Nonlinear Diffusion in Irregular Domains

Ugur G. Abdulla

Max-Planck Institute for Mathematics in the Sciences,

Leipzig 04103, Germany

We investigate the Dirichlet problem for the parablic equation $u_t = \Delta u^m, m > 0$, in a non-smooth domain $\Omega \subset I\!R^{N+1}, N \geq 2$. In a recent paper [*U.G.Abdulla, J. Math. Anal. Appl., 260, 2 (2001), 384-403.*] existence and boundary regularity results were established. In this paper we present uniqueness, comparison and stability theorems.

1 Introduction and Statement of Main Results

Consider the equation

$$u_t = \Delta u^m \tag{1.1}$$

where $u = u(x,t), x = (x_1, \dots, x_N) \in I\!R^N, N \geq 2, t \in I\!R_+, \Delta = \sum_{i=1}^{N} \partial^2/\partial x_i^2, m > 0, m \neq 1$. In this paper we continue our study of the Dirichlet problem (DP) for the equation (1.1) in a general domain $\Omega \subset I\!R^{N+1}$. In the recent paper [1] (see also [2]) existence and boundary regularity results were established (see Theorem 2.1 of [1]). The purpose of this paper is to establish uniqueness, comparison and stability theorems. Let Ω be bounded open subset of $I\!R^{N+1}, N \geq 2$. Let the boundary $\partial\Omega$ of Ω consists of the closure of open domain $B\Omega$ lying on $t = 0$, an open domain $D\Omega$ lying on $t = T \in (0, \infty)$ and a (not necessarily connected) manifold $S\Omega$ lying in the strip $0 < t \leq T$. Denote $\Omega(\tau) = \{(x,t) \in \Omega : t = \tau\}$ and assume that $\Omega(t) \neq \emptyset$ for $t \in (0,T)$. The set $\mathcal{P}\Omega = \overline{B\Omega} \bigcup S\Omega$ is called a parabolic boundary of Ω. Furthermore the class of domains with described structure will be denoted by $\mathcal{D}_{0,T}$. Let $\Omega \in \mathcal{D}_{0,T}$ is given and

ψ is an arbitrary continuous nonnegative function defined on $\mathcal{P}\Omega$. DP consists in finding a solution to equation (1.1) in $\Omega \bigcup D\Omega$ satisfying initial-boundary condition

$$u = \psi \quad \text{on } \mathcal{P}\Omega. \tag{1.2}$$

We shall follow the following notion of weak solutions (super- or subsolutions):

Definition 1.1 We shall say that the function $u(x,t)$ is a solution (respectively super- or subsolution) of DP (1.1), (1.2), if

(a) u is nonnegative and continuous in $\overline{\Omega}$, locally Hölder continuous in $\Omega \cup D\Omega$, satisfying (1.2) (respectively satisfying (1.2) with $=$ replaced by $\geq$ or $\leq$),

(b) for any t_0, t_1 such that $0 < t_0 < t_1 \leq T$ and for any domain $\Omega_1 \in \mathcal{D}_{t_0,t_1}$ such that $\overline{\Omega}_1 \subset \Omega \cup D\Omega$ and $\partial B\Omega_1, \partial D\Omega_1, S\Omega_1$ being sufficiently smooth manifolds, the following integral identity holds

$$\int\limits_{D\Omega_1} uf dx = \int\limits_{B\Omega_1} uf dx + \int\limits_{\Omega_1} (uf_t + u^m \Delta f) dxdt - \int\limits_{S\Omega_1} u^m \frac{\partial f}{\partial \nu} dxdt, \tag{1.3}$$

(respectively (1.3) holds with $=$ replaced by $\geq$ or $\leq$), where $f \in C^{2,1}_{x,t}(\overline{\Omega}_1)$ is an arbitrary function (respectively nonnegative function) that equals to zero on $S\Omega_1$ and ν is the outward-directed normal vector to $\Omega_1(t)$ at $(x,t) \in S\Omega_1$.

