First Boundary Value Problem for the Diffusion Equation

I. Iterated Logarithm Test for the Boundary Regularity and Solvability

UGUR G. ABDULLA

Max-Planck Institute for Mathematics in the Sciences,
Leipzig 04103, Germany
1 Introduction

This paper establishes a precise sufficient condition for the regularity of a boundary point of an arbitrary open subset of \( \mathbb{R}^{N+1} \) \((N \geq 2)\), and for the solvability of the first boundary value problem (FBVP) for the diffusion equation in general domains.

Let \( \Omega \subset \mathbb{R}^{N+1} \) \((N \geq 2)\) denote any bounded open subset and \( \partial \Omega \) its topological boundary. We write a typical point as \( z = (x, t) = (x_1, \overline{x}, t), x = (x_1, \overline{x}) \in \mathbb{R}^N, \overline{x} = (x_2, \ldots, x_N) \in \mathbb{R}^{N-1}, t \in \mathbb{R} \). For a given point \( z_0 = (x_0, t_0) \) and a positive number \( \epsilon \) define the cylinder

\[
Q(z_0, \epsilon) = \{ z : |x - x_0| < \epsilon, t_0 - \epsilon < t < t_0 \},
\]

We split \( \partial \Omega \) as \( \partial \Omega = \mathcal{P}\Omega \cup \mathcal{D}\Omega \), where \( \mathcal{P}\Omega \) is the set of all points \( z_0 \in \partial \Omega \) such that for any \( \epsilon > 0 \), the cylinder \( Q(z_0, \epsilon) \) contains points not in \( \Omega \). The set \( \mathcal{P}\Omega \) is called the parabolic boundary of \( \Omega \). The set \( \mathcal{D}\Omega \) is naturally called the top boundary of \( \Omega \). We split also \( \mathcal{P}\Omega \) as \( \mathcal{P}\Omega = \mathcal{S}\Omega \cup \mathcal{B}\Omega \), where \( \mathcal{B}\Omega \) is the set of all points \( z_0 \in \mathcal{P}\Omega \) such that for some \( \epsilon > 0 \), the cylinder \( Q(z_0, \epsilon) \) lies outside \( \Omega \). The set \( \mathcal{B}\Omega \) is naturally called the bottom boundary of \( \Omega \), while \( \mathcal{S}\Omega \) will be called the lateral boundary of \( \Omega \).

For \( u \in C^{2,1}_{x,t}(\Omega) \), we define the diffusion operator

\[
\mathbf{D}u = u_t - \Delta u = u_t - \sum_{i=1}^{N} u_{x_ix_i}, \ z \in \Omega.
\]

A function \( u \in C^{2,1}_{x,t}(\Omega) \) is called parabolic in \( \Omega \) if \( \mathbf{D}u = 0 \) for \( z \in \Omega \). Let \( f : \mathcal{P}\Omega \to \mathbb{R} \) be a bounded function. First boundary value problem (FBVP) may be formulated as follows:

Find a function \( u \) which is parabolic in \( \Omega \) and satisfies the conditions

\[
f_* \leq u_* \leq u^* \leq f^* \text{ for } z \in \mathcal{P}\Omega,
\]

where \( f_* \), \( u_* \) (or \( f^*, u^* \)) are lower (or upper) limit functions of \( f \) and \( u \) respectively.

In particular, if \( f \in C(\mathcal{P}\Omega; \mathbb{R}) \), from (1.1) it follows that \( u \) takes continuously the given values of \( f \) on \( \mathcal{P}\Omega \). The strategy for solving the FBVP may be well expressed by the citation from the classical paper [W] on the Dirichlet problem (DP) for Laplace equation. As pointed out by Lebesgue and independently by Wiener "the DP divides itself into two parts, the first of which
is the determination of a harmonic function corresponding to certain boundary conditions, while the second is the investigation of the behaviour of this function in the neighbourhood of the boundary.” The same strategy, obviously replacing harmonic function with parabolic function and boundary with the parabolic boundary, is applicable to the FBVP for diffusion equation. As in the case of Laplace equation, generalised solution to the FBVP for diffusion equation may be constructed by Perron’s super- or subsolutions method (see, Section 2). However, in general the generalised solution doesn’t satisfy (1.1).

We say that a point \( z_0 \in \mathcal{P} \Omega \) is regular if, for any bounded function \( f : \mathcal{P} \Omega \to \mathbb{R} \), the generalised solution of the FBVP constructed by Perron’s method satisfies (1.1) at the point \( z_0 \). It is well-known that the boundary points \( z_0 \in \mathcal{B} \Omega \) are always regular.

The principal result of this paper is the characterization of the regularity of the boundary points \( z_0 \in \mathcal{S} \Omega \) via local geometry of the lateral boundary near this point.

