Evolution of Interfaces and Explicit Asymptotics at Infinity for the Reaction-Diffusion Equations.

UGUR G ABULLA\textsuperscript{*}

Faculty of Applied Mathematics and Cybernetics
Baku State University, Baku, Azerbaijan

and

Max-Planck Institute for Mathematics in the Sciences
Inselstr.22-26, Leipzig 04103, Germany

Abstract. The evolution of interfaces and the local behaviour of solutions near the interface in problems for the one-dimensional fast diffusion equation with a reaction term are studied. In all cases explicit formulae for the interface, with accuracy up to constant coefficients, are found together with the local solution. Explicit asymptotics of solutions at infinity are derived in cases in which interfaces are absent. The methods used are based on scale of variables and on a barrier technique using special comparison theorems in irregular domains.

Key words. reaction-diffusion equations, evolution of interfaces, fast diffusion, nonlinear degenerate parabolic equations

AMS subject classifications. 35K55, 35K65

\textsuperscript{*}E-mail address: Ugur.Abdulla@mis.mpg.de
1 Introduction

Consider the Cauchy problem (CP) for the reaction-diffusion equation

\[ Lu \equiv u_t - (u^m)_{xx} + bu^\beta = 0, \quad x \in \mathbb{R}, \quad 0 < t < T, \]  

with

\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \]  

where \( 0 < m < 1, \ b > 0, \ \beta > 0, \ 0 < T \leq +\infty \) and \( u_0 \) is non-negative and continuous. In this paper an investigation of the evolution of interfaces and the local structure of solutions near the interface is continued. As in the previous paper [4], without loss of generality it is assumed that \( \eta(0) = 0, \) where

\[ \eta(t) = \sup\{x : u(x, t) > 0\}. \]

More precisely, we are interested in the short-time behaviour of the interface function \( \eta(t) \) and of the local solution near the interface. Furthermore, unless otherwise stated, we shall assume that

\[ u_0 \sim C(-x)^\alpha_+ \quad \text{as} \quad x \to -\infty, \quad \text{for some} \quad C > 0, \ \alpha > 0. \]  

A full description of the small-time behaviour of \( \eta(t) \) and of the local solution near \( \eta(t) \) for all relevant values of parameters \( m, b, \beta, C, \alpha \) in the case of slow diffusion \( (m > 1) \) is presented in [4] (see Remark 1.1 in [4]). In this paper a similar description is presented for the fast diffusion \( (0 < m < 1) \) case; we also give a full description of local solutions in cases in which interfaces are absent. The behaviour of \( u_0 \) as \( x \to -\infty \) has no influence on our results. Accordingly, we may suppose that \( u_0 \) is either bounded or even unbounded without any conditions as \( x \to -\infty, \) which is suitable for existence, uniqueness and comparison results (see Section 3). In some cases in addition we shall consider the special case

\[ u_0(x) = C(-x)^\alpha_+, \quad x \in \mathbb{R}, \]  

namely when the solution to the problem (1.1), (1.4) is of exactly self-similar form. In these cases our estimations will be global in time.
Besides the recent paper [4], there was previously a complete picture of the above-mentioned problem only in the case of the semilinear equation \((m = 1)\), given in [8,9]. Accordingly, we are not directly interested in the case of \(m = 1\), although when our results contain as a particular case the semilinear equation, it will be mentioned. It should be noted that sometimes this may not be the case. For instance, if \(0 < \beta < 1, m > \beta, C > 0, 0 < \alpha < 2/(m - \beta)\), then the interface initially expands, and if \(m > 1\) then \([4]\)

\[
\eta(t) \sim C_1 t^{1/(2 - \alpha(m - 1))} \quad \text{as} \quad t \to 0+,
\]

while if \(m < 1\), we prove below that

\[
\eta(t) \sim C_2 t^{(m - \beta)/2(1 - \beta)} \quad \text{as} \quad t \to 0+.
\]

Formally, as \(m \to 1\) both of these estimates yield

\[
\eta(t) = O(t^{1/2}) \quad \text{as} \quad t \to 0+,
\]

while as a matter of fact from [9] it follows that if \(m = 1\) then

\[
\eta(t) \sim C_3 (t \log 1/t)^{\frac{1}{2}} \quad \text{as} \quad t \to 0+
\]

\((C_i, i = 1, 3\) are positive constants\), so that the case \(m = 1\) is in some respects a singular limit.

The methods of this paper are similar to those of [4]. As in [4], the results of the recent paper [3] on the general theory of initial-boundary value problems for reaction-diffusion equations in non-cylindrical domains with non-smooth and characteristic boundary curves will play a crucial role. The organisation of the paper is as follows. In Section 2 we outline the main results. In Section 3 we then apply scale of variables methods for some preliminary estimations which are necessary for using our barrier technique. Finally in Section 4 we prove the results of Section 2. To avoid difficulties for the reader we give explicit values of the constants which appear in Sections 2 and 4 in Appendix I.

## 2 Description of Main Results

We divide the results into the two different cases: I \(b > 0\) and II \(b = 0\).

I. In this case there are five different sub-cases, as shown in Figure 1.
Figure 1. Classification of different cases in the \((\alpha, \beta)\) plane for interface development in the problem (1.1)–(1.4).

(1) Suppose that \(0 < \beta < m\), \(0 < \alpha < 2/(m - \beta)\). In this case the interface initially expands and for arbitrary \(\rho \in \mathbb{R}\), there exists a positive number \(f(\rho)\) depending on \(C\), \(m\) and \(\alpha\) such that

\[
\lim_{t \to 0^+} u(x,t) \sim f(\rho)t^{\alpha/(2+\alpha(1-m))}
\]

along the curve \(x = \xi_\rho(t) = \rho t^{1/(2+\alpha(1-m))}\). Actually, \(f\) is a self-similar solution to the problem (1.1), (1.4) with \(b = 0\) (see Lemma 3.1) and

\[
f(\rho) = C^{2/(2+\alpha(1-m))} f_0 \left( (1-m)/(2+\alpha(1-m)) \rho \right), \quad f_0(\rho) = \omega(\rho,1),
\]

where \(\omega\) is a solution of the CP (1.1), (1.4) with \(b = 0\), \(C = 1\). Lower and upper estimations for \(f\) are given in (2.22), (2.23). Moreover, there exists a number \(\delta > 0\) such that

\[
C_1 t^{1/(1-\beta)} \left( \zeta_1 - \zeta \right)^{2/(m-\beta)} \leq u \leq C_* t^{1/(1-\beta)} \left( \zeta_2 - \zeta \right)^{2/(m-\beta)}, \quad 0 < t \leq \delta,
\]

(2.3a)

\[
\zeta_1 t^{\frac{m-\beta}{2(1-\beta)}} \leq \eta(t) \leq \zeta_2 t^{\frac{m-\beta}{2(1-\beta)}}, \quad 0 \leq t \leq \delta,
\]

(2.3b)

where \(\zeta = x t^{-\frac{m-\beta}{2(1-\beta)}}\) and the left-hand side of (2.3a) is valid for \(0 \leq x < +\infty\), while the right-hand side is valid for \(x \geq \ell_0 t^{\frac{m-\beta}{2(1-\beta)}}\) and the constants \(C_1\), \(\zeta_1\), \(\zeta_2\) and \(\ell_0\) are positive and depend only on \(m\), \(\beta\) and \(b\) (see Appendix 1 for a description of their explicit values). If \(u_0\) is defined as in (1.4), then the right-hand sides of (2.3) are valid for \(0 < t < +\infty\).
The explicit formula \((2.1)\) means that the local behaviour of solution along the curves \(x = \xi_\rho(t)\) approaching the origin coincides with that of the problem \((1.1), (1.4)\) with \(b = 0\). In other words, diffusion completely dominates in this region. However, domination of diffusion over the reaction fails along the curves \(x = \zeta_\rho(t) = \rho t^{\frac{m-\beta}{2(1-\beta)}}\), \(\rho > 0\) approaching the origin and the balance between diffusion and reaction in this region governs the interface, as expressed in estimations \((2.3)\). We stress the fact that the constants \(C_1, \zeta_1, \zeta_2\) and \(\ell_0\) in \((2.3)\) do not depend on \(C\) and \(\alpha\).

