3.6) and (3.7) we easily derive that $u_n \geq \tilde{w}_n$ in $\overline{V}_n$, for $n \geq n_1$. In the limit as $n' \to +\infty$, we have

$$u \geq \tilde{w} \text{ in } \overline{V}, \tag{3.11}$$

where

$$\begin{gathered}\tilde{w} = max(w; 0), \ \text{ in } \overline{V} \\ w(x,t) = f(h(\mu) + \phi(\overline{x}^0, T) - x_1 - \mu[T - t + \varepsilon_0^{-2}|\overline{x} - \overline{x}^0|^2]), \\ V = \{(x,t) : \phi(\overline{x}, t) < x_1 < \phi_1(\overline{x}, t), |\overline{x} - \overline{x}^0| < \varepsilon_0[\delta_* + t - T]^{\frac{1}{2}}, T - \delta_* < t < T\}, \\ \phi_1(\overline{x}, t) = \phi(\overline{x}, t) + (1 + M_3)\mu^{-1}F(\mu^{-2}) - \mu[T - t + \varepsilon_0^{-2}|\overline{x} - \overline{x}^0|^2].\end{gathered}$$

Obviously, we have

$$\lim_{z \to z_0, z \in \overline{V}} \tilde{w} = \lim_{z \to z_0, z \in \overline{\Omega}} \tilde{w} = \psi(z_0) - \varepsilon$$

Hence, from (3.11), (3.4) follows. Let us now prove (3.5) for an arbitrary $\varepsilon > 0$ such that $\psi(z_0) + \varepsilon < M$. For arbitrary $\mu > 0$ consider a function $w_n(x,t) = f_1(\xi) \equiv [\overline{M}^{\frac{1}{\alpha}} + \xi h^{-1}(\mu)(M_4^{\frac{1}{\alpha}} - \overline{M}^{\frac{1}{\alpha}})]^{\alpha}$, where $\overline{M} = \psi^1(T)$, ξ is defined as before and

$$h(\mu) = M_6\mu^{-1}F(\mu^{-2}), M_4 = \psi(z_0) + \varepsilon, M_5 = \psi(z_0) + \varepsilon/2, M_6 = (\overline{M}^{\frac{1}{\alpha}} - M_4^{\frac{1}{\alpha}})(M_4^{\frac{1}{\alpha}} - M_5^{\frac{1}{\alpha}})^{-1},$$

and α is an arbitrary number such that $0 < \alpha < min\,(1; m^{-1})$. Similarly, consider the domains V_n with M_3 replaced by M_6 in the expression of $\phi_{1n}(\overline{x}, t)$ (see Lemma 3.1). We then construct an upper barrier function as follows: $\tilde{w}_n = \ min\,(w_n; \psi^1(t))$. Obviously, $\tilde{w}_n = \psi^1(t)$ for $x_1 \geq \theta_n(\overline{x}, t)$; $\tilde{w}_n = w_n$ for $x_1 < \theta_n(\overline{x}, t)$, where $\theta_n(\overline{x}, t) = M_* h(\mu) + \phi_n(\overline{x}^0, T) - \mu[T - t + \varepsilon_0^{-2}|\overline{x} - \overline{x}^0|^2]$, $\ M_* = 1 - [\overline{M}^{\frac{1}{\alpha}} - M_4^{\frac{1}{\alpha}}]^{-1}[\overline{M}^{\frac{1}{\alpha}} - (\psi^1(t))^{\frac{1}{\alpha}}]$.

Lemma 3.4 If $\mu > 0$ is chosen large enough, then

$$u_n \leq \tilde{w}_n \text{ on } \overline{SV}_n, \text{ for } n \geq n_1(\epsilon), \tag{3.12}$$

The proof of Lemma 3.4 is similar to that of Lemma 3.4 from [4].