We shall use the same notation as in [1]: $z = (x,t) = (x_1, \ldots, x_N, t) \in I\!R^{N+1}, N \geq 2, x = (x_1, \overline{x}) \in I\!R^N, \overline{x} = (x_2, \ldots, x_N) \in I\!R^{N-1}, |x|^2 = \sum_{i=1}^{N} |x_i|^2, |\overline{x}|^2 = \sum_{i=2}^{N} |x_i|^2$. For a point $z = (x,t) \in I\!R^{N+1}$ we denote by $B(z;\delta)$ an open ball in $I\!R^{N+1}$ of radius $\delta > 0$ and with center being in z.

Let $\Omega \in \mathcal{D}_{0,T}$ be a given domain. Assume that for arbitrary point $z_0 = (x^0, t_0) \in S\Omega$ (or $(z_0 = (x^0, 0) \in \overline{S\Omega})$ there exists $\delta > 0$ and a continuous function ϕ such that, after a suitable rotation of x-axes, we have $\overline{S\Omega} \bigcap B(z_0, \delta) = \{z \in B(z_0, \delta) : x_1 = \phi(\overline{x}, t)\}$ and sign $(x_1 - \phi(\overline{x}, t)) = 1$ for $z \in B(z_0, \delta) \bigcap \Omega$.

Furthermore, **we always suppose in this paper that the conditions of the Theorem 2.1 of [1] are satisfied.** We are going now to formulate another pointwise restriction at the point $z_0 = (x^0, t_0) \in S\Omega, 0 < t_0 < T$, which plays a crucial role in the proof of uniqueness of the constructed solution. For an arbitrary sufficiently small $\delta > 0$, consider a domain

$$Q(\delta) = \{(\overline{x}, t) : |\overline{x} - \overline{x}^0| < (\delta + t_0 - t)^{\frac{1}{2}}, t_0 < t < t_0 + \delta\}, \tag{1.4}$$

Assumption $\mathcal{M}$ Let for all sufficiently small positive δ we have

$$\phi(\overline{x}^0, t_0) - \phi(\overline{x}, t) \leq [t - t_0 + |\overline{x} - \overline{x}^0|^2]^\mu \text{ for } (\overline{x}, t) \in \overline{Q(\delta)}. \tag{1.5}$$

where $\mu > \frac{1}{2}$ if $0 < m < 1$, and $\mu > \frac{m}{m+1}$ if $m > 1$.

Furthermore we denote $\nu = \mu - 1$ assuming without loss of generality that $\nu \in (-\frac{1}{2}, 0)$ if $0 < m < 1$ and $\nu \in (-\frac{1}{m+1}, 0)$ if $m > 1$.

Definition 2.3 Let $[c, d] \subset (0, T)$ be a given segment and $S\Omega_{[c,d]} = S\Omega \bigcap \{(x, t) : c \leq t \leq d\}$. We shall say that assumption $\mathcal{M}$ is satisfied uniformly in $[c, d]$, if there exists $\delta_0 > 0$ and $\mu > 0$ as in (1.5) such that for $0 < \delta \leq \delta_0$, (1.5) is satisfied for all $z_0 \in S\Omega_{[c,d]}$ with the same μ.

Our main theorems read:

Theorem 2.2 (Uniqueness) Let there exists a finite number of points $t_i, i = 1, \dots, k$ such that $t_1 = 0 < t_2 < \dots < t_k < t_{k+1} = T$ and for the arbitrary compact subsegment $[\delta_1, \delta_2] \subset (t_i, t_{i+1}), i = 1, \dots, k$, assumption $\mathcal{M}$ is uniformly satisfied in $[\delta_1, \delta_2]$. Then the solution of the DP is unique.

Theorem 2.3 (Comparison) Let u be a solution of DP and g be a supersolution (respectively subsolution) of DP. Assume that the assumption of Theorem 2.2 is satisfied. Then $u \leq$ (respectively $\geq$) g in $\overline{\Omega}$.