(2) Suppose that \(0 < \beta < m\), \(\alpha = 2(m - \beta)^{-1}\) (here we describe the results for the case \(\beta < m < 2 - \beta\); these overlap with some of the results from [4] for the case \(1 \leq m < 2 - \beta\)).

In this case the behaviour of the interface depends on the constant \(C\), the critical value being \(C = C_*\). First, assume that \(u_0\) is defined by \((1.4)\). If \(C = C_*\) then \(u_0\) is a stationary solution to \((1.1), (1.4)\).

If \(C > C_*\) then the interface expands, \(f_1(0) = A_1 > 0\) (see Lemma 3.3) and

\[
C' \left( \zeta' t^{\frac{m-\beta}{2(1-\beta)}} - x \right)^{\frac{2}{m-\beta}} \leq u \leq C'' \left( \zeta'' t^{\frac{m-\beta}{2(1-\beta)}} - x \right)^{\frac{2}{m-\beta}}, \quad 0 < t < +\infty,
\]

where \(0 \leq x < +\infty\) and \(C' = C_2, C'' = C_3, \zeta' = \zeta_3, \zeta'' = \zeta_4\) (see Appendix I).

If \(0 < C < C_*\) then the interface shrinks. There exists a constant \(\ell_1 > 0\) such that for arbitrary \(\ell \leq -\ell_1\), there exists a \(\lambda > 0\) such that

\[
u \left( \ell t^{\frac{m-\beta}{2(1-\beta)}}, t \right) = \lambda t^{\frac{1}{3-\beta}}, \quad t \geq 0.
\]

Moreover, \(u\) and \(\zeta_*\) satisfy \((2.6)\) with \(C' = C_*\), \(C'' = C_3, \zeta' = -\zeta_5 = -\ell_1 + (\lambda/C_*)^{\frac{m-\beta}{2(1-\beta)}} < 0, \zeta'' = -\zeta_6\) and the left-hand side of \((2.6a)\) is valid for \(x \geq -\ell_1 t^{\frac{m-\beta}{2(1-\beta)}}\), while the right-hand side is valid for \(x \geq -\ell_2 t^{\frac{m-\beta}{2(1-\beta)}}\) (see Appendix I, Lemma 3.3 and \((3.7)\)). In general the precise value \(\zeta_*\) can be found only by solving the similarity ODE \(\mathcal{L}^0 f_1 = 0\) (see...
(4.3b) below) and by calculating \( \zeta_* = \sup \{ \zeta : f_1(\zeta) > 0 \} \). It may easily be shown that the estimation (2.6), together with existence, uniqueness and comparison results for the original Cauchy problem (1.1), (1.4) (see Section 3), imply the unique solvability of the relevant boundary value problems for the similarity ODE, as well as the existence and uniqueness of \( \zeta_* \). Respective lower and upper bounds for \( \zeta_* \) are given in (2.6b).

Now assume that \( u_0 \) satisfies (1.3) with \( \alpha = 2/(m - \beta) \). Then if \( C \neq C_* \) we have

\[
\eta(t) \sim \zeta_* t^{\frac{m-\beta}{2(1-\beta)}} \quad \text{as} \quad t \to 0+,
\]

and for arbitrary \( \rho < \zeta_* \)

\[
u(x, t) \sim f_1(\rho) t^{1/(1-\beta)} \quad \text{for} \quad x = \zeta_\rho(t) = pt^{\frac{m-\beta}{2(1-\beta)}}, \quad t \to 0+,
\]

where the right-hand side of (2.9) (respectively (2.8)) relates to the self-similar solution (2.4), for which we have lower and upper bounds via (2.6). If \( u_0 \) satisfies (1.3) with \( \alpha = 2/(m - \beta) \), \( C = C_* \) then the small-time behaviour of the interface and the local solution depends on the terms smaller than \( C_* (-x)^{2/(m-\beta)} \) in the expansion of \( u_0 \) as \( x \to 0_- \).

It should be noted that if \( C > C_* \) then the estimation (2.6) coincides with the one proved in [4], for the case \( 1 \leq m < 2 - \beta \) (see (2.11) in [4]). If \( 0 < C < C_* \) then the right-hand side of the estimation (2.6) coincides with that proved in [4] for the case \( 1 \leq m < 2 - \beta \) (see (2.14) in [4]), while the left-hand side of (2.6) is new. It should also be noted that the left-hand side of the estimation (2.14) from [4], proved there for the case \( 1 \leq m < 2 - \beta \), is still valid if \( m \geq 1 - \beta \).

(3) Suppose that \( 0 < \beta < 1, \alpha > 2/(m - \beta) \). In this case the interface initially shrinks and

\[
\eta(t) \sim -\ell_* t^{1/\alpha(1-\beta)} \quad \text{as} \quad t \to 0+,
\]

where \( \ell_* = C^{-1/\alpha} \left( b(1 - \beta) \right)^{1/\alpha(1-\beta)} \). For arbitrary \( \ell > \ell_* \) we have

\[
u(x, t) \sim \left[ C^{1-\beta} (-x)^{\alpha(1-\beta)} - b(1 - \beta) t \right]^{1/(1-\beta)} \quad \text{as} \quad t \to 0+,
\]
along the curve \( x = \eta(t) = -\ell t^{1/\alpha (1-\beta)} \). These results mean that the interface initially coincides with that of the solution

\[
\tilde{u}(x,t) = \left[ C^{1-\beta}(-x)^{\alpha (1-\beta)} - b(1-\beta)t \right]^{\frac{1}{1-\beta}}
\]

to the problem

\[
\tilde{u}_t + b\tilde{u}^\beta = 0, \quad \tilde{u}(x,0) = C(-x)^\alpha.
\]

(4) Let \( b > 0, \ 0 < m = \beta < 1, \ \alpha > 0 \). In this case there is an infinite speed of propagation and (2.1), (2.2) are valid (see Lemma 3.2). For arbitrary \( M > 0 \) there exists a number \( \delta = \delta(M) > 0 \) such that

\[
t^{\frac{1}{1-m}} \phi_M(x) \leq u(x,t) \leq t^{\frac{1}{1-m}} \phi_\infty(x) \quad \text{for} \quad 0 < x < \infty, \quad 0 \leq t \leq \delta,
\]

where

\[
\phi_M(x) = \left( b(2m)^{-1}(1-m^2) \right)^{\frac{1}{1-m}} \sinh^{\frac{2}{m-1}} \left[ b^{\frac{1}{2}}(2m)^{-1}(1-m) \left( x + \psi(M) \right) \right],
\]

\[
\psi(M) = b^{-\frac{1}{2}} m(1-m)^{-1} \ln \left( \frac{1+2mb^{-1}(1-m)^{-1}M^{1-m}}{1+2mb^{-1}(1-m)^{-1}M^{1-m}} \right)^{\frac{1}{2}} - 1, \quad \psi(\infty) = 0.
\]

The function \( \phi = \phi_M(x), \ 0 < M \leq +\infty \) is the solution of the stationary problem

\[
(\phi^m)' = b\phi^m + (1-m)^{-1}\phi, \quad 0 < x < +\infty,
\]

(2.13a)

\[
\phi(0) = M, \quad \phi(+\infty) = 0.
\]

(2.13b)

It follows from (2.12) that

\[
\lim_{t \to 0^+} \lim_{x \to +\infty} u \exp \left( b^\frac{1}{2} m^{-1} x \right) t^{\frac{1}{m-1}} = \left( 2bm^{-1}(1-m^2) \right)^{\frac{1}{1-m}}.
\]

(2.14)

It should be pointed out that the right-hand side of (2.12) is in fact valid for \( 0 < t < +\infty \) for the solution of the general problem (1.1), (1.2), whenever \( u_0 \) vanishes for \( x \geq 0 \).

(5) Suppose that \( \beta > m \). Here we describe also the results for the case \( b < 0, \ \beta \geq 1 \), since they coincide, with small modifications, with those of the case \( b > 0, \ \beta \geq 1 \). As in the
previous case there is an infinite speed of propagation and (2.1), (2.2) are valid.