Consider the following domains

\[
\mathcal{G}_1^\rho = \{z : x_1^2 < 4\xi \log \rho(\xi), (\overline{\tau}, t) \in P(\delta)\},
\]

\[
\mathcal{G}_2^\rho = \{z : -2(\xi \log \rho(\xi))^{1/2} < x_1 < 2(-\delta \log \rho(-\delta))^{1/2}, (\overline{\tau}, t) \in P(\delta)\},
\]

\[
P(\delta) = \{(\overline{\tau}, t) : -\delta < \xi < 0, -\delta < \alpha t < 0\},
\]

where \( \delta > 0 \) is a sufficiently small positive number, \( \xi = \alpha t - \beta |\overline{\tau}|^2 \); \( \alpha \) and \( \beta \) are given positive numbers. Throughout this paper we assume that a function \( \rho = \rho(\xi), -\delta \leq \xi < 0 \) is positive and continuously differentiable function satisfying the following condition:

\[
\rho(\xi) \downarrow 0, \ \xi \rho^{-1}(\xi) \rho'(\xi) \to 0 \text{ as } \xi \uparrow 0. \tag{1.2}
\]

Applying de l’Hospital’s rule from (1.2) it follows that \( \xi \log \rho(\xi) \to 0 \) as \( \xi \uparrow 0 \). In Fig. 1 and Fig. 2 the domains \( \mathcal{G}_1^\rho \) and \( \mathcal{G}_2^\rho \) are described when \( N = 2 \). Parabolic boundary \( \mathcal{P} \mathcal{G}_1^\rho \) consists of two manifolds

\[
\mathcal{L}_\pm = \{z \in \overline{\mathcal{G}_1^\rho} : x_1 = \pm 2(\xi \log \rho(\xi))^{1/2}, t > -\delta \alpha^{-1}\},
\]

and the cylindrical hypersurface
\{z \in \overline{G}_\rho^1 : \xi = -\delta, t > -\delta \alpha^{-1}\}.

Bottom boundary \(BG_1\) consists of a line segment \(\{z : x_1^2 < -4\delta \log \rho(-\delta), |\overline{x}| = 0, t = -\delta \alpha^{-1}\}\).

Figure 1. The domain \(G_\rho^1\) when \(N = 2\).

Parabolic boundary \(PG_\rho^2\) differs from \(PG_\rho^1\) by replacing the manifold \(L_+\) with

\(\{z \in \overline{G}_\rho^2 : x_1 = 2(-\delta \log \rho(-\delta))^\frac{1}{2}, (\overline{x}, t) \in \overline{P(\delta)}\}\).

Figure 2. The domain \(G_\rho^2\) when \(N = 2\).

Our main theorem reads:
Theorem 1.1 Let \( \alpha + 2(N - 1)\beta \leq 1 \) and

\[
\lim_{\epsilon \uparrow 0} \int_{-\delta}^{\epsilon} \frac{\rho(\eta)}{\eta} d\eta = -\infty.
\]

(1.3)

Then the origin \((O)\) is a regular point for \(G^1_\rho\) (or \(G^2_\rho\)) and the FBVP is solvable in \(G^1_\rho\) (or \(G^2_\rho\)).

Some examples of functions \(\rho\) that satisfy (1.2),(1.3) are the following:

\[
\rho(\xi) = |\log|\xi||^{-1}, \quad \rho(\xi) = \{|\log|\xi|| \prod_{k=2}^{n} \log k|\xi|\}^{-1}, \quad n = 2, 3, \ldots,
\]

(1.4)

where we use the following notation

\[
\log_2|\xi| = \log|\log|\xi||, \quad \log_n|\xi| = \log\log_{n-1}|\xi|, n \geq 3.
\]

Theorem 1.1 provides a general sufficient condition for the regularity of the boundary points \(z_0 \in S\Omega\) and for the solvability of the FBVP in \(\Omega\).

Let \(A_\rho = G \setminus \overline{G^2_\rho}\), where

\[
G = \{z : x_1^2 < -4\delta\log\rho(-\delta), (\bar{x}, t) \in P(\delta)\}.
\]

In Fig. 3 the domain \(A_\rho\) is described when \(N = 2\). We call the origin the vertex of \(A_\rho\). Consider the rigid body displacements of \(A_\rho\) composed of translations and (or) rotations in \(x\)-space and shift along the \(t\)-axis.

Figure 3. The domain \(A_\rho\) when \(N = 2\).
Definition 1.1 We shall say that $\Omega$ satisfies exterior $A_\rho$-condition at the point $z_0 \in S\Omega$, if after the above mentioned displacement the vertex of $A_\rho$ coincides with $z_0$ and for all sufficiently small $\delta$, $A_\rho$ lies in the exterior of $\Omega$.

Theorem 1.1 implies the following more general result:

Theorem 1.2 Boundary point $z_0 \in S\Omega$ is regular, if $\Omega$ satisfies exterior $A_\rho$-condition at this point. The FBVP is solvable in $\Omega$, which satisfies exterior $A_\rho$-condition at every point $z_0 \in S\Omega$.

In the case when the lateral boundary is locally a continuous graph, exterior $A_\rho$-condition may be expressed in terms of modulus of lower semicontinuity of the lateral boundary manifold. To make this precise, assume that for $z_0 = (x_0, t_0) \in S\Omega$ there exists $\epsilon > 0$ and a continuous function $\phi$ such that, after a suitable rotation of $x$-axes, we have

$$
\overline{S\Omega} \cap Q(z_0, \epsilon) = \{ z \in Q(z_0, \epsilon) : x_1 = \phi(\overline{x}, t) \},
$$

$$
\text{sign } (x_1 - \phi(\overline{x}, t)) = 1 \text{ for } z \in Q(z_0, \epsilon) \cap \Omega.
$$