Theorem 2.4 (Stability or L_1-contraction) Let the assumption of Theorem 2.2 be satisfied. Let g_1 and g_2 are solutions of DP with inital boundary data ψ_1 and ψ_2 respectively. If $\psi_1 = \psi_2$ on $S\Omega$, then for arbitrary $t \in [0, T]$ we have $||g_1 - g_2||_{L_1(\Omega(t))} \leq ||\psi_1 - \psi_2||_{L_1(B\Omega)}$.

2 Geometric Meaning of the Assumption $\mathcal{M}$

Assumption $\mathcal{M}$ is of geometric nature. To explain its meaning, for simplicity assume that $N = 2, d(z_0) = 1$ and rewrite (1.5) as follows: $x_1^0 - x_1 \leq [t - t_0 + (x_2 - x_2^0)^2]^\mu$ for $(x_2, t) \in \overline{Q(\delta)}$, where $x_1^0 = \phi(x_2^0, t_0)$ and $x_1 = \phi(x_2, t)$ for $(x_2, t) \in \overline{Q(\delta)}$. Consider the hyperbolic paraboloid

$x_1^2 = t + x_2^2$ (Figure 1) in the x_1x_2t-space. Let M_δ be the piece of it lying in the half-space $\{t \geq 0\}$, between the planes $\{x_1 = 0\}$ and $\{x_1 = -\delta^{\frac{1}{2}}\}$ (see Figure 2). The projection of M_δ to the plane $\{x_1 = 0\}$ is $\overline{Q_0(\delta)}$, where as $Q_0(\delta)$ we denote $Q(\delta)$ with $N = 2, x_2^0 = 0, t_0 = 0$. The surface M_δ has the following representation: $x_1 = \phi(x_2, t) \equiv -\sqrt{x_2^2 + t}, (x_2, t) \in \overline{Q_0(\delta)}$.

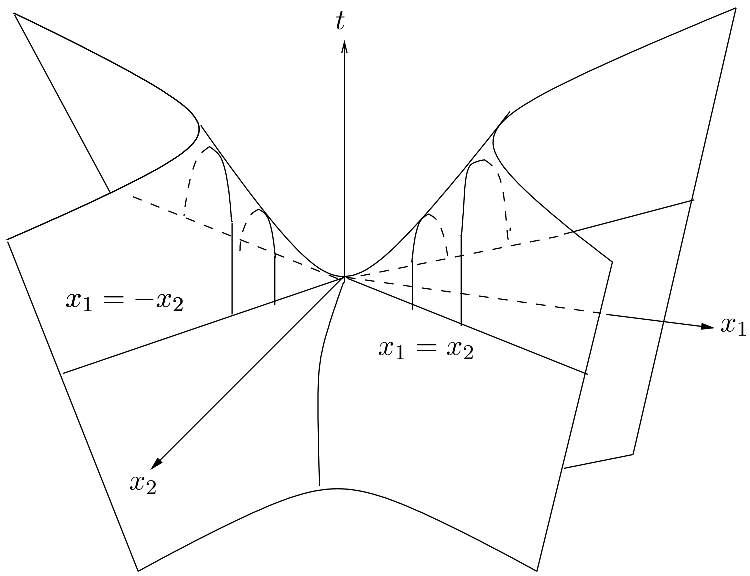

Figure 1: Hyperbolic paraboloid $x_1^2 = t + x_2^2$

Obviously, the function ϕ satisfies (1.5) with $=$ instead of $\leq$ in the critical case when $\mu = \frac{1}{2}$ (we also replace $Q(\delta)$ with $Q_0(\delta)$ in (1.5)). Consider the displacements of M_δ, while it is moved on the x_1x_2-plane and shifted along the t-axis.

Let us now consider the critical case of the assumption $\mathcal{M}$ with $\mu = \frac{1}{2}$. Namely, we take $\frac{1}{2}$ in (1.5). Equivalent formulation of this critical assumption may be given as follows:

Assume that after the displacement of the above type M_δ occupies such a position that its vertex coincides with the point $z_0 = (x^0, t_0) \in S\Omega$, and for δ being positive and sufficiently small it has no common point with Ω.