Let $\beta \geq 1$. Then for arbitrary sufficiently small $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$C_5 t^{\frac{2+\alpha}{2+\alpha(1-m)}} (\xi_1 + \xi) \frac{2}{m-1} \leq u \leq C_6 t^{\frac{2+\alpha}{2+\alpha(1-m)}} (\xi_2 + \xi) \frac{2}{m-1}, \quad x \geq 0, \quad 0 \leq t \leq \delta,$$

(2.15)

where $\xi = xt^{-1/(2+\alpha(1-m))}$ (see Appendix I for the relevant constants).

If $b > 0, \beta \geq 1$, then the following upper estimation is also valid

$$u(x,t) \leq Dt^\frac{1}{1-m} x^{\frac{2}{m-1}} \quad 0 < x < +\infty, \quad 0 < t < +\infty,$$

(2.16)

with

$$D = \left(2m(m+1)(1-m)^{-1}\right)^{\frac{1}{1-m}}.$$

From (2.15) and (2.16) it follows that if $\beta \geq (3-m)/2$, then there exists a $\delta > 0$ (which does not depend on $\varepsilon$) such that for $\forall$ fixed $t \in (0,\delta]$

$$u(x,t) \sim D t^\frac{1}{1-m} x^{\frac{2}{m-1}} \quad \text{as} \quad x \to +\infty.$$

(2.17)

However, if $1 \leq \beta < (3-m)/2$, then from (2.15), (2.16) it follows only that

$$\lim_{t \to 0^+} \lim_{x \to +\infty} u(x,t) t^{\frac{1}{m-1}} x^{\frac{2}{m-1}} = D.$$

(2.18)

Let $b < 0, \beta \geq 1$. Then for arbitrary sufficiently small $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$u(x,t) \leq D(1-\varepsilon)^{\frac{1}{m-1}} t^\frac{1}{1-m} x^{\frac{2}{m-1}} \quad \text{for} \quad \mu t^{\frac{1}{2+\alpha(1-m)}} < x < +\infty, \quad 0 < t \leq \delta,$$

(2.19)

with

$$\mu = \left(D^{-1}(A_0 + \varepsilon)\right)^{\frac{m-1}{2}} (1-\varepsilon)^{-\frac{1}{2}}.$$

From (2.15) and (2.19), (2.17) again follows.

Let $b > 0, m < \beta < 1$. Then there exists a number $\delta > 0$ such that

$$C_*(1-\varepsilon)^{\frac{1}{m-1}} (\zeta_1 + \zeta) \frac{2}{m-1} \leq u(x,t) \leq C_1 x^{\frac{2}{m-1}}, \quad 0 < x < +\infty, \quad 0 < t \leq \delta,$$

(2.20)
where $\varepsilon > 0$ is an arbitrary sufficiently small number. From (2.20) it follows that for arbitrary fixed $t \in (0, \delta]$

$$u(x,t) \sim C_*x^{\frac{2}{m-\beta}} \text{ as } x \to +\infty. \quad (2.21)$$

As in the case I(1), the explicit formulae (2.1) expresses the domination of diffusion over the reaction. If $\beta \geq 1$, then from (2.15), (2.17), (2.18) it follows that domination of diffusion is the case for $x \gg 1$ as well, and the asymptotic behaviour as $x \to +\infty$ coincides with that of the solution to problem (1.1), (1.4) with $b = 0$ (see the case II below). However, if $m < \beta < 1$ then domination of diffusion fails for $x \gg 1$ and there is a full balance between diffusion and reaction as $x \to +\infty$, expressed by the stationary solution of the equation (1.1) on the right-hand side of (2.21).

II. $b = 0$, $0 < m < 1$, $\alpha > 0$

**Remark 2.1.** It should be noted that this case has been widely investigated in the past [7,10,13, etc.] (see also the review article [11]). Nevertheless, using the same techniques as in the case $b \neq 0$, we derive new global estimations. The new element here is that we have constructed lower and upper solutions to the similarity ordinary differential equation for the function $f(\xi)$ in (2.22).

In this case there is an infinite speed of propagation. First, assume that $u_0$ is defined by (1.4). Then the solution to (1.1), (1.4) has the self-similar form

$$u(x,t) = t^{\frac{\alpha}{2+\alpha(1-m)}}f(\xi), \quad \xi = x^t \frac{t^{1+\alpha(1-m)}}{2+\alpha(1-m)}, \quad (2.22)$$

where $f$ satisfies (2.2). Moreover we have

$$Dt^{\frac{\alpha}{2+\alpha(1-m)}}(\xi_3+\xi)^{\frac{2}{m-1}} \leq u \leq Ct^{2+\alpha(1-m)}(\xi_4+\xi)^{\frac{2}{m-1}}, \quad 0 \leq x < +\infty, \quad 0 < t < +\infty \quad (2.23)$$

(see Appendix I). The right-hand side of (2.23) is not in fact sharp enough as $x \to +\infty$ and the required upper estimation is provided by an explicit solution to equation (1.1), as in (2.16). From (2.23) and (2.16) it follows that, for arbitrary fixed $0 < t < +\infty$, the asymptotic result (2.17) is valid.
Now assume that \( u_0 \) satisfies (1.3) with \( \alpha > 0 \). Then (2.1) is valid and for arbitrary sufficiently small \( \varepsilon > 0 \) there exists a \( \delta = \delta(\varepsilon) > 0 \) such that the estimation (2.23) is valid for \( 0 < t \leq \delta \), except that in the left-hand side (respectively in the right-hand side) of (2.23) the constant \( A_0 \) should be replaced by \( A_0 - \varepsilon \) (respectively \( A_0 + \varepsilon \)). Moreover, there exists a number \( \delta > 0 \) (which does not depend on \( \varepsilon \)) such that, for arbitrary \( t \in (0, \delta] \), the asymptotic result (2.17) is valid. It should be noted that the result (2.17) is already known when \( b = 0 \) [13].

3 Preliminary Results

Throughout this paper we shall follow the definitions of the generalized solution and super-(or sub-) solutions of the initial or initial-boundary value problems for equation (1.1) given in [3]. Boundary value problems for (1.1) in non-cylindrical domains with non-smooth boundaries have recently been investigated in [3]. The results of [3] are widely used in the remainder of the paper.

Suppose that \( b \geq 0 \) and that \( u_0 \) may have unbounded growth as \( |x| \to +\infty \). In the case of the fast diffusion equation \( (b = 0, \ 0 < m < 1 \ \text{in} \ (1.1)) \) it follows from the results of [10] that there exists a unique global solution to CP (1.1), (1.2) without any restriction on the growth rate of initial data at infinity. Consider now the more general problem

\[
\mathcal{L}u \equiv u_t - \Delta \left( |u|^{m-1}u \right) + b|u|^\beta - 1 u = 0 \quad \text{on} \quad \mathbb{R}^N \times (0, \infty),
\]

\[
u(\cdot, 0) = u_0,
\]

where \( 0 < m < 1, \ b > 0, \ \beta > 0, \ u_0 \in L^1_{\text{loc}}(\mathbb{R}^N) \). By a solution of (3.1), we mean a function \( u \) satisfying

\[
u \in C([0, \infty); L^1_{\text{loc}}(\mathbb{R}^N)),
\]

\[
\mathcal{L}u = 0 \quad \text{in} \quad \mathcal{D}'\left( \mathbb{R}^N \times (0, \infty) \right),
\]

where \( \mathcal{D}'(A) \) is the space of distributions on \( A \).
Theorem 3.1 Let \( u_0 \in L^1_{\text{loc}}(\mathbb{R}^N) \). Then there exists \( u \) satisfying (3.3), (3.4) and (3.2). The solution is unique if

\[
\frac{\partial u}{\partial t} \in L^1_{\text{loc}}\left(\mathbb{R}^N \times (0, \infty)\right).
\] (3.5)

This theorem was proved in [10] for the case \( b = 0 \). By employing exactly the same techniques as in [10], it may easily be proved in the case of \( b > 0 \) as well. The existence part of Theorem 3.1, together with standard regularity results, implies the existence of a generalized solution to CP (1.1), (1.2). Hence, if (3.5) is valid, then Theorem 3.1 implies uniqueness and comparison results for the problem (1.1), (1.2). To prove (3.5), take an arbitrary \( a > 0, \tau > 0, T_1 > \tau \) and let \( D^a = \{(x,t) : |x| < a, \tau < t \leq T_1\} \). Consider the Dirichlet problem

\[
\mathcal{L}v = 0 \quad \text{in} \quad D^a_{\tau/2}; \quad v = u \quad \text{on} \quad \bar{D}^a_{\tau/2} \setminus D^a_{\tau/2}.
\] (3.6)

Methods to prove existence of a weak solution (in the sense of [12]) to the problem (3.6), such that \( \partial v/\partial t \in L^2(D^a_{\tau}) \), can be found in [12]. Methods to prove that a weak solution of the problem (3.6) is also a generalized solution can be found in [6] (see Theorem 3.2, p. 383). By Theorem 3.2 of [3], \( u(x,t) \) is a unique generalized solution to (3.6). Hence, \( v = u \) and, since \( a, \tau \) and \( T_1 \) are arbitrary, the desired regularity of \( \partial u/\partial t \) follows.