Exterior $A_\rho$-condition is equivalent to the following one-side inequality for the function $\phi$:

$$
\phi(\overline{x}^0, t_0) - \phi(\overline{x}, t) \leq 2(\xi' \log \rho(\xi'))^{\frac{1}{2}} \text{ for } (\overline{x}, t) \in P'(\delta).
$$

(1.5)

where $\delta > 0$ be sufficiently small number, $\xi' = \alpha(t - t_0) - \beta|x - \overline{x}^0|^2$; the domain $P'(\delta)$ coincides with $P(\delta)$ by replacing $\xi$ with $\xi'$ and $t$ with $t - t_0$, the numbers $\alpha > 0, \beta > 0$ and the function $\rho$ satisfy the conditions of Theorem 1.1. The equivalence easily follows from the fact that after the displacement according to the exterior $A_\rho$-condition, the boundary manifold which is a common boundary of the translated domains $A_\rho$ and $G^1_\rho$ has a representation $x_1 = \phi(\overline{x}, t), (\overline{x}, t) \in P'(\delta)$, where $\phi$ satisfies (1.5) with ",=" instead of ",\leq\". Inequality (1.5) means that at the point $z_0 = (x^0, t_0) \in S\Omega$, the lateral boundary manifold is allowed to be "slightly worth" than lower Lipschitz in $x$-direction and "slightly worth" than lower $\frac{1}{2}$-Hölder in $-t$-direction. "Slightly worth" means that the related Lipschitz (or respectively Hölder) coefficient may converge to infinity as $(\overline{x}, t) \rightarrow (x^0, t_0)$, but not faster than $2(-\beta \log \rho(-\beta|x - \overline{x}^0|^2))^{\frac{1}{2}}$ (or respectively $2(-\alpha \log \rho(\alpha(t - t_0)))^{\frac{1}{2}}$). In the particular case when both coefficients are constant, we get the parabolic analog of the well-known exterior cone condition for Laplace equation.
Let us formulate this condition geometrically in the spirit of our exterior $A_\rho$-condition. For simplicity take $N = 2$ and consider the hyperbolic paraboloid (Fig. 4)

\[ x_1^2 = M(-t + x_2^2), \ M > 0. \]

Figure 4. Hyperbolic paraboloid $x_1^2 = M(-t + x_2^2)$.

Let $\delta > 0$ is given and consider the subsurface of the hyperbolic paraboloid which is situated in the half space \( \{ t \leq 0 \} \), between two planes \( \{ x_1 = 0 \} \) and \( \{ x_1 = -\delta^2 \} \) (Fig. 5). Consider the open domain $\mathcal{M}_\delta$ which is bounded by this subsurface and with the planes \( \{ t = 0 \} \) and \( \{ x_1 = -\delta^2 \} \) (Fig. 5).

Figure 5. The domain $\mathcal{M}_\delta$.

We call the origin the vertex of $\mathcal{M}_\delta$. Consider the rigid body displacements of $\mathcal{M}_\delta$ composed of translations and (or) rotations in $x$-space and shift along the $t$-axis. If after such a displacement
the vertex of $\mathcal{M}_\delta$ coincides with the point $z_0 \in \mathcal{S}\Omega$ and for all sufficiently small $\delta$, $\mathcal{M}_\delta$ lies in the exterior of $\Omega$, then $z_0$ is a regular point. This fact is an easy consequence of the Theorem 1.2. Similar condition is obviously true when $N > 2$, just by replacing $x_2^2$ with $|x|^2$. This is exactly the parabolic analog of the exterior cone condition for Laplace equation and it is natural to call it an exterior hyperbolic paraboloid condition.

It should be mentioned that the boundary regularity result of the Theorem 1.1 has a probabilistic meaning in the context of the short-time behaviour of the Brownian motion trajectories for the high dimensional diffusion processes. Without going into the details, let us just formulate the probabilistic analog of this result taking a simplest example $\rho(\xi) = |log|\xi||^{-1}$. Consider the standard $N$-dimensional Brownian motion in which the coordinates of the sample path are standard 1-dimensional Brownian motions. The intuitive meaning of the Theorem 1.1 is that the Brownian path that starts at the origin (assuming for a while that a process goes in $-t$-direction) with the probability 1 will reach the exterior of $\mathcal{G}_\rho^1$ within arbitrarily short time. From the classical iterated logarithm law it easily follows that with the probability 1 the same trajectory will remain in the domain $\mathcal{G}_\rho^1$ within some positive time, if $\alpha > 1, \beta > 0$. Important open problem is whether the same is true if $0 < \alpha \leq 1$, but $\alpha + 2(N - 1)\beta > 1$. The related open problem in the context of the FBVP consists in the derivation of the precise sufficient condition for the irregularity of the boundary points. We address this issue in the next paper.

We prove the main theorems in Section 3, after some preliminaries in Section 2. Section 4 contains some final remarks.