Lemma 3.5 If $\mu > 0$ is chosen large enough, then at the points of V_n with $x_1 < \theta_n(\overline{x}, t)$, we have

$$Lw_n > 0, \tag{3.13}$$

Proof. In view of our construction of V_n, we have $w_n \geq M_5$ in $\overline{V}_n$ (see (3.17) in [4]). Hence, at the points of V_n with $x_1 < \theta_n(\overline{x}, t)$, we have (taking into account that $\epsilon_n < \overline{M}$)

$$Lw_n = -\mu h^{-1}(\mu)\alpha(\overline{M}^{\frac{1}{\alpha}} - M_4^{\frac{1}{\alpha}}) f_1^{\frac{\alpha-1}{\alpha}} + m\alpha(1-\alpha m)h^{-2}(\mu)(\overline{M}^{\frac{1}{\alpha}} - M_4^{\frac{1}{\alpha}})^2 f_1^{\frac{\alpha m-2}{\alpha}}(1+4\mu^2\varepsilon_0^{-4}|\overline{x}-\overline{x}^0|^2)$$

$$-2\mu h^{-1}(\mu)\alpha m\varepsilon_0^{-2}(N-1)(\overline{M}^{\frac{1}{\alpha}} - M_4^{\frac{1}{\alpha}}) f_1^{\frac{\alpha m-1}{\alpha}} + bf^\beta - b\theta_b\epsilon_n^\beta \geq h^{-2}(\mu)\alpha(\overline{M}^{\frac{1}{\alpha}} - M_4^{\frac{1}{\alpha}})S, \quad (3.14)$$

$$S = -M_6 M_5^{\frac{\alpha-1}{\alpha}} F(\mu^{-2}) + m(1-\alpha m)(\overline{M}^{\frac{1}{\alpha}} - M_4^{\frac{1}{\alpha}})\overline{M}^{\frac{\alpha m-2}{\alpha}}$$

$$-2M_6 m\varepsilon_0^{-2}(N-1)M_5^{\frac{\alpha m-1}{\alpha}} F(\mu^{-2}) - |b|\overline{M}^\beta \alpha^{-1}(\overline{M}^{\frac{1}{\alpha}} - M_4^{\frac{1}{\alpha}})^{-1} h^2(\mu).$$

Hence, if μ is chosen large enough, from (3.14), (3.13) follows. Lemma is proved.

Thus $\tilde{w}_n$ is the minimum of two smooth supersolutions of the equation (3.1) in V_n. By the standard maximum principle, from Lemma 3.1, (3.12) and (3.13) we easily derive that $u_n \leq \tilde{w}_n$ in $\overline{V}_n$, for $n \geq n_1$. In the limit as $n' \to \infty$, we have

$$u \leq \tilde{w} \text{ in } \overline{V}, \quad (3.15)$$

where $\tilde{w} = min(w; \psi^1(t))$, $w(x,t) = f_1(h(\mu) + \phi(\overline{x}^0, T) - x_1 - \mu[T - t + \varepsilon_0^{-2}|\overline{x} - \overline{x}^0|^2])$ and the domain V being defined as in (3.11). Obviously, we have

$$\lim_{z \to z_0,\, z \in \overline{V}} \tilde{w} = \lim_{z \to z_0,\, z \in \overline{\Omega}} \tilde{w} = \psi(z_0) + \varepsilon$$

Hence, from (3.15), (3.5) follows. Thus we have proved (3.3) for $z_0 = (x^0, T) \in S\Omega$ when $d(z_0) = 1$. The proof is similar when $d(z_0) = -1$. Suppose now that $z_0 = (x^0, t_0) \in S\Omega$ with $t_0 < T$. Clearly, the same proof given in the case $t_0 = T$, implies the regularity of z_0 regarding subdomain $\Omega_- = \Omega \bigcap \{t < t_0\}$. Namely, (3.3) is valid for $z \in \overline{\Omega}_-$. Hence, it is enough to prove (3.3) for $z \in \Omega_+, \Omega_+ = \Omega \bigcap \{t > t_0\}$. The proof of this latter, however, is equivalent to the proof of regularity of the point $z_0 = (x^0, 0) \in \overline{S\Omega}$ under the assumption $\mathcal{B}$. That easily follows from the fact that assumption $\mathcal{B}$ (with redefined $\overline{\phi}(\overline{x}) \equiv \phi(\overline{x}, t_0)$) is a consequence of the assumption $\mathcal{A}$. Thus, to complete the proof, it remains just to prove (3.3) for $z_0 = (x^0, 0) \in \overline{S\Omega}$. The proof is similar to that given in [4]. The main difference consists in choosing appropriately the function $x_1 = \phi_{1n}(\overline{x}, t)$. To prove an analog of (3.4), we choose it similarly just by replacing n^{-1} with the relevant lower bound ϵ_n of u_n. While to prove an analog of (3.5), we choose it such that $f_1(\xi) = \psi^1(t)$ for $x_1 = \phi_{1n}(\overline{x}, t)$.

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