In the next two lemmas we establish some preliminary estimations of the solution to CP, the proof of these estimations being based on scale of variables.

Lemma 3.1 If \( b = 0 \) and \( 0 < m < 1, \alpha > 0 \), then the solution \( u \) of the CP (1.1), (1.4) has the self-similar form (2.22), where the self-similarity function \( f \) satisfies (2.2). If \( u_0 \) satisfies (1.3) then the solution to the CP (1.1), (1.2) satisfies (2.1).

The first assertion of the Lemma and the resulting properties of the self-similarity function \( f \) are well known (see Remark 2.1). The proof of the second assertion coincides with that given in Lemma 3.1 from [4].

Lemma 3.2 Let \( u \) be a solution of the CP (1.1), (1.2) and let \( u_0 \) satisfy (1.3). Let one of the following conditions be valid:

(a) \( b > 0, \ 0 < \beta < m < 1, \ 0 < \alpha < 2/(m - \beta) \);

(b) \( b > 0, \ 0 < m < 1, \ \beta \geq m, \ \alpha > 0 \);
(c) \( b < 0, \ \beta \geq 1, \ 0 < m < 1, \ \alpha > 0. \)

Then \( u \) satisfies (2.1) with the same function \( f \) as in Lemma 3.1.

The proof for cases (a) and (b) coincides with the proof for cases (a) and (b) with \( b > 0 \) in Lemma 3.2 of [4]. The proof for (c) coincides (with some modifications) with the proof for case (b) with \( b < 0 \) in the Lemma 3.2 for [4]; namely, instead of zero boundary condition on the line \( x = -x_\varepsilon \) and \( x = -\kappa^{1/\alpha}x_\varepsilon \) (see (3.14 and (3.15) in [4]) we take

\[
\begin{align*}
  u_{\pm\varepsilon}(-x_\varepsilon, t) &= u(-x_\varepsilon, t), \quad 0 \leq t \leq \delta \\
  u_{\pm\varepsilon}^{\pm}(\kappa^{1/\alpha}x_\varepsilon, t) &= \kappa u\left(-x_\varepsilon, \kappa^{\alpha(m-1)/\alpha} t\right), \quad 0 \leq t \leq \kappa^{2+\alpha(1-m)/\alpha} \delta
\end{align*}
\]

which are used to imply (3.7) from [4]. Moreover, if \( \beta > 1 \) then to prove uniform boundedness of the sequence \( \{ u_{\pm\varepsilon}^{\pm}\} \) we choose

\[
g = (C + 1)(1 + x^2)^{\frac{\alpha}{2}} (1 - \nu t)^{\frac{1}{1-\beta}}, \quad x \in \mathbb{R}, \quad 0 \leq t \leq t_0 = \nu^{-1}/2,
\]

where \( \nu, h_* \) are chosen as in [4] and

\[
h(x) = (\beta - 1)(C + 1)^{m-1} \alpha m (1 + x^2)^{\frac{\alpha(m-1)}{2}} - 2 \left(1 + (\alpha m - 1)x^2\right),
\]

while if \( \beta = 1 \) we take

\[
g = (C + 1) \exp(\nu t) \left(1 + x^2\right)^{\alpha/2},
\]

where

\[
\nu = 1 + \max_{x \in \mathbb{R}} h^\dagger(x),
\]

\[
h^\dagger(x) = (C + 1)^{m-1} \alpha m (1 + x^2)^{\frac{\alpha(m-1)}{2}} - 2 \left(1 + (\alpha m - 1)x^2\right).
\]

**Lemma 3.3** Let \( u \) be a solution to the CP (1.1), (1.4) with \( b > 0, \ 0 < \beta < 1, \ m > \beta, \ \alpha = 2/(m - \beta), \ C > 0 \). Then the solution \( u \) has the self-similar form (2.4). There is a constant \( \ell_1 > 0 \) such that for arbitrary \( \ell \in (-\infty, -\ell_1] \) there exists \( \lambda > 0 \) such that (2.7) is valid. If \( 0 < C < C_* \) then

\[
0 < \lambda < C_* (-\ell)^{2/(m-\beta)}.
\]

(3.7)
If $C > C_*$ then $f_1(0) = A_1 > 0$, where $A_1$ depends on $m$, $\beta$, $C$ and $b$.

**Proof.** The first assertion of the lemma is known when $m \geq 1$ [5] (see also Lemma 3.3 of [4]).

The proof is similar if $\beta < m < 1$. If we consider a function

$$u_\kappa(x,t) = \kappa u \left( \kappa \frac{m-\beta}{2} x, \kappa^{\beta-1} t \right), \quad \kappa > 0,$$

(3.8)

it may easily be checked that this satisfies (1.1), (1.4). Since under the conditions of the lemma there exists a unique global solution to (1.1), (1.4) we have

$$u(x,t) = \kappa u \left( \kappa - \frac{m-\beta}{2} x, \kappa^{\beta-1} t \right), \quad \kappa > 0.$$  

(3.9)

If we choose $\kappa = t^{1/(1-\beta)}$ then (3.9) implies (2.4) with $f_1(\zeta) = u(\zeta, 1)$.

To prove the second assertion of the lemma, take an arbitrary $x_1 < 0$. Since $u$ is continuous, there exists $\delta_1 > 0$ such that

$$\left( \frac{C}{2} \right) \left( -x_1 \right)^{\frac{2}{m-\beta}} \leq u(x, \delta) \quad \text{for} \quad \delta \in [0, \delta_1].$$

(3.10a)

If $C \in (0, C_*)$ then we also choose $\delta_1 > 0$ such that

$$u(x, \delta) < C_*( -x_1)^{\frac{2}{m-\beta}} \quad \text{for} \quad \delta \in [0, \delta_1].$$

(3.10b)

Choosing $\kappa = (t/\delta)^{1/(1-\beta)}$ in (3.9) and then taking

$$x = -\ell t^{\frac{m-\beta}{2(1-\beta)}}, \quad \ell = \ell(\delta) = x_1 \delta^{-\frac{m-\beta}{2(1-\beta)}}, \quad \delta \in (0, \delta_1],$$

we obtain (2.7) with

$$\ell_1 = -x_1 \delta_1^{-\frac{m-\beta}{2(1-\beta)}}, \quad \lambda = \lambda(\delta) = \delta^{\frac{1}{\beta-1}} u(x_1, \delta), \quad \delta \in (0, \delta_1].$$

If $0 < C < C_*$, then (3.7) follows from (3.10b). Let $C > C_*$; to prove that $f_1(0) = A_1 > 0$ it is enough to prove that there exists a $t_0 > 0$ such that

$$u(0, t_0) > 0.$$  

(3.11)

If $m \geq 1$, (3.11) is a known result [5] (see also Lemma 3.3 of [4]). To prove (3.11) when $\beta < m < 1$, consider the function

$$g(x,t) = C_1(-x + t)^{2/(m-\beta)},$$

where $C_1$ depends on $m$, $\beta$, $C$, and $b$. [Note: The text contains a typographical error; $C_1$ should not be present in the exponent.]
where \( C_1 \in (C_*, C) \). If \( x < t \) we have
\[
Lg = bg^\beta S, \quad S = 1 - (C_1/C_*)^{m-\beta} + \left( b(m - \beta) \right)^{-1} C_1^{1-\beta} (-x + t)^{\frac{2-m-\beta}{m-\beta}}.
\]
We can choose \( x_1 < 0 \) and \( t_1 > 0 \) such that
\[
S \leq 0 \quad \text{if} \quad x_1 \leq x \leq t, \quad 0 \leq t \leq t_1.
\]
Since \( u \) is continuous, we can also choose \( t_1 > 0 \) sufficiently small that
\[
g(x_1, t) \leq u(x_1, t) \quad \text{for} \quad 0 \leq t \leq t_1.
\]
Moreover
\[
g(x, 0) \leq u_0(x) \quad \text{for} \quad x \geq x_1.
\]
Applying Lemma 2.1 of [3] and Theorem 3.4 of [3] we have
\[
u(x, t) \geq g(x, t) \quad \text{for} \quad x \geq x_1, \quad 0 \leq t \leq t_1,
\]
which implies (3.11). The lemma is proved.