**Historical Comments** In 1935, Petrovsky ([P]) presented complete results on the FBVP for the one-dimensional diffusion equation $u_t = u_{xx}$ in a plane domain whose lateral boundary is given by two continuous curves $x = \phi_1(t)$ and $x = \phi_2(t)$. Petrovsky’s paper was motivated with the proof of the so called Kolmogorov test for the distinction between the upper and the lower functions of the one-dimensional space-time Brownian motion trajectories (see [IM]). If we take $N = 1$, then our Theorem 1.1 coincides with the regularity result from [P, Section 2]. Moreover, the analog of our domain $\mathcal{G}_\rho^1$ is a plane domain between the curve $x_1^2 = 4tlog\rho(t)$ and the line $t = -\delta < 0$. Similar domain we get intersecting $\mathcal{G}_\rho^1$ with the hyperplane $\{x_1 = 0\}$. 
As it is proved in [P, Section 3], even for the particular example \( \rho(t) = |\log|t||^{-1} \) the result is close to be an optimal in the sense that the origin is irregular point, if we replace the boundary curve with \( x_1^2 = -4\alpha t \log_2|t|, \alpha > 1 \). In the context of the space-time Brownian motion this result repeats Khinchin’s iterated logarithm law. As a direct implication of the one-dimensional results, in [P, Section 4] the case \( N = 2 \) was also shortly considered. It is shown that the origin is irregular point for the bounded domain lying beneath the plane \( \{ t = 0 \} \) and bounded on its sides by the surface of revolution

\[
x^2 + y^2 = -4(1 + \epsilon)t \log_2|t|,
\]

where \( \epsilon > 0 \) is arbitrary small number. From another side from the regularity condition formulated in [P, Section 4] it follows that the origin is a regular point for the same domain, if we replace the surface of revolution with the following one:

\[
x^2 + y^2 = 4t \log \rho(t),
\]

where \( \rho(t) \) satisfies the conditions of [P, Section 2] (or our Theorem 1.1). Both results are true for the case \( N \geq 3 \) as well. The meaning of these results is that the conditions for the regularity or irregularity of the boundary point on the top of the radially symmetric domain formed with the rotation of the plane curve \( x_1^2 = 4t \log \rho(t) \) around the \( t \)-axis are the same as in the one-dimensional case. Probabilistic interpretation of this fact may be given in the context of the space-time Brownian motion.

Starting from 1954-55, Wiener’s ideas ([W]) concerning the regularity of the boundary points for Laplace equation was adapted to the case of diffusion equation. In [L], Wiener’s type necessary and sufficient conditions for the regularity of the boundary points in the FBVP for diffusion equation was published. The analog of Wiener’s condition, namely necessary and sufficient condition which is a quasigeometric characterization for a boundary point of an arbitrary bounded open subset of \( \mathbb{R}^{N+1} \) to be regular for the diffusion equation, was established in [EG]. However, it should be mentioned that Wiener’s criterion does not resolve the natural geometric and analytic question which we impose in this paper. Despite its generality, it seems impossible to derive from Wiener’s condition, say Petrovsky’s one-dimensional results.
Another sufficient condition for the regularity of the boundary points in the FBVP for diffusion equation, the so called exterior tusk condition which is an analog of the exterior cone condition for Laplace equation, was established in [EK].

2 Preliminary Results

In this section we present some facts about the Perron’s solution of the FBVP. Lemma 2.1 is standard and demonstrates the role of barriers for the regularity of the origin for $G^1_\rho$ or $G^2_\rho$. Lemma 2.2 proves the equivalence of the regularity (or irregularity) of the origin for $G^1_\rho$ and $G^2_\rho$, which allows to prove Theorem 1.1 only for $G^1_\rho$.

It should be mentioned that the results of this section are general and we do not need to assume that neither the conditions of the Theorem 1.1 or the second condition from (1.2) are satisfied. However, we need to assume that $\xi \log \rho(\xi) \to 0$ as $\xi \to 0$.

A bounded open subset $U \in \mathbb{R}^{N+1}$ is called regular if for each continuous function $\phi \in C(\partial U; \mathbb{R})$ there exists one (and only one) function $H^U_\phi$, which is parabolic in $U$, and

$$
\lim_{z \to z_0, z \in U} H^U_\phi = \phi(z_0) \text{ for all } z_0 \in \partial U.
$$

A function $u \in C(\Omega)$ is called superparabolic in $\Omega$ if the following conditions are satisfied:

(a) $u$ is lower semicontinuous; $-\infty < u \leq +\infty$, $u < +\infty$ on a dense subset of $\Omega$;

(b) if $U \subset \overline{U} \subset \Omega$ is a regular open set, $\phi \in C(\partial U; \mathbb{R})$ and $\phi \leq u$ on $\partial U$, then $H^U_\phi \leq u$ in $U$.

A function $v$ is called a subparabolic, if $-v$ is a superparabolic. For example, any function $u \in C^2_{x,t}(\Omega)$ satisfying $D_u \geq 0$ (or $D_u \leq 0$) for $z \in \Omega$ is superparabolic (or subparabolic). The classical theory defines the Perron’s solution of the FBVP to be, for each $z \in \Omega$,

$$
H^\Omega_f = \inf \{u(z)\},
$$

where the infimum is taken over all superparabolic functions $u$ in $\Omega$, such that

$$
u_*(z_0) \geq f^*(z_0) \text{ for all } z_0 \in \mathcal{P} \Omega.
$$
According to the classical theory (see, for example [D,B] for the most general framework) $H^\Omega_f$ is parabolic in $\Omega$. However, in general it does not satisfy (1.1). Boundary point $z_0 \in P\Omega$ is called regular, if for arbitrary bounded boundary function $f$, $H^\Omega_f$ satisfies (1.1) at this point. It is well-known that bottom boundary points $z_0 \in B\Omega$ are always regular (see, for example [P]). It is a standard fact in the classical theory that the boundary point $z_0 \in S\Omega$ is regular, if there exists a so called ”regularity barrier” $\overline{u}$ with the following properties:

(a) $\overline{u}$ is superparabolic in $U = Q(z_0, \epsilon) \cap \Omega$ for some $\epsilon > 0$;

(b) $\overline{u}$ is continuous and nonnegative in $\overline{U}$, vanishing only at $z_0$.