Similar geometric reformulation of the assumption $\mathcal{M}$ may be given just modifying subsurface M_δ according to the lower restriction imposed on μ. Thus if $0 < m < 1$, then the exterior touching surface is slightly more regular at the vertex point than related subsurface M_δ of the hyperbolic paraboloid. Otherwise speaking, it is slightly more regular than $C_{x,t}^{1,\frac{1}{2}}$ at the vertex point. When m changes from 1 to $+\infty$, the regularity of M_δ increases continuously, for each m

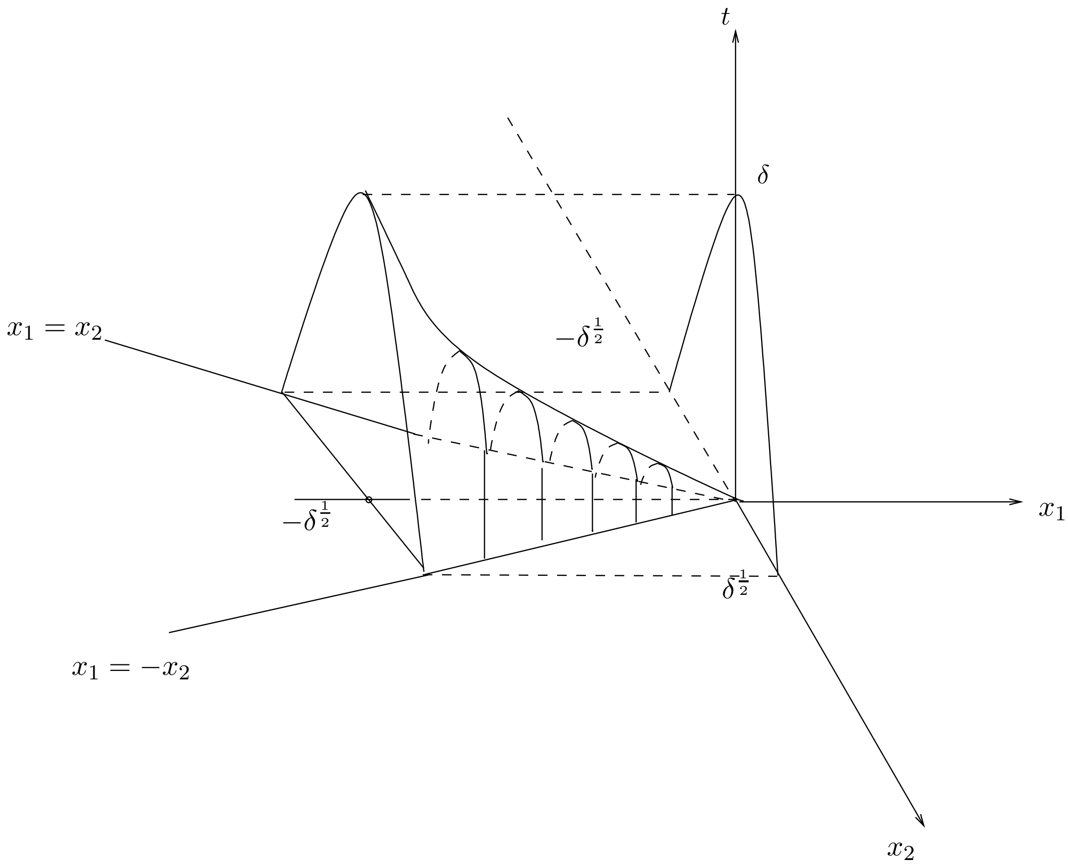

Figure 2: Piece M_δ of the hyperbolic paraboloid from Fig. 1 lying in the half-space $\{t \geq 0\}$, between the planes $\{x_1 = 0\}$ and $\{x_1 = -\delta^{\frac{1}{2}}\}$.

being slightly more regular than $C_{x,t}^{\frac{2m}{m+1},\frac{m}{m+1}}$ at the vertex point. Another limit position of M_δ as $m \to +\infty$ (or $\mu \to 1-$) is the upper paraboloid frustrum with vertex at the origin.