**Lemma 3.4** Let \( u \) be a solution to the CP (1.1)–(1.3) with \( b > 0, \ 0 < \beta < 1, \ m > \beta, \alpha = 2/(m - \beta), \ C > 0 \). Then for arbitrary \( \ell \in (-\infty, -\ell_1] \) we have
\[
u \left( \ell t^{\frac{m-\beta}{m(1-\beta)}}, t \right) \sim \lambda t^{\frac{1}{1-\beta}} \quad \text{as} \quad t \to 0+,
\]
where \( \ell_1 > 0, \ \lambda > 0 \) are the same as in Lemma 3.3 and if \( 0 < C < C_* \) then (3.7) is also valid. If \( C > C_* \) then \( u \) satisfies
\[
u(0, t) \sim A_1 t^{\frac{1}{1-\beta}} \quad \text{as} \quad t \to 0+,
\]
where \( A_1 = f_1(0) > 0 \) (see Lemma 3.3). The results of Lemma 3.4 may be proved by localization of the proof given in Lemma 3.3, exactly as similar local results were proved in Lemma 3.1 of [4].

**Lemma 3.5** Let \( u \) be a solution to the CP (1.1)–(1.3) with \( b > 0, \ 0 < \beta < 1, \ m > \beta, \alpha > 2/(m - \beta), \ C > 0 \). Then for arbitrary \( \ell_1 \) (see (2.10)), the asymptotic formula (2.11) is valid, with \( x = \eta_\ell(t) = -\ell t^{1/\alpha(1-\beta)} \).

The proof of Lemma 3.5 coincides with the proof of Lemma 3.4 from [4] (see Remark 3.2 from [4]).
4 Proofs of the Main Results

In this section we prove the main results described in Section 2.

I. $b \neq 0$

(1) Suppose that $0 < \beta < m$, $0 < \alpha < 2/(m - \beta)$. The asymptotic estimations (2.1) and (2.2) follow from Lemma 3.2. Take an arbitrary sufficiently small number $\varepsilon > 0$; from (2.1) it follows that there exists a number $\delta_1 = \delta_1(\varepsilon) > 0$ such that

$$ (A_0 - \varepsilon)t^{\alpha/(2 + \alpha(1 - m))} \leq u(0, t) \leq (A_0 + \varepsilon)t^{\alpha/(2 + \alpha(1 - m))}, \quad 0 \leq t \leq \delta_1, \quad (4.1) $$

where $A_0 = f(0) > 0$. Consider a function

$$ g(x, t) = t^{1/(1 - \beta)}f_1(\zeta), \quad \zeta = xt^{-\frac{m - \beta}{m - \beta}}. \quad (4.2) $$

We then have

$$ Lg = t^{\beta/(1 - \beta)}L^0 f_1, \quad (4.3a) $$

$$ L^0 f_1 = \frac{1}{1 - \beta}f_1 - (f_1^{m})'' - \frac{m - \beta}{2(1 - \beta)}\zeta f_1' + bf_1^\beta. \quad (4.3b) $$

For the function $f_1$ we take

$$ f_1(\zeta) = C_0(\zeta_0 - \zeta)^{\frac{2}{m - \beta}}, \quad 0 < \zeta < +\infty, $$

where $C_0$, $\zeta_0$ are some positive constants. From (4.3b) we then have

$$ L^0 f_1 = bC_0^\beta(\zeta_0 - \zeta)^{\frac{2\beta}{m - \beta}} \left\{ 1 - \left( \frac{C_0}{C_1} \right)^{\frac{m - \beta}{2(1 - \beta)}} + \frac{C_0^{1 - \beta} \zeta \zeta_0 - \zeta_1^{2 - m - \beta}}{b(1 - \beta)\zeta_1^{2 - m - \beta}} \right\}. \quad (4.4) $$

To prove a lower estimation we take $C_0 = C_1$, $\zeta_0 = \zeta_1$ (see Appendix I). Then we have

$$ L^0 f_1 \leq bC_1^\beta(\zeta_0 - \zeta)^{\frac{2\beta}{m - \beta}} \left\{ 1 - \left( \frac{C_1}{C_1} \right)^{\frac{m - \beta}{2(1 - \beta)}} + \frac{C_1^{1 - \beta} \zeta \zeta_1 - \zeta_1^{2 - m - \beta}}{b(1 - \beta)\zeta_1^{2 - m - \beta}} \right\} = 0. \quad (4.5) $$

From (4.3) it follows that

$$ Lg \leq 0 \quad \text{for} \quad 0 < x < \zeta_1 t^{\frac{m - \beta}{2(1 - \beta)}}, \quad 0 < t < +\infty, \quad (4.6a) $$

$$ Lg = 0 \quad \text{for} \quad x > \zeta_1 t^{\frac{m - \beta}{2(1 - \beta)}}, \quad 0 < t < +\infty. \quad (4.6b) $$
Lemma 2.1 of [3] implies that \( g \) is a subsolution of equation (1.1) in \( \{(x, t) : x > 0, \ t > 0\} \).

Since \( 1/(1 - \beta) > \alpha/(2 + \alpha(1 - m)) \), it follows from (4.1) that there exists a \( \delta_2 > 0 \), which does not depend on \( \varepsilon \), such that

\[
g(0, t) \leq u(0, t) \quad \text{for} \quad 0 \leq t \leq \delta_2. \tag{4.7a}
\]

We also have

\[
u(x, 0) = g(x, 0) = 0 \quad \text{for} \quad 0 \leq x < +\infty. \tag{4.7b}
\]

Now we can fix a particular value of \( \varepsilon = \varepsilon_0 \) and take \( \delta = \min(\delta_1, \delta_2) \). From (4.6), (4.7) and Comparison Theorem 2.4 of [3], the left-hand side of (2.3) follows. To prove an upper estimation, first we use the following rough estimation (see (2.16))

\[
u(x, t) \leq Dt_{1 \to m}^{-\frac{1}{m-1}} \cdot x_{m-1}^{-\frac{1}{m-1}} \quad 0 < x < +\infty, \quad 0 < t < +\infty. \tag{4.8}
\]

The estimation (4.8) is obvious, since by Comparison Theorem 2.4 of [3] \( u(x, t) \) may be upper estimated by the solution of equation (1.1) with \( b = 0 \). Using (4.8), we can now establish a more accurate estimation (2.3). For that, consider a function \( g \) with \( C_0 = C_* \), \( \zeta_0 = \zeta_2 \) in \( G_{\ell_0, \delta} \), where

\[
G_{\ell, \delta} = \left\{(x, t) : \zeta_\ell(t) = \ell t_{\frac{m-\beta}{1-m}} \leq x < +\infty, \ 0 < t < \delta\right\}.
\]

From (4.3), (4.4) it follows that

\[
Lg \geq 0 \quad \text{for} \quad x < \zeta_2 t_{\frac{m-\beta}{1-m}}, \quad 0 < t < +\infty, \tag{4.9a}
\]

\[
Lg = 0 \quad \text{for} \quad x > \zeta_2 t_{\frac{m-\beta}{1-m}}, \quad 0 < t < +\infty. \tag{4.9b}
\]

Moreover, from (4.8) we have

\[
u(\zeta_{\ell_0}(t), t) \leq DL_{\ell_0}^{-\frac{2}{m-1}} t^{-\frac{1}{m-1}} = C_*(\zeta_2 - \ell_0) t_{\frac{2}{m-1}}^{-\frac{1}{m-1}} = g(\zeta_{\ell_0}(t), t) \quad \text{for} \quad 0 \leq t \leq \delta \tag{4.10}
\]

By applying the Comparison Theorem 2.4 of [3] in \( G_{\ell_0, \delta} \), the right-hand side of (2.2) follows from (4.9), (4.10) and (4.7b).
If \( u_0 \) is defined as in (1.4), then the CP (1.1), (1.4) has a global solution and from a Comparison Theorem it follows that the solution may be globally upper estimated by the solution to the CP (1.1), (1.4) with \( b = 0 \). Hence (4.8), (4.10) and the right-hand side of (2.2) are valid for \( 0 < t < +\infty \).