In particular, concerning the regularity of the origin for $G^1_\rho$ or $G^2_\rho$ we have

**Lemma 2.1.** The origin ($O$) is regular for $G^1_\rho$ (or $G^2_\rho$) if and only if there exists a regularity barrier $\overline{u}$ for $O$ regarded as a boundary point of $G^1_\rho$ (or $G^2_\rho$) for sufficiently small $\delta$.

Proof. The proof of the ”if ” part is standard. To prove the ”only if ” part, take $f = -t + |x|^2$ and let $\overline{u} = H^\Omega_f \overline{G}^1_\rho$ be Perron’s solution. Since $\rho(\xi)$ is $C^1$ for $\xi < 0$, from the classical theory it follows that all the boundary points $z_0 \in P\overline{G}^1_\rho, z_0 \neq O$ are regular points. But $O$ is regular by our assumption. Therefore, $\overline{u} \in C(\overline{G}^1_\rho)$. From the strong maximum principle it follows that $\overline{u}$ is nonnegative in $\overline{G}^1_\rho$ and vanishes only at $O$. Thus $\overline{u}$ is a regularity barrier for $O$. The proof for the domain $G^2_\rho$ is similar. The lemma is proved.

The next lemma is the high-dimensional analog of the Theorem III from [P, p.389].

**Lemma 2.2.** The origin is simultaneously regular or irregular for $G^1_\rho$ and $G^2_\rho$.

Proof. Assume that $O$ is regular for $G^2_\rho$. Then by Lemma 2.1 there exists a regularity barrier for $O$ regarded as a boundary point of $G^2_\rho$. Obviously, it will be also a regularity barrier for $O$ regarded as a boundary point of $G^1_\rho$. From Lemma 2.1 it follows that $O$ is regular for $G^1_\rho$.

Conversely, assume now that $O$ is regular for $G^1_\rho$. Let $u = H^\Omega_f \overline{G}^2_\rho$, where $f = -t + |x|^2$. Since $\rho(\xi)$ is $C^1$ for $\xi < 0$, all the boundary points $z_0 \in P\overline{G}^2_\rho, z_0 \neq O$ are regular points. Accordingly, $u \in C(\overline{G}^2_\rho \setminus \{O\})$. Denote

$$L = \limsup_{z \to O, x_1 = \phi(x,t)} u,$$
where $\phi(x, t) = 4 \xi \log \rho(\xi)$. Obviously, we have $0 \leq L < +\infty$. Let $f_1$ be an arbitrary function which is defined and continuous in $\mathcal{P}G^1_\rho \setminus \{O\}$ satisfying

$$f_1(x_1, \overline{x}, t) = -f_1(-x_1, \overline{x}, t) \quad (2.1)$$

and

$$\lim_{z \to O, x_1 = \phi(\overline{x}, t)} f_1 = \frac{L}{2}, \quad \lim_{z \to O, x_1 = -\phi(\overline{x}, t)} f_1 = -\frac{L}{2}.$$

Choose a function $f_2$ in such a way that

$$f_1 + f_2 = u \text{ for } z \in \mathcal{P}G^1_\rho \cap G^2_\rho,$$

$$f_1 + f_2 = f \text{ on the rest of } \mathcal{P}G^1_\rho.$$ 

Since all the points $z_0 \in \mathcal{P}G^1_\rho$ are regular points, Perron’s solutions $u_1 = H_{f_1}^{G^1_\rho}$, $u_2 = H_{f_2}^{G^1_\rho}$ and $H_{f_1+f_2}^{G^1_\rho}$ are continuous functions in $\overline{G^1_\rho} \setminus \{O\}$. Applying the maximum principle in $G^1_\rho \cap \{z : t \leq -\epsilon\}$ for arbitrary sufficiently small $\epsilon > 0$ and passing to the limit as $\epsilon \downarrow 0$ we easily derive that

$$u = H_{f_1+f_2}^{G^1_\rho} = H_{f_1}^{G^1_\rho} + H_{f_2}^{G^1_\rho} = u_1 + u_2. \quad (2.2)$$

Applying the same arguments, from (2.1) we deduce that

$$u_1(x_1, \overline{x}, t) = -u_1(-x_1, \overline{x}, t),$$

and hence

$$u_1(0, \overline{x}, t) = 0. \quad (2.3)$$

We have

$$\limsup_{z \to O, x_1 = \pm \phi(\overline{x}, t)} f_2 \leq \frac{L}{2}.$$

Since $O$ is a regular point for $G^1_\rho$, it follows that

$$\limsup_{\xi \to 0} u_2(0, \overline{x}, t) \leq \frac{L}{2}. \quad (2.4)$$

From (2.2)-(2.4) we have
\[ \limsup_{\xi \to 0} u(0, \overline{\mathbf{x} t}) \leq \frac{L}{2}. \]

Since \( \mathcal{O} \) is a regular point regarded as a boundary point of \( \mathcal{G}_3 = \{z \in \mathcal{G}_\rho^2 : x_1 > 0\} \), we have \( L \leq L/2 \), which implies that \( L = 0 \). Thus \( u \) is continuous in \( \overline{\mathcal{G}_\rho^2} \) and by the strong maximum principle vanishes only at \( \mathcal{O} \). Hence, \( u \) is a regularity barrier for \( \mathcal{O} \) regarded as a boundary point of \( \mathcal{G}_\rho^2 \). From the Lemma 2.1 it follows that \( \mathcal{O} \) is a regular point for \( \mathcal{G}_\rho^2 \) as well. The lemma is proved.

