Necessary and Sufficient Condition for Uniqueness of Solution to the First Boundary Value Problem for the Diffusion Equation in Unbounded Domains.

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On the occasion of the 81st birthday of V. Lakshmikantham

Abstract. This paper introduces a notion of regularity of \( t = -\infty \) for the diffusion (or heat) equation and establishes a necessary and sufficient condition for the existence of a unique bounded solution to the first boundary value problem for the diffusion equation in a general domain \( \Omega \subset \mathbb{R}^{N+1} \) which extends up to \( t = -\infty \).

Key words and phrases. Diffusion (or heat) equation, first boundary value problem, regularity of infinity, uniqueness.
1 Introduction and Main Result

Let $\Omega \subset \mathbb{R}^{N+1}$ $(N \geq 1)$ denote an open subset and $\partial \Omega$ its topological boundary. We write a typical point as $z = (x, t), x = (x_1, \ldots, x_N) \in \mathbb{R}^N, t \in \mathbb{R}$. Assume that $\Omega \subset \{z : t \leq T\}, T < \infty$ and $\Omega(\tau) = \{z \in \Omega : t = \tau\}$ is bounded for any $\tau \leq T$. For a given point $z_0 = (x^0, t_0)$ and a positive number $\epsilon$ define the cylinder

$$Q(z_0, \epsilon) = \{z : |x - x_0| < \epsilon, t_0 - \epsilon < t < t_0\}.$$ 

Let $P$ be the set of all points $z_0 \in \partial \Omega$ such that for any $\epsilon > 0$, the cylinder $Q(z_0, \epsilon)$ contains points not in $\Omega$. The set $P$ is called the parabolic boundary of $\Omega$. A function $u \in C^{2,1}_x(\Omega)$ is called parabolic in $\Omega$ if $u_t - \Delta u = 0$ for $z \in \Omega$. Let $f : P \rightarrow \mathbb{R}$ be a bounded function.

First boundary value problem (FBVP) may be formulated as follows: Find a function $u$ which is parabolic in $\Omega$ and satisfies the conditions

$$f_* \leq u_* \leq u^* \leq f^* \text{ for } z \in P \Omega, \quad (1.1)$$

$$f_* \leq u_* \leq u^* \leq f^* \text{ as } t \downarrow -\infty, \quad (1.2)$$

where $f_*, u_*$ (or $f^*, u^*$) are lower (or upper) limit functions of $f$ and $u$ respectively.

In particular, if $f \in C(P \Omega; \mathbb{R})$, from (1.1) it follows that $u$ takes continuously the given values of $f$ on $P \Omega$. The condition (1.2) is the new ingredient which should be taken into account for well-posedness in unbounded domains which extend up to $t = -\infty$. To construct a solution of FBVP consider a sequence of expanded bounded domains $\Omega_n = \Omega \cap \{z : t > -n\}$ and let $f = \overline{f}$ on $P \Omega_n \cap \Omega$, where $\overline{f}$ is some fixed constant. Assume that $H_f^{\Omega_n}$ be a generalized solution of FBVP in $\Omega_n$ constructed as $H_f^{\Omega_n}(z) = \inf\{u(z)\}$, where the infimum is taken over all superparabolic functions $u$ in $\Omega_n$, such that $u_*(z_0) \geq f^*(z_0)$ for all $z_0 \in P \Omega_n$. According to the classical theory $H_f^{\Omega_n}$ is parabolic in $\Omega_n$ and satisfies (1.1) at regular boundary points. Obviously, $H_f^{\Omega_n}$ is a uniformly bounded sequence and by the standard diagonalization argument one can easily derive that for some subsequence $n'$ a pointwise limit $u = \lim H_f^{\Omega_n'}$ is a parabolic function in $\Omega$. It is natural to call it a generalized Perron-Wiener-Brelot-Bauer solution of FBVP in $\Omega$. As in the case of the FBVP in bounded domains, the question whether generalized
solution satisfies (1.1) is a matter of regularity of the boundary points ([1],[2],[4],[7]). The important difference between the bounded and unbounded cases is the matter of uniqueness of the generalized solution. The generalized solution in bounded domains is always unique, even if there are irregular boundary points where (1.1) fails. However this is not the case in unbounded domains and there are, in general, infinitely many solutions even if all the boundary points \( z \in \partial \Omega \) are regular points. The next definition plays a significant role in that context.