(2) \( b > 0, \ 0 < \beta < 1, \ \beta < m < 2 - \beta, \ \alpha = 2(m - \beta)^{-1} \)

First, assume that \( u_0 \) is defined by (1.4). As mentioned in Section 3 (for \( m < 1 \)) and earlier in Section 3 of [4] (for \( m \geq 1 \)) the problem (1.1), (1.4) has a unique global solution and for this class of initial data a comparison principle is valid. The self-similar form (2.4) follow from Lemma 3.3. The proof of the estimation (2.6a) when \( C > C_* \) (and of the corresponding local one when \( u_0 \) satisfies (1.3)) completely coincides with the proof given in [4] for the case \( 1 \leq m < 2 - \beta \) (see (2.11) in [4]).

Let \( 0 < C < C_* \). Formula (2.7) follows from Lemma 3.3. The proof of the right-hand side of (2.6a) (and of the corresponding local one when \( u_0 \) satisfies (1.3)) completely coincides with the proof given in [4] for the case \( 1 \leq m < 2 - \beta \) (see (2.14) in [4]). To prove the left-hand side of (2.6a), consider a function \( g \) from (4.2) with

\[
f_1(\zeta) = C_*(-\zeta_5 - \zeta \frac{2}{2 + m - \beta}, \ -\infty < \zeta < +\infty
\]

and hence (see (4.3), (4.4))

\[
L g \leq 0 \quad \text{in} \quad G_{-\ell_1, \infty}.
\]

Moreover, we have

\[
u \left( -\ell_1 t^{\frac{m - \beta}{m - 2}} \right, t \right) = \lambda t^{\frac{1}{2 - \beta}} = C_* (\ell_1 - \zeta_5) \frac{2}{2 + m - \beta} t^{\frac{1}{2 - \beta}} = g \left( -\ell_1 t^{\frac{m - \beta}{m - 2}}, t \right), \\
0 \leq t < +\infty,
\]

\[
u(x, 0) = g(x, 0) = 0, \quad 0 \leq x \leq x_0,
\]

\[
u(x_0, t) = g(x_0, t) = 0, \quad 0 \leq t < +\infty,
\]

where \( x_0 > 0 \) is an arbitrary fixed number. By using (4.11) and (4.12), we can apply
Comparison Theorem 3.4 from [3] in

\[ G'_{-\ell_1,\infty} = G_{-\ell_1,\infty} \cap \{ x < x_0 \} . \]

Since \( x_0 > 0 \) is an arbitrary number the desired lower estimation from (2.6a) follows.

Suppose that \( u_0 \) satisfies (1.3) with \( \alpha = 2/(m - \beta) \), \( 0 < C < C_* \). Then from (3.12) it follows that for arbitrary sufficiently small \( \varepsilon > 0 \) there exists a number \( \delta = \delta(\varepsilon) > 0 \) such that

\[ (\lambda - \varepsilon)t^{\frac{1}{1+m}} \leq u(-\ell_1t^{\frac{m-\beta}{2(1-\beta)}},t) \leq (\lambda + \varepsilon)t^{\frac{1}{1+m}}, \quad 0 \leq t \leq \delta. \]

Using this estimation, the left-hand side of (2.6a) may be established locally in time. The proof completely coincides with the proof given above for the global estimations, except that \( \lambda \) should be replaced by \( \lambda - \varepsilon \). As in [4], (2.4) and (2.6a) easily imply (2.5) and (2.6b).

(3) Suppose that \( b > 0 \), \( 0 < \beta < 1 \), \( \beta < m < 2 - \beta \), \( \alpha > 2/(m - \beta) \), \( C > 0 \). The asymptotic estimation (2.11) follows from Lemma 3.5. The proof of the asymptotic estimation (2.10) coincides with the proof given in [4]. In particular, the estimations (4.19) and (4.20) from [4] are true in this case as well.

(4) Suppose that \( b > 0 \), \( 0 < \beta = m < 1 \), \( \alpha > 0 \). The asymptotic estimations (2.1) and (2.2) follow from Lemma 3.2. From (2.1), (4.1) follows, where we fix a particular value of \( \varepsilon = \varepsilon_0 \). For arbitrary given \( M > 0 \), the function \( g(x,t) = t^{\frac{1}{1+m}}\phi_M(x) \) is a solution of equation (1.1). Since \((1 - m)^{-1} > \alpha(2 + \alpha(1 - m))^{-1}\), there exists \( \delta = \delta(M) > 0 \) such that

\[ u(0,t) \geq g(0,t), \quad 0 \leq t \leq \delta. \]

Hence, from the Comparison Theorem 2.4 of [3], the left-hand side of (2.12) follows. Let us prove the right-hand side of (2.12). As it was mentioned in Section 2, the right-hand
side of (2.12) is valid for $0 < t < +\infty$ if the initial data $u_0$ from (1.2) vanishes for $x \geq 0$.

Take an arbitrary $M > 0$ and consider a function

$$g_M(x, t) = (t + M^{-\gamma})^{\frac{1}{1-m}} \phi_M(x), \quad \gamma \in (0, 1 - m).$$

Since $u(0, t)$ is bounded in an arbitrary bounded segment $0 \leq t \leq T$, we have

$$g_M(0, t) \geq M^{1-\gamma/(1-m)} \geq u(0, t) \quad \text{for} \quad 0 \leq t \leq T(M),$$

where $T(M) \to +\infty$ as $M \to +\infty$. Since $g_M$ is a solution of the equation (1.1), from the Comparison Theorem 2.4 of [3] it follows that

$$u(x, t) \leq g_M(x, t) \quad \text{for} \quad 0 \leq x < +\infty, \quad 0 \leq t \leq T(M).$$

In the limit as $M \to +\infty$, we derive the right-hand side of (2.12) for $0 \leq t < +\infty$. From (2.12) it follows that for $\forall$ fixed $t \in (0, \delta]$

$$L g = t^{m \alpha - 2}/(2 + \alpha(1-m)) L_t f$$

$$L_t f = \frac{\alpha}{2 + \alpha(1-m)} f - \frac{1}{2 + \alpha(1-m)} \xi f' - (f^m)' + bt^{2-\alpha(m-1)} f^\beta.$$
As a function \( f \) we take
\[
f(\xi) = C_0(\xi_0 + \xi)^{-\gamma_0}, \quad 0 \leq \xi < +\infty, \quad (4.15)
\]
where \( C_0, \xi_0, \gamma_0 \) are some positive constants. Taking \( \gamma_0 = 2/(1 - m) \) from (4.14b) we have
\[
\mathcal{L}_t f = (2 + \alpha(1 - m))^{-1} C_0(\xi_0 + \xi)^{\frac{2}{m}}\frac{2 + \alpha(\beta - m)}{2 + \alpha(1 - m)} C_0^{-1}(\xi_0 + \xi)^{\frac{2(1 - \beta)}{1 - m}}.
\]
\[\quad R(\xi) = \left[\alpha - 2m(m + 1)(2 + \alpha(1 - m))(1 - m)^{-2}C_0^{-1}\right] + 2(1 - m)^{-1}\xi(\xi_0 + \xi)^{-1}. \quad (4.16a)
\]
To prove an upper estimation we take \( C_0 = C_6, \xi_0 = \xi_2 \) (see Appendix I). Then we have
\[
R(\xi) \geq \alpha(\mu_b - 1)\mu_b^{-1} \quad \text{for} \quad \xi \geq 0. \quad (4.17)
\]
From (4.16), (4.17) it follows that
\[
\mathcal{L}_t f \geq 0 \quad \text{for} \quad \xi \geq 0, \quad 0 \leq t \leq \delta_2,
\]
where \( \delta_2 = \delta_1 \) if \( b > 0, \delta_2 = \min(\delta_1, \delta_3) \) if \( b < 0 \) and
\[
\delta_3 = \left[\alpha \varepsilon (A_0 + \varepsilon)^{1 - \beta}(-(2 + \alpha(1 - m))(1 + \varepsilon))^{-1}\right]^\frac{2 + \alpha(1 - m)}{2 + \alpha(\beta - m)}. \]
Hence, from (4.14) we have
\[
Lg \geq 0 \quad \text{for} \quad 0 \leq x < +\infty, \quad 0 < t \leq \delta_2. \quad (4.18)
\]
From (4.1) and Comparison Theorem 3.3 of [4], the right-hand side of (2.15) follows with \( \delta = \delta_2 \). To prove a lower estimation in this case we take \( C_0 = C_5, \xi_0 = \xi_1 \). If \( b > 0 \) and \( \beta < (3 - m)/2 \) we derive from (4.16) that
\[
R(\xi) \leq \alpha + 2(1 - m)^{-1} - 2m(m + 1)(2 + \alpha(1 - m))(1 - m)^{-2}C_5^{-1}
\]
\[
= -(2 + \alpha(1 - m))(1 - \varepsilon)^{-1} \varepsilon, \quad (4.19a)
\]
\[
\mathcal{L}_t f \leq 0 \quad \text{for} \quad \xi \geq 0, \quad 0 \leq t \leq \delta_4. \quad (4.19b)
\]
where $\delta_4 = \min(\delta_1, \delta_5)$ and