**Corollary 2.1.** Let \( \mathcal{G}_0 \) be given open set in \( \mathbb{R}^{N+1} \) and \( \mathcal{O} \in \partial \mathcal{G}_0, \mathcal{G}_0^- \neq \emptyset \), where \( \mathcal{G}_0^- = \{z \in \mathcal{G}_0 : t < 0\} \). If \( \mathcal{G}_0^- \subset \mathcal{G}_\rho^2 \), then from the regularity of \( \mathcal{O} \) for \( \mathcal{G}_\rho^1 \) or \( \mathcal{G}_\rho^2 \) it follows that \( \mathcal{O} \) is regular for \( \mathcal{G}_0 \). Otherwise speaking, from the irregularity of \( \mathcal{O} \) for \( \mathcal{G}_0 \) or \( \mathcal{G}_0^- \) it follows that \( \mathcal{O} \) is irregular for \( \mathcal{G}_\rho^1 \) and \( \mathcal{G}_\rho^2 \).

Since bottom boundary points are always regular, the assertion of the Corollary 2.1 easily follow from Lemmas 2.1, 2.2 and from the fact that the regularity barrier for \( \mathcal{O} \) regarded as a boundary point of \( \mathcal{G}_\rho^2 \) is at the same time a regularity barrier for \( \mathcal{O} \) regarded as a boundary point of \( \mathcal{G}_0^- \).

Obviously, the assertion of the Corollary 2.1 is true if we take an arbitrary boundary point \( z_0 = (x^0, t_0) \in \partial \mathcal{G}_0 \), assuming that \( \mathcal{G}_0^- = \{z \in \mathcal{G}_0 : t < t_0\} \neq \emptyset \), and by replacing \( \mathcal{G}_\rho^1 \) and \( \mathcal{G}_\rho^2 \) with their translations after rigid body displacement composed of a translation in \( x \)-space and shift along the \( t \)-axis, in such a way that \( \mathcal{O} \) coincides with \( z_0 \) after this displacement.

### 3 Proofs of the Main Results

**Proof of Theorem 1.1** The proof is based on the construction of the regularity barrier for \( \mathcal{O} \) regarded as a boundary point of \( \mathcal{G}_\rho^1 \). We construct the regularity barrier assuming that \( \delta > 0 \) is sufficiently small. This makes no loss of generality, since boundary regularity is a local property. Without loss of generality we also assume that the positive numbers \( \alpha \) and \( \beta \) satisfy

\[ \alpha + 2(N - 1)\beta = 1. \quad (3.1) \]
Indeed, if $\alpha + 2(N - 1)\beta < 1$, then we can take $\bar{\beta} > \beta$ such that $\alpha + 2(N - 1)\bar{\beta} = 1$ and consider the domain $\bar{G}^1_\rho$ by replacing $\beta$ with $\bar{\beta}$ in $G^1_\rho$. It may be easily seen that $\bar{G}^1_\rho$ contains $G^1_\rho$ if we replace $\delta$ in $G^1_\rho$ with $\beta \bar{\beta}^{-1}\delta$. Therefore, if we construct a regularity barrier for $\bar{G}^1_\rho$ for all $\delta \leq \delta_1$, the latter will be a regularity barrier for $G^1_\rho$ for all $\delta \leq \beta \bar{\beta}^{-1}\delta_1$.

Without loss of generality we may also assume that $\rho(\xi)$ is twice continuously differentiable for $\xi < 0$ and satisfies

$$\xi^2 \rho^{-1}(\xi)\rho''(\xi) \to 0, \text{ as } \xi \uparrow 0. \tag{3.2}$$

Indeed, otherwise we can choose a monotonically decreasing and twice continuously differentiable function $\rho_1(\xi)$, $-\delta \leq \xi < 0$ which satisfies the following conditions:

$$\frac{1}{2} \rho(\xi) < \rho_1(\xi) < \rho(\xi), \tag{3.3}$$

$$\min(\rho'(\xi); -\rho(\xi)) < \rho_1'(\xi) < \frac{1}{2} \rho'(\xi). \tag{3.4}$$

From (3.4) and (1.2) it follows that

$$0 < \xi \rho_1^{-1}(\xi)\rho_1'(<\xi) < 2 \max(\xi \rho^{-1}(\xi)\rho'(\xi); -\xi) \to 0 \text{ as } \xi \uparrow 0. \tag{3.5}$$

Hence, form (3.3)-(3.5) it follows that $\rho_1$ satisfies (1.2) and (1.3). Applying de l’Hospital’s rule from (3.5) and (3.4) we have

$$\frac{\xi \rho_1''(\xi)}{\rho_1'(\xi)} \to -1 \text{ as } \xi \uparrow 0. \tag{3.6}$$

From (3.4)-(3.6) it easily follows that $\rho_1$ satisfies (3.2). Hence, $\rho_1$ satisfies all the required conditions and in view of (3.3) we have $G^1_\rho \subset G^1_{\rho_1}$. Accordingly, the regularity barrier for $O$ regarded as a boundary point of $G^1_{\rho_1}$ will be a regularity barrier for $O$ regarded as a boundary point of $G^1_\rho$.