3 Proofs of the Main Theorems

Proof of Theorem 2.2 Suppose that g_1 and g_2 are two solutions of DP. We shall prove uniqueness by proving that

$$g_1 \equiv g_2 \text{ in } \overline{\Omega} \bigcap \{(x,\tau) : t_j \leq \tau \leq t_{j+1}\}, j = 1, \ldots, k \tag{3.1}$$

We present the proof of (3.1) only for the case $j = 1$. The proof for cases $j = 2, \ldots, k$ coincides with the proof for the case $j = 1$. We prove (3.1) with $j = 1$ by proving that for some limit solution $u = \lim u_n$ the following inequalities are valid

$$\int\limits_{\Omega(t)} (u(x,t) - g_i(x,t))\omega(x)dx \leq 0, i = 1, 2 \tag{3.2}$$

for every $t \in (0, t_2)$ and for every $\omega \in C_0^\infty(\Omega(t))$ such that $|\omega| \leq 1$. Obviously, from (3.2) it follows that

$$g_1 = u = g_2 \text{ in } \overline{\Omega} \bigcap \{(x,\tau) : t_1 \leq \tau < t_2\}, \tag{3.3}$$

which implies (3.1) with $j = 1$ in view of continuity of u, g_1 and g_2 in $\overline{\Omega}$. Since the proof of (3.2) is similar for each i, we shall henceforth let $g = g_i$. Let $t \in (0, t_2)$ be fixed and let $\omega \in C_0^\infty(\Omega(t))$ be an arbitrary function such that $|\omega| \leq 1$. To construct the required limit solution, as in [1], we approximate Ω and ψ with a sequence of smooth domains $\Omega_n \in \mathcal{D}_{0,T}$ and smooth positive functions ψ_n. We make a slight modification to the construction of Ω_n and ψ_n. Let Ψ be a nonnegative and continuous function in $I\!R^{N+1}$, which coincides with ψ on $\mathcal{P}\Omega$. This continuation is always possible. Let ψ_n be a sequence of smooth functions such that

$$max(\Psi; n^{-1}) \leq \psi_n \leq (\Psi^m + Cn^{-m})^{\frac{1}{m}}, n = 1, 2, \dots, \tag{3.4}$$

where $C > 1$ is a fixed constant. For arbitrary subset $G \subset I\!R^{N+1}$ and $\rho > 0$, we define

$$O_\rho(G) = \bigcup_{z \in G} B(z, \rho).$$

Since g and Ψ are continuous functions in $\overline{\Omega}$ and $g = \psi$ on $\mathcal{P}\Omega$, for arbitrary n there exists $\rho_n > 0$ such that

$$|g^m(z) - \Psi^m(z)| \leq n^{-m} \text{ for } z \in O_{\rho_n}(\overline{S\Omega}) \bigcap \overline{\Omega}. \tag{3.5}$$

We then assume that Ω_n satisfies the following:

$$\Omega_n \in \mathcal{D}_{0,T}, \overline{\Omega}_n \subseteq \Omega \bigcup D\Omega, S\Omega_n \subseteq O_{\rho_n}\overline{(S\Omega)} \tag{3.6}$$

We now formulate assumptions on $S\Omega_n$ near its point z_n, which are direct implications of the assumption $\mathcal{M}$ at the point $z_0 \in S\Omega$. Without loss of generality assume that $d(z_0) = 1$. Assume that $S\Omega_n$ in some neighbourhood of its point $z_n = (x_1^{(n)}, \overline{x}^0, t_0)$ is represented by the function $x_1 = \phi_n(\overline{x}, t)$, where $\{\phi_n\}$ is a sequence of sufficiently smooth functions and $\phi_n \to \phi$

as $n \to +\infty$, uniformly in $\overline{Q(\delta_0)}$, where $\delta_0 > 0$ be a sufficiently small fixed number, which does not depend on n. Obviously, we can assume that ϕ_n satisfies assumption $\mathcal{M}$ (namely (1.5)) at the point $(\overline{x}^0, t_0)$, uniformly with respect to n and with the same exponent μ. Let $\{\delta_n\}$ be some sequence of positive real numbers such that $\delta_n \to 0$ as $n \to +\infty$. Assume also that the sequence $\{\phi_n\}$ is chosen such that, for n being large enough, the following inequality is satisfied.