$$\delta_5 = \left[ (A_0 - \varepsilon)^{1-\beta} (b(1-m)(1-\varepsilon))^{-1} \varepsilon \right]^{\frac{2+\alpha(1-m)}{2+\alpha(\beta-m)}}.$$  

From (4.19) it follows that

$$Lg \leq 0 \text{ for } 0 \leq x < +\infty, 0 < t \leq \delta_4. \quad (4.20)$$

If either $b > 0$, $\beta \geq (3-m)/2$ or $b < 0$, $\beta \geq 1$, from (4.16) we have

$$Ltf = (2 + \alpha (1-m))^{-1} C_5 (\xi + \xi)^{\frac{3m}{m-1}}$$

$$\left[ R_1(\xi) + b (2 + \alpha (1-m)) t^{\frac{2+\alpha(1-m)}{2+\alpha(\beta-m)}} C_5^{\beta-1} (\xi + \xi)^{\frac{3m-2\beta}{m-1}} \right], \quad (4.21a)$$

$$R_1(\xi) = \left[ \alpha - 2m(m+1) (2 + \alpha (1-m)) (1-m)^{-2} C_5^{m-1} \right] (\xi + \xi)$$

$$+ 2(1-m)^{-1} \xi = -2(1-m)^{-1} \xi_1, \quad (4.21b)$$

which again imply (4.19b), where $\delta_4 = \delta_1$ if $b < 0$, $\delta_4 = \min(\delta_1, \delta_5)$ if $b > 0$ and

$$\delta_5 = \left[ 2 (b (2 + \alpha (1-m)) (1-m))^{-1} (A_0 - \varepsilon)^{1-\beta} \right]^{\frac{2+\alpha(1-m)}{2+\alpha(\beta-m)}}.$$  

As before, (4.20) follows from (4.21b). From (4.1) and Comparison Theorem 2.4 of [3], the left-hand side of (2.15) follows with $\delta = \delta_4$. Thus we have proved (2.16) with $\delta = \min(\delta_2, \delta_4)$.

Let $b > 0$, $\beta \geq 1$. The upper estimation (2.16) is an easy consequence of Comparison Theorem 2.4 of [3], since the right-hand side of it is a solution of the equation (1.1) with $b = 0$. Let $b > 0$ and $\beta \geq (3-m)/2$. Now we can fix a particular value of $\varepsilon = \varepsilon_0$ and take $\delta = \delta(\varepsilon_0) > 0$ in (2.15). Then from the left-hand side of (2.15) and (2.16), the asymptotic result (2.17) follows. However, if $b > 0$, $1 \leq \beta < (3-m)/2$, from (2.15) and (2.16) it follows that for $\forall$ fixed $t \in (0, \delta(\varepsilon))$

$$D(1-\varepsilon)^{\frac{1}{m-1}} \leq \liminf_{x \to +\infty} ut^{\frac{1}{m-1}} x^{\frac{2}{m-1}} \leq \limsup_{x \to +\infty} ut^{\frac{1}{m-1}} x^{\frac{2}{m-1}} \leq D,$$

which easily implies (2.18) in view of arbitrariness of $\varepsilon$.

We now let $b < 0$, $\beta \geq 1$ and prove (2.19). Consider a function

$$\bar{g}(x, t) = D(1-\varepsilon)^{\frac{1}{m-1}} t^{\frac{1}{m-1}} x^{\frac{2}{m-1}}.$$
in $G = \{(x,t) : \mu t^{\frac{1}{2+\alpha(1-m)}} < x < +\infty, \ 0 < t \leq \delta\}$, where $\mu$ is defined as in (2.19). Let $g(x, t) = \bar{g}(x, t)$ for $(x, t) \in \bar{G} \setminus (0,0)$ and $g(0,0) = 0$. We have

$$Lg = D(1-m)^{-1}(1-\varepsilon)\frac{1}{m-1} t^{\frac{m}{m-1}} x^{\frac{2}{m-1}} S \quad \text{in} \quad G,$$

$$S = \varepsilon + b(1-m)D^{\beta-1}(1-\varepsilon)\frac{\beta-1}{m-1} t^{\frac{\beta-m}{m-1}} x^{\frac{2(\beta-1)}{m-1}}.$$

We then derive

$$S \geq \varepsilon + b(1-m)D^{\beta-1}(1-\varepsilon)\frac{\beta-1}{m-1} \mu^{\frac{2(\beta-1)}{2+\alpha(1-m)}} \quad \text{in} \quad G.$$

Hence,

$$S \geq 0 \quad \text{in} \quad G, \quad \text{for} \quad \delta \in (0, \delta_0],$$

$$\delta_0 = \left[(-b(1-m))^{-1} D^{1-\beta}(1-\varepsilon)\frac{1-\beta}{m-1} \mu^{\frac{2(1-\beta)}{2+\alpha(1-m)}} \varepsilon^{\frac{2+\alpha(1-m)}{2+\alpha(1-m)}} \right],$$

which implies

$$Lg \geq 0 \quad \text{in} \quad G. \quad \quad (4.22a)$$

Moreover, we have

$$g|_{x=\mu t^{\frac{1}{2+\alpha(1-m)}}} = (A_0 + \varepsilon) t^{\frac{\alpha}{2+\alpha(1-m)}} \quad \text{for} \quad 0 \leq t \leq \delta.$$ 

From (2.15) it follows that

$$u|_{x=\mu t^{\frac{1}{2+\alpha(1-m)}}} \leq C_0 (\xi_2 + \mu)^{\frac{2}{m-1}} t^{\frac{\alpha}{2+\alpha(1-m)}} \leq (A_0 + \varepsilon) t^{\frac{\alpha}{2+\alpha(1-m)}} \quad \text{for} \quad 0 \leq t \leq \delta.$$ 

Therefore we have

$$g \geq u \quad \text{on} \quad \bar{G} \setminus G, \quad \quad (4.22b)$$

From (4.22) and Comparison Theorem 2.4 of [3], the desired estimation (2.19) follows.

Since $\varepsilon > 0$ is arbitrary, from the left-hand side of (2.15) and (2.19) the asymptotic result (2.17) follows as before.

Let $b > 0$, $m < \beta < 1$. The left-hand side of (2.20) may be proved as the left-hand side
of (2.3a) was earlier. The only difference is that we take $f_1(\zeta) = C_*(1 - \varepsilon)(\zeta_8 + \zeta)^{2\beta}x^{\beta-m}$ in (4.2), (4.3). The right-hand side of (2.20) is almost trivial, since $C_*x^{\beta-m}$ is a stationary solution of the equation (1.1). The important point in (2.20) is that $\delta > 0$ does not depend on $\varepsilon > 0$. This is clear from the analysis involved in the proof of the similar estimation (2.3a). From (2.20) it follows that $\forall$ fixed $t \in (0, \delta]$, we have

$$C_*(1 - \varepsilon) \leq \liminf_{x \to +\infty} u x^{\beta-m} \leq \limsup_{x \to +\infty} u x^{\beta-m} \leq C_*.$$  

Since $\varepsilon > 0$ is arbitrary, (2.21) easily follows.