Thus it is enough to construct a regularity barrier for $O$ regarded as a boundary point of $G^1_\rho$, assuming additionally that $\rho$ is $C^2$ for $\xi < 0$ and satisfies (3.2). By the way, all the examples of $\rho$ from (1.4) satisfy these conditions as well.

We prove that the following function is the required regularity barrier:

$$u(x, t) = g(\xi)exp\left(-\frac{x^2}{4\xi}\right) + \phi(\xi),$$
where $\phi$ is defined via

$$9 \log \phi = \int_{\xi_0}^{\xi} \frac{\rho(\eta)}{\eta} d\eta, \quad \xi_0 < \xi < 0,$$

with $\xi_0$ being a fixed negative number with sufficiently small $|\xi_0|$, and

$$g(\xi) = -\frac{1}{2} \rho(\xi) \phi(\xi).$$

From (1.2) and (1.3) it follows that

$$\phi(\xi) > 0, \quad 9 \phi'(\xi) = \xi^{-1} \rho(\xi) \phi(\xi) < 0; \quad \phi(\xi) \downarrow 0 \text{ as } \xi \uparrow 0. \quad (3.7)$$

$$g(\xi) < 0, \quad g'(\xi) > 0; \quad g(\xi) \uparrow 0 \text{ as } \xi \uparrow 0. \quad (3.8)$$

The equation of the level hypersurface $u(x, t) = 0$ is given by

$$x_1^2 = 4\xi \left[ \log \rho(\xi) - \log 2 \right].$$

Moreover, we have

$$u > 0 \text{ in } G' = \{ z : x_1^2 < 4\xi [\log \rho(\xi) - \log 2], (\pi, t) \in P(2\delta) \}. $$

Since $G'_{\rho} \subset G'$, we derive that $u$ is positive and continuous in $\overline{G'_{\rho}} \setminus \{O\}$. The function $u$ is symmetric with respect to the $x_1$-variable and for arbitrary fixed $(\bar{\pi}, t) \in \overline{P(\delta)}$, $(\bar{\pi}, t) \neq (0, 0)$, $u$ attains its maximum at $x_1 = 0$. Hence, we have

$$0 < u(x_1, \bar{\pi}, t) \leq u(0, \bar{\pi}, t) = \phi(\xi) \left( 1 - \frac{1}{2} \rho(\xi) \right) \rightarrow 0 \text{ as } \xi \uparrow 0.$$

Thus $u$ has a removable singularity at the point $O$ and prescribing $u = 0$ at $O$, we have $u \in C(\overline{G'_{\rho}})$. To complete the proof, we need to show that $u$ is superparabolic in $G'_{\rho}$. Taking into account (3.1), we derive

$$D u = \exp \left( -\frac{x_1^2}{4\xi} \right) S, \quad (3.9)$$

where

$$S = \frac{g(\xi)}{\xi} \left[ \frac{1}{2} + 2\beta^2 |\pi|^2 x_1^2 \xi^{-2} - \frac{1}{4} \beta^2 |\pi|^2 x_1^4 \xi^{-3} \right] + g'(\xi) \left[ 1 - 2\beta^2 |\pi|^2 x_1^2 \xi^{-2} \right].$$
\[ +\phi'(\xi)\exp\left(\frac{x^2}{4\xi}\right) - 4\beta^2|\vec{x}|^2\left[g''(\xi) + \phi''(\xi)\exp\left(\frac{x^2}{4\xi}\right)\right]. \]  

(3.10)

Assuming that $|\xi|$ is sufficiently small, from (1.2) and (3.7) it follows that

\[ \frac{g(\xi)}{\xi} - g'(\xi) > \frac{1}{2}\phi(\xi)\left[\frac{\rho'(\xi)}{2\xi} - \frac{\rho(\xi)}{2\xi}\right] > 0. \]  

(3.11)

Therefore, from (3.8), (3.10) and (3.11) we derive that

\[ S > \frac{1}{4}\frac{g(\xi)}{\xi} + \phi'(\xi)\exp\left(\frac{x^2}{4\xi}\right) > \frac{1}{4}\frac{g(\xi)}{\xi} + \phi'(\xi) = \frac{\rho(\xi)\phi(\xi)}{72\xi} > 0. \]  

(3.12)

Using (3.7), we estimate the first two terms on the right-hand side of (3.12) as follows:

\[ \frac{1}{4}\frac{g(\xi)}{\xi} + \phi'(\xi)\exp\left(\frac{x^2}{4\xi}\right) \geq \frac{1}{4}\frac{g(\xi)}{\xi} + \phi'(\xi) = -\frac{\rho(\xi)\phi(\xi)}{72\xi} > 0. \]

Therefore, from (3.12) it follows that

\[ S > \frac{1}{4}\frac{g(\xi)}{\xi} - 4\beta^2|\vec{x}|^2\left[g''(\xi) + \phi''(\xi)\exp\left(\frac{x^2}{4\xi}\right)\right]. \]  