$$\phi_n(\overline{x}^0, t_0) - \phi_n(\overline{x}, t) \leq \delta_n^{\nu}[t - t_0 + |\overline{x} - \overline{x}^0|^2] \text{ for } (\overline{x}, t) \in \overline{Q(\delta_n)}. \tag{3.7}$$

Obviously, this is possible in view of uniform convergence of ϕ_n to ϕ. For example, if $\phi(\overline{x}, t)$ coincides with its lower bound $\tilde{\phi}(\overline{x}, t) = \phi(\overline{x}^0, t_0) - [t - t_0 + |\overline{x} - \overline{x}^0|^2]^{\mu}$, for $(\overline{x}, t) \in \overline{Q(\delta_0)}$ (namely (1.5) is satisfied with $=$ instead of $\leq$), then for all large n such that $\delta_n < \delta_0$ we first choose $\tilde{\phi_n}$ as follows:

$$\tilde{\phi_n}(\overline{x}, t) = \begin{cases} \phi(\overline{x}^0, t_0) - \delta_n^{\nu}[t - t_0 + |\overline{x} - \overline{x}^0|^2] \text{ for } (\overline{x}, t) \in \overline{Q(\delta_n)}, \\ \tilde{\phi}(\overline{x}, t) \text{ for } (\overline{x}, t) \in \overline{Q(\delta_0)} \backslash \overline{Q(\delta_n)}. \end{cases}$$

Obviosly, $\tilde{\phi_n}$ satisfies (3.7) and converges to ϕ uniformly in $Q(\delta_0)$. Then we easily construct ϕ_n by smoothing $\tilde{\phi_n}$ at the boundary points of $Q(\delta_n)$ satisfying $t - t_0 + |\overline{x} - \overline{x}^0|^2 = \delta_n$. In general, we can do similar construction by taking instead of $\tilde{\phi_n}(\overline{x}, t)$ the function $\tilde{\tilde{\phi_n}}(\overline{x}, t) = max(\tilde{\phi_n}(\overline{x}, t); \phi(\overline{x}, t))$, which satisfies (3.7) and converges to $\phi(\overline{x}, t))$ as $n \to +\infty$, uniformly in $\overline{Q(\delta_0)}$. Furthermore we will assume that the sequence δ_n is chosen as follows:

$$\delta_n = n^{\frac{\frac{1}{\gamma} - m - \epsilon}{1 + 2\nu}} \text{ with } 0 < \epsilon < (1 + \nu)^{-1}\Big[\nu\Big(m + \frac{1}{\gamma}\Big) + \frac{1}{\gamma}\Big], \tag{3.8}$$

where $\gamma = 1$ if $m > 1$, while if $0 < m < 1$ then γ is chosen such that

$$\frac{1}{m} < \gamma < -\frac{1 + \nu}{\nu m}. \tag{3.9}$$

Let u_n be a classical solution to the following problem:

$$u_t = \Delta u^m, \text{ in } \Omega_n \bigcup D\Omega_n \tag{3.10}$$

$$u = \psi_n \text{ on } \mathcal{P}\Omega_n. \tag{3.11}$$

This is a nondegenerate parabolic problem and classical theory [3,4] implies the existence of a unique $C_{2+\alpha}$ solution. From maximum principle and (3.4) it follows that

$$n^{-1} \leq u_n \leq M \text{ in } \overline{\Omega}_n, n = 1, 2, \ldots \tag{3.12}$$

where M is some constant which do not depend on n and $M > max(\sup_{\mathcal{P}\Omega}\psi, \sup_{\mathcal{P}\Omega_n}\psi_n)$. As in [1], we then prove that for some subsequence n', $u = \lim_{n'\to\infty} u'_n$ is a solution of DP (1.1), (1.2). Furthermore, without loss of generality we write n instead of n'. Take an arbitrary sequence of real numbers $\{\alpha_l\}$ such that

$$0 < \alpha_{l+1} < \alpha_l < t, \alpha_l \downarrow 0 \text{ as } l \to +\infty. \tag{3.13}$$