II. $b = 0$

First assume that $u_0$ is defined by (1.4). The self-similar form (2.22) and the formula (2.2) follow from Lemma 3.1. To prove (2.23), consider a function $g$ from (4.13), which satisfies (4.14) with $b = 0$. As a function $f$ we take (4.15) with $\gamma_0 = 2/(1 - m)$. Then we derive (4.16) with $b = 0$. To prove an upper estimation we take $C_0 = C_7$, $\xi_0 = \xi_4$ and from (4.16b) we have

$$R(\xi) \geq \left[ \alpha - 2m(\alpha + 1) (2 + \alpha(1 - m))(1 - m)^{-2}C_7^{\beta-m-1} \right] = 0,$$

which implies (4.18) with $\delta_2 = +\infty$. As before, from (4.18) and Comparison Theorem 2.4 of [3], the right-hand side of (2.23) follows. The left-hand side of (2.25) may be established similarly if we take $C_0 = D$, $\xi_0 = \xi_3$.

To prove the estimation (2.16), consider

$$g_\mu(x,t) = D(t + \mu)^{1/(1-m)}(x + \mu)^{2/(m-1)}, \quad \mu > 0,$$

which is a solution of equation (1.1) for $x > 0$, $t > 0$. Since

$$g_\mu(0,t) \geq D\mu^{1/(m-1)} \geq u(0,t) \quad \text{for} \quad 0 \leq t \leq T(\mu) = \left[ DA_0^{-1} \mu^{1/(m-1)} \right]^{2+\alpha(1-m)},$$

the Comparison Theorem 2.4 of [3] implies

$$u(x,t) \leq g_\mu(x,t) \quad \text{for} \quad 0 < x < +\infty, \quad 0 \leq t \leq T(\mu).$$

In the limit as $\mu \to 0+$, we can easily derive (2.16). Finally, from (2.23) and (2.16) it easily follows that for arbitrary fixed $0 < t < +\infty$ the asymptotic formula (2.17) is valid. If $u_0$ satisfies
(1.3) with $\alpha > 0$, then (2.1) and (4.1) follow from Lemma 3.1. Similarly, we can then prove that for arbitrary sufficiently small $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that (2.23) is valid for $0 \leq t \leq \delta(\varepsilon)$, except that in the left-hand side (respectively in the right-hand side) of (2.23) the constant $A_0$ is replaced by $A_0 - \varepsilon$ (respectively by $A_0 + \varepsilon$). Then we can fix a particular value of $\varepsilon = \varepsilon_0$ and let $\delta = \delta(\varepsilon_0) > 0$. Obviously, from the local analog of (2.23) and (2.16) it follows that, for arbitrary fixed $t \in (0, \delta]$, the asymptotic formula (2.17) is valid.

Acknowledgements

This paper was mostly accomplished while the author was visiting the Department of Theoretical Mechanics at Nottingham University. This visit was supported by the Royal Society of Britain and by the British Council. The author thanks Professor J R King for his invitation and for many discussions. The final draft of this paper was written while the author was an Alexander von Humboldt Research Fellow at the University of Paderborn, Germany.

References


Appendix I

We give here explicit values of the constants used in Section 2 in the outline of the results and later in Section 4 during the proof of these results.

I (1) $0 < \beta < m$, $0 < \alpha < 2(m - \beta)$

$$C_* = \left[ b(m - \beta)^2 / \left(2m(m + \beta)\right) \right]^{1/(m - \beta)}, \quad C_1 = \left(1 - \beta / (1 - m)\right)^{1/(m - \beta)} C_*,$$

$$\zeta_1 = \frac{m - \beta}{(1 - \beta)^2} (2m)^{\frac{1}{2}} \left(1 - \beta\right)^{\frac{1}{2}} (m - \beta)^{\frac{1}{2}} C_2,$$

$$\zeta_2 = b^{\frac{m - \beta}{2(1 - \beta)}} (m - \beta)^{\frac{1}{2}} (1 + m)^{\frac{1}{2}} (1 - m)^{\frac{1}{2}} (1 - \beta)(m - \beta)^{-1},$$

$$\ell_0 = \frac{m - \beta}{1 - \beta} \zeta_2.$$

(2) $0 < \beta < m$, $\alpha = 2(m - \beta)^{-1}$

$$\zeta_3 = A_1^{\frac{m - \beta}{2}} \left[ (m - \beta)^{\frac{1}{2}} \left(1 + b(1 - \beta)A_1^{\beta - 1}\right)^{\frac{1}{2}} (2m + \beta) \right]^{\frac{1}{2}} (m - \beta)^{-1},$$

$$\zeta_4 = (A_1 / C_*)^{\frac{m - \beta}{2}}, \quad C_2 = A_1^{\frac{\beta - m}{2}},$$

$$\zeta_5 = \ell_1 - (\lambda / C_*)^{\frac{m - \beta}{2}} > 0 \quad \text{(see Lemma 3.3 and (3.7))},$$

$$\ell_2 = C_3^{\frac{\beta - m}{4}} b(1 - \beta)(\delta_\Gamma)^{-1} \left(1 - \delta_\Gamma - (1 - \delta_\Gamma)^{-1} (C/C_*)^{m - \beta} \right) \left[ m - \beta \right]^{\frac{m - \beta}{2(1 - \beta)}},$$

$$\zeta_6 = \delta_\Gamma \ell_2, \quad \Gamma = 1 - (C/C_*)^{\frac{m - \beta}{2}}, \quad C_3 = C(1 - \delta_\Gamma)^{\frac{m - \beta}{2}},$$

where $\delta_\Gamma \in (0, 1)$ satisfies

$$g(\delta_\Gamma) = \max_{[0,1]} g(\delta), \quad g(\delta) = \delta^{\frac{2 - m - \beta}{m - \beta}} \left[ 1 - \delta \Gamma - (1 - \delta \Gamma)^{-1} (C/C_*)^{m - \beta} \right].$$
(5) \( \beta > m \)

\[
D = \left[2m(m+1)(1-m)^{-1}\right]^{\frac{1}{1-m}},
\]

\[
\xi_1 = (A_0 - \varepsilon)^{\frac{m-1}{2}} (1-\varepsilon)^{\frac{1}{2}} D^{\frac{1-m}{2}} \text{ if } b > 0, \ 1 \leq \beta < (3-m)/2,
\]

\[
\xi_1 = (A_0 - \varepsilon)^{\frac{m-1}{2}} D^{\frac{1-m}{2}} \text{ if either } b > 0, \ \beta \geq (3-m)/2 \text{ or } b < 0, \ \beta \geq 1,
\]

\[
C_5 = (A_0 - \varepsilon)^{\frac{2}{1-m}} \xi_1^{\frac{1}{1-m}},
\]

\[
A_0 = f(0) > 0 \quad \text{(see (2.2) and Lemma 3.1)},
\]

\[
\xi_2 = (A_0 + \varepsilon)^{\frac{m-1}{2}} \left[ \frac{2m(m+1)(2 + \alpha(1-m)\mu_b)}{\alpha(1-m)^2} \right]^{\frac{1}{2}},
\]

\[
C_6 = \left[ \frac{2m(m+1)(2 + \alpha(1-m)\mu_b)}{\alpha(1-m)^2} \right]^{\frac{1}{1-m}},
\]

\[
\mu_b = 1 \text{ if } b > 0, \ \mu_b = 1 + \varepsilon \text{ if } b < 0,
\]

\[
\zeta_8 = \left[ b(1-\beta)C_5^{\beta-1}(1-\varepsilon)^{\beta-1}\left(1-(1-\varepsilon)^{m-\beta}\right) \right]^{\frac{m-\beta}{2\alpha-1}}.
\]

II \( b = 0 \)

\[
\xi_3 = (A_0/D)^{\frac{m-1}{2}}, \quad \xi_4 = \xi_3 \left(1 + 2/\alpha(1-m)\right)^{\frac{1}{2}},
\]

\[
C_7 = D \left(1 + 2/\alpha(1-m)\right)^{\frac{1}{1-m}}.
\]