**Definition 1.1** \( t = -\infty \) is called regular for \( \Omega \) if for any bounded function \( f : \partial \Omega \rightarrow \mathbb{R} \) the generalized solution of FBVP satisfies (1.2).

The principal result of this paper is the characterization of the regularity of \( t = -\infty \) and at the same time the uniqueness of solution to FBVP via the growth of diameter of \( \Omega(t) \) as \( t \downarrow -\infty \). Consider the model domain

\[
\mathcal{L}_H = \{ z \in \mathbb{R}^{N+1} : |x| < 2(t \log \rho(t))^{\frac{1}{2}}, H < t < 0 \},
\]

where \(-\infty \leq H < 0, \rho \in C(-\infty, 0), 0 < \rho(t) < 1 \) for \(-\infty < t < 0; \rho(t) \downarrow 0 \) as \( t \downarrow -\infty \) and

\[
\log \rho(t) = o(\log(\log |t|)) \text{ as } t \downarrow -\infty \quad (1.3)
\]

(see Remark 2 concerning this condition). The main result reads as follows.

**Theorem 1.1** \( t = -\infty \) is regular or irregular for \( \mathcal{L}_{-\infty} \), that is to say there exists a unique solution or infinitely many generalized solutions of the FBVP in \( \mathcal{L}_{-\infty} \) according as

\[
\int_{-\infty}^{-H} \frac{\rho(t)|\log \rho(t)|^{\frac{N}{2}}}{t} dt
\]

diverges or converges.

For example, (1.4) diverges for each of the following functions:

\[
\rho(t) = \log^{-1}|t|, \rho(t) = \{\log|t| \cdot \log^{\frac{N+2}{2}}|t|\}^{-1}, \rho(t) = \{\log|t| \cdot \log^{\frac{N+2}{2}}|t| \cdot \prod_{k=3}^{n} \log |t|\}^{-1}, n \geq 3,
\]

where we use the notation \( \log_n |t| = \log \log_{n-1} |t|, n \geq 2 \). From another side, (1.4) converges for each function

\[
\rho(t) = \log^{-1-\epsilon}|t|, \rho(t) = \{\log|t| \cdot \log^{\frac{N+2}{2}+\epsilon}|t|\}^{-1}, \rho(t) = \{\log|t| \cdot \log^{\frac{N+2}{2}}|t| \cdot \log^{1+\epsilon}|t|\}^{-1},
\]
\[ \rho(t) = \{ \log|t| \cdot \log^2 \frac{N+2}{t} \cdot \log_3|t| \cdot \log_4^{1+\epsilon}|t| \}^{-1} \text{ etc.}, \text{where } \epsilon > 0 \text{ is sufficiently small number.} \]

Theorem 1.1 implies precise regularity/irregularity or uniqueness/non-uniqueness tests for general domains. Consider the rigid body displacements of \( L_{-\infty} \) composed of translations in \( x \)-space and shift along the \( t \)-axis.

**Definition 1.1** We shall say that \( \Omega \) satisfies the exterior (respectively interior) \( M_\rho \) condition at \( t = -\infty \) if for some \( \delta < 0 \) after the above mentioned displacement \( L_{-\infty} \) contains (respectively is contained in) \( \Omega \cap \{ z : t < \delta \} \).

**Corollary 1.1** The regularity (respectively irregularity) assertion of Theorem 1.1 is true for a domain \( \Omega \) which satisfies the exterior (respectively interior) \( M_\rho \) condition.