Let $\Omega_n^l = \Omega_n \bigcap \{(x,\tau) : \alpha_l < \tau < t\}, \Omega_n^0 = \Omega_n \bigcap \{(x,\tau) : 0 < \tau < t\}, S\Omega_n^l = S\Omega_n \bigcap \{(x,\tau) : \alpha_l < \tau < t\}, S\Omega_n^0 = S\Omega_n \bigcap \{(x,\tau) : 0 < \tau < t\}$. Since u_n is a classical solution of (3.10), it satisfies

$$\int\limits_{\Omega_n(t)} u_n f dx = \int\limits_{\Omega_n(\alpha_l)} u_n f dx + \int\limits_{\Omega_n^l} (u_n f_\tau + u_n^m \Delta f) dx d\tau - \int\limits_{S\Omega_n^l} u_n^m \frac{\partial f}{\partial \nu} dx d\tau, \tag{3.14}$$

for arbitrary $f \in C_{x,t}^{2,1}(\overline{\Omega}_n^l)$ that equals to zero on $S\Omega_n^l$, and $\nu = \nu(x,\tau)$ is the outward-directed normal vector to $\Omega_n(\tau)$ at $(x,\tau) \in S\Omega_n^l$. Since g is the weak solution of the DP (1.1), (1.2), we also have

$$\int\limits_{\Omega_n(t)} g f dx = \int\limits_{\Omega_n(\alpha_l)} g f dx + \int\limits_{\Omega_n^l} (g f_\tau + g^m \Delta f) dx d\tau - \int\limits_{S\Omega_n^l} g^m \frac{\partial f}{\partial \nu} dx d\tau. \tag{3.15}$$

Substracting (3.15) from (3.14), we derive

$$\int\limits_{\Omega_n(t)} (u_n - g) f dx = \int\limits_{\Omega_n(\alpha_l)} (u_n - g) f dx - \int\limits_{S\Omega_n^l} (u_n^m - g^m) \frac{\partial f}{\partial \nu} dx d\tau + \int\limits_{\Omega_n^l} (u_n^{\frac{1}{\gamma}} - g^{\frac{1}{\gamma}})[C_n f_\tau + A_n \Delta f] dx d\tau, \tag{3.16}$$

where $C_n = 1$ if $m > 1$ (accordingly $\gamma = 1$) and $C_n = B_n$ if $0 < m < 1$, and

$$A_n = m\gamma \int\limits_0^1 (\theta u_n^{\frac{1}{\gamma}} + (1-\theta) g^{\frac{1}{\gamma}})^{m\gamma - 1} d\theta, B_n = \gamma \int\limits_0^1 (\theta u_n^{\frac{1}{\gamma}} + (1-\theta) g^{\frac{1}{\gamma}})^{\gamma - 1} d\theta.$$

The functions A_n and B_n are Hölder continuous in $\overline{\Omega}_n$. From (3.12) and Definition 1.1 we derive

$$n^{\frac{1-m\gamma}{\gamma}} \leq A_n \leq \overline{A}, n^{\frac{1-\gamma}{\gamma}} \leq B_n \leq \overline{B} \text{ for } (x,\tau) \in \overline{\Omega}_n, \tag{3.17}$$

where $\overline{A}$, $\overline{B}$ are some positive constants which do not depend on n. To choose the test function $f = f(x,\tau)$ in (3.16), consider the following problem:

$$C_n f_\tau + A_n \Delta f = 0 \text{ in } \Omega_n^0 \bigcup B\Omega_n \tag{3.18a}$$

$$f = 0 \text{ on } S\Omega_n^0 \text{ and } f = \omega(x) \text{ on } \Omega_n(t). \tag{3.18b}$$