Using (3.7), we easily derive that

\[ 9\phi'' = \frac{2}{\xi}\left[\frac{g(\xi)}{\xi} - g'(\xi)\right] < 0. \]  

(3.14)

Since $G^1_\rho \subset G'$, from (3.13), (3.14) and (3.7) we have

\[ S > \frac{1}{4}\frac{g(\xi)}{\xi} - 4\beta^2|\vec{x}|^2\left[g''(\xi) + \phi''(\xi)\right] = \frac{1}{4}\frac{g(\xi)}{\xi} \]

\[ +4\beta^2|\vec{x}|^2\left[\frac{1}{2}\phi(\xi)\rho''(\xi) + \rho'(\xi)\phi'(\xi)\right] > \frac{1}{4}\frac{g(\xi)}{\xi} + 2\beta^2|\vec{x}|^2\phi(\xi)\rho''(\xi). \]  

(3.15)

If $|\vec{x}| = 0$ then from (3.15) and (3.8) it follows that $S > 0$. Otherwise, from (3.15) we derive that

\[ S > 2\beta^2|\vec{x}|^2\phi(\xi)\left[\frac{1}{16\beta}\frac{\rho(\xi)}{\xi^2} + \phi''(\xi)\right]. \]  

(3.16)

Assuming that $|\xi|$ is sufficiently small, from (3.2) and (3.16) we have $S > 0$. Hence, from (3.9) it follows that $u$ is superparabolic in $G^1_\rho$ with sufficiently small $\delta$. Accordingly, $u$ is a required regularity barrier. The regularity of $O$ for $G^2_\rho$ is a consequence of Lemma 2.2. The theorem is proved.
The first assertion of the Theorem 1.2 easily follows from the Theorem 1.1 and Lemma 2.1. Indeed, since the space-time transformations due to rigid body displacements in the exterior $A_\rho$-condition preserve the diffusion equation, the regularity barrier for $G^2_\rho$ after the transformations according to exterior $A_\rho$-condition will be a regularity barrier for $z_0 \in S\Omega$. The second assertion of the Theorem 1.2 is a consequence of the classical theory (see Section 2).

4 Conclusions

The following natural question arises for each example of $\rho$ from (1.4):

How sharp is the condition $\alpha + 2(N - 1)\beta \leq 1$ for the regularity of $O$ for $G^1_\rho$ or $G^2_\rho$?

Let
\[
\rho(\xi) = |\log|\xi||^{-1}.
\] (4.1)

In [P] it is proved that $O$ is irregular point for the bounded domain $G_0$ lying in the strip $\{z : -\delta_1 < t < 0\}$ and bounded on its sides by the hypersurface of revolution
\[
|x|^2 = -4\alpha_1 t \log|t|, \; \alpha_1 > 1.
\]

It is easy to see that for sufficiently small $\delta_1$, $G_0 \subset G^1_\rho$ if $1 < \alpha_1 < \alpha$. Therefore, from the Corollary 2.1 it follows that $O$ is irregular for $G^1_\rho$ and $G^2_\rho$ if $\alpha > 1, \beta > 0$. Obviously, the same result is true, if we take any other particular example of $\rho$ from (1.4).

Hence, the following natural question arises:

Is $O$ regular or irregular for $G^1_\rho$ and $G^2_\rho$, if $\rho(\xi) = |\log|\xi||^{-1}$, $0 < \alpha \leq 1$, $\beta > 0$ and $\alpha + 2(N - 1)\beta > 1$?

The probabilistic analog of this question was formulated at the end of the Section 1. This issue is addressed in a subsequent paper.

Another important conclusion says that $O$ may be regular for
\[
u_t = \Delta u
\] (4.2)
and at the same time irregular for
\[
u_t = a\Delta u, \; 0 < a < 1,
\] (4.3)
regarded as a boundary point of $G_1^1$ or $G_2^2$. Let us check this fact by considering again the simplest case (4.1). Consider $G_1^1$ and $G_2^2$ with $\alpha = 1 - \frac{\varepsilon}{2}, \beta = \frac{\varepsilon}{4(N-1)}$, where $\varepsilon$ be arbitrary number satisfying $0 < \varepsilon \leq 1 - a$. From the Theorem 1.1 it follows that $O$ is regular for (4.2), regarded as a boundary point of $G_1^1$ or $G_2^2$. If $u(x,t)$ solves (4.3) in $G_1^1$, then after the transformation $x = x, \tau = at$, the function $\tilde{u}(x,\tau) = u(x,t)$ satisfies
\[\tilde{u}_\tau = \Delta \tilde{u},\] (4.4)
while the domain $G_1^1$ is transformed to the domain
\[\tilde{G}_1^1 = \{z = (x,\tau) : x_1^2 < 4\xi_1 log\rho(\xi_1), -\delta < \xi_1 < 0, -\delta < a^{-1}a^{-1}\tau < 0\},\]
where $\xi_1 = a^{-1}a^{-1} - \beta|x|^2$. We have
\[\alpha a^{-1} > (1 - \varepsilon)a^{-1} \geq 1.\]
Hence, $O$ is irregular for (4.4), regarded as a boundary point of $\tilde{G}_1^1$. Accordingly, $O$ is irregular for (4.3) regarded as a boundary point of $G_1^1$ or $G_2^2$.

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References