## 2 Sketch of the proof and conclusions

Let \( u = u_H(x, t), H > -\infty \) is a classical solution of the FBVP in \( L_H \) satisfying \( u_H(x, H) = 1 \) on the bottom boundary of \( L_H \) and \( u_H = 0 \) on the lateral boundary of \( L_H \). From the classical maximum principle it easily follows that \( 0 \leq u_H \leq 1 \) and \( u_{H_1} \geq u_{H_2} \) for \( H_1 > H_2 \). Accordingly, there exists a limit \( u(x, t) = \lim_{H \to -\infty} u_H(x, t) \) for \( (x, t) \in L_{-\infty} \). Let us fix any point \( (x, t) \in L_{-\infty} \).

The following problem plays an essential role in the proof of the Theorem 1.1.

**Kolmogorov problem**: Is \( u(x, t) > 0 \) or \( = 0 \)?

**Lemma 2.1** The following three assertions are equivalent

(1) \( t = -\infty \) is regular (respectively irregular) for \( L_{-\infty} \).

(2) There exist a unique solution (respectively infinitely many generalized solutions) of the FBVP in \( L_{-\infty} \).

(3) Kolmogorov problem has zero (respectively positive) solution.

Proof of Lemma 2.1. (1) \( \Rightarrow \) (2): Assume that \( u_1 \) and \( u_2 \) are two solutions of FBVP. Then \( v = u_1 - u_2 \) is a solution of FBVP with zero boundary data. From (1) it follows that \( \lim v = 0 \) as \( t \downarrow -\infty \) uniformly for all \( x \). By the maximum principle one can easily conclude that \( v \equiv 0 \).
(2) ⇒ (1): Assume that $t = -\infty$ is irregular, that is to say there is a bounded function $f$ such that either $u^* > f^*$ or $u_* < f_*$ as $t \downarrow -\infty$. However, by choosing a number $\bar{f}$ satisfying $f_* \leq \bar{f} \leq f^*$ in the above described procedure we can always construct a generalized solution of FBVP which satisfies (1.2). Contradiction with uniqueness.

(3) ⇒ (2): Let $u_1$ and $u_2$ are two bounded solutions of the FBVP. By using the maximum principle one can easily show that $|u_1 - u_2| < Mu_H$ in $L_{H,H} > -\infty$, where $M = \sup|u_1| + \sup|u_2|$. Passing to the limit $H \downarrow -\infty$, from (3) it follows that $u_1 \equiv u_2$. Since (2) ⇒ (3) is immediate, the direct assertions of lemma are proved. To prove the alternative assertions of the lemma first note that (1) ⇔ (3) follows from the proof given for direct assertions. Then note that if $u$ is a positive solution of the Kolmogorov problem, then for arbitrary constant $C$, a function $Cu$ is a generalized solution of the FBVP with zero boundary data. This immediately implies that (3) ⇔ (2) for the alternative assertions as well and the lemma is proved.

Theorem 1.1 is a consequence of Lemma 2.1 and the following Theorem 2.1 concerning Kolmogorov problem.

**Theorem 2.1 ([3],[7])**. $u(x,t) > 0$ or $0 = 0$ according as (1.4) converges or diverges. Moreover, either $u(x,t) \equiv 0$ or $\lim_{t \to -\infty} u(0,t) = 1$ according as (1.4) diverges or converges.

Corollary 1.1 is a consequence of Theorem 1.1 and the following remark.

**Remark 1 ([3])**. Assume that $\rho_1 \geq \rho_2$ for $t \leq H_0$ for some $H_0 < 0$. We denote by $L_{H,i}^i$, $i = 1,2$ the domain $L_H$ with $\rho$ replaced by $\rho_i$, $i = 1,2$. Similarly, denote as $u_{H,i}^i$ and $u^i$, $i = 1,2$ the related functions $u_H$ and $u$ respectively in the context of the Kolmogorov problem. From the maximum principle it follows that

$$0 \leq u^1(x,t) \leq u^2(x,t), \text{ for } (x,t) \in L_{-\infty}^1 \cap \{t \leq H_0\}.$$ 

Accordingly, if the positivity assertion of Theorem 2.1 is true for $\rho = \rho_1$ then it is also true for $\rho = \rho_2$. From another side if the alternative is true for $\rho = \rho_2$ then it is also true for $\rho = \rho_1$.

**Remark 2.** It should be mentioned that we do not need the condition (1.3) for the proof of the irregularity (or nonuniqueness) assertion of Theorem 1.1. For the alternative assertion of Theorem 1.1 the assumption (1.3) makes almost no loss of generality. First of all, this condition is satisfied for all examples given above. Secondly, note that the class of functions satisfying
(1.3) contains the class of functions satisfying the following inequality
\[
\rho(t) \geq \rho_C^M = |\log(Ct)|^{-M},
\]
for all \(t\) with large \(|t|\) and for some \(C < 0, M > 1\). Since the integral (1.4) is divergent, the function \(\rho(t)\) may not satisfy (2.1) with reversed inequality and for all large \(|t|\), because (1.4) is convergent for each function \(\rho^M_C(t)\). Accordingly, the condition (2.1) together with divergence of (1.4) excludes only pathological functions with the property that in any interval \(-\infty < t < T\) they intersect infinitely many times all the functions \(\rho^M_C\), with \(C < 0, M > 1\).

It should be mentioned that the results of this paper has a probabilistic meaning in the context of the large-time behaviour of the diffusion processes. The intuitive meaning of the notion of regularity (respectively irregularity) of \(t = \infty\) is that the N-dimensional Brownian motion trajectory that starts at the origin (assuming that the process goes in the \(-t\)-direction) with the probability 1 will leave (respectively will stay within) the domain \(L_{-\infty}\) as \(t \downarrow \infty\). The probabilistic counterpart of Theorem 1.1 is a well-known N-dimensional Kolmogorov-Petrovsky test for the asymptotic behaviour of the Brownian motion trajectory (or Wiener processes) as \(t \uparrow +\infty\) ([5],[6]).

Another important conclusion says that the well-posedness criteria for \(u_t = \Delta u\) and \(u_t = a\Delta u\), \(a \neq 1\) is completely different. For example, consider a domain \(L_{-\infty}\) with \(\rho(t) = \log^{-1}|t|\). Theorem 1.1 implies that \(t = \infty\) is regular for \(L_{-\infty}\) regarding \(u_t = \Delta u\) and accordingly, FBVP is uniquely solvable. However, \(t = \infty\) is irregular for the same domain but regarding \(u_t = a\Delta u\) with any \(0 < a < 1\) and accordingly, the related FBVP has infinitely many generalized solutions.

Indeed, after the transformation \(\tau = at\), the function \(\tilde{u}(x, \tau) = u(x, t)\) satisfies the equation \(\tilde{u}_\tau = \Delta \tilde{u}\) and \(L_{-\infty}\) is transformed to \(\tilde{L}_{-\infty} = \{(x, \tau) : |x| < 2(-a^{-1}\tau \log|a^{-1}\tau|)^{\frac{1}{2}}, -\infty < \tau < 0\}\). For arbitrary \(\epsilon < a^{-1} - 1\), \(\tilde{L}_{-\infty} \cap \{(x, \tau) : t < \delta\}\) contains \(\{(x, \tau) : |x| < 2(-1 + \epsilon)\tau \log|\tau|)^{\frac{1}{2}}, -\infty < \tau < \delta\}\), provided that \(\delta\) is small enough. By Theorem 1.1, \(t = \infty\) is irregular for \(L_{-\infty}\) with \(\rho(t) = \log^{-1-\epsilon}|t|\) regarding \(u_t = \Delta u\) and we just need to apply the Corollary 1.1.
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References