This is the linear non-degenerate backward-parabolic problem. From the classical parabolic theory ([3,4]) it follows that there exists a unique classical solution $f_n \in C^{2+\beta,1+\beta/2}_{x,\tau}(\overline{\Omega^0_n})$ with some $\beta > 0$. From the maximum principle it follows that

$$|f_n| \leq 1 \quad \text{in } \overline{\Omega^0_n} \tag{3.19}$$

By the condition of theorem assumption $\mathcal{M}$ is satisfied uniformly on every compact subsegment of $(0,t]$. We prove that for every fixed l (see (3.13)) there exists a positive constant $C(l)$, which does not depend on n, such that

$$\sup_{z\in S\Omega^l_n} |\nabla f_n(z)| \leq C(l) n^{\frac{(1+\nu)(m+\epsilon-\frac{1}{\gamma})}{1+2\nu}}. \tag{3.20}$$

To prove (3.20), we use the modification of the method proposed in [1] for proving the boundary regularity of the solution to Dirichlet problem. We use (3.20), in order to estimate the right-hand side of (3.16) with $f = f_n(x,\tau)$, which is a solution of the problem (3.18). We have

$$\int\limits_{\Omega_n(t)} (u_n - g)\omega(x)dx = \int\limits_{\Omega_n(\alpha_l)} (u_n - g) f dx - \int\limits_{S\Omega^l_n} (u^m_n - g^m)\frac{\partial f}{\partial \nu} dx d\tau \equiv \mathcal{I}_1 + \mathcal{I}_2 \tag{3.21}$$

By using (3.4) - (3.6), we have

$$|\mathcal{I}_2| \leq \sup_{z\in S\Omega^l_n} |\nabla f(z)| \int\limits_{S\Omega^l_n} (|\psi^m_n - \Psi^m| + |\Psi^m - g^m|) dx d\tau \leq (C+1)n^{-m} \sup_{z\in S\Omega^l_n} |\nabla f(z)|. \tag{3.22}$$

From (3.20), (3.21) we derive

$$|\mathcal{I}_2| \leq (C+1)C_1(l) n^{\frac{\epsilon(1+\nu)-\frac{1}{\gamma}-\nu(m+\frac{1}{\gamma})}{1+2\nu}}. \tag{3.23}$$

where ϵ and γ are chosen as in (3.8),(3.9). Applying (3.19), we have

$$|\mathcal{I}_1| \leq \int\limits_{\Omega_n(\alpha_l)} |u_n - g| dx. \tag{3.24}$$

To estimate the right-hand side, introduce a function

$$u^l_n(x) = \begin{cases} u_n(x,\alpha_l), & x \in \overline{\Omega_n(\alpha_l)} \\ \psi_n(x,\alpha_l), & x \in \overline{\Omega(\alpha_l)} \backslash \overline{\Omega_n(\alpha_l)}. \end{cases}$$

Obviously $u^l_n(x), x \in \overline{\Omega(\alpha_l)}$ is bounded uniformly with respect to n, l. From (3.24), we have

$$|\mathcal{I}_1| \leq \int\limits_{\Omega(\alpha_l)} |u^l_n - g| dx. \tag{3.25}$$

Since $\lim\limits_{n\to+\infty} u_n^l(x) = u(x,\alpha_l)$ for $x \in \overline{\Omega(\alpha_l)}$, from Lebesgue's theorem it follows that

$$\lim_{n\to+\infty} \int\limits_{\Omega(\alpha_l)} |u_n^l - g|dx = \int\limits_{\Omega(\alpha_l)} |u(x,\alpha_l) - g(x,\alpha_l)|dx \tag{3.26}$$

Hence, by using (3.22) - (3.25) in (3.21) and passing to the limit $n \to +\infty$ we have

$$\int\limits_{\Omega(t)} (u-g)\omega(x)dx \le \int\limits_{\Omega(\alpha_l)} |u-g|dx. \tag{3.27}$$

Let

$$U_l(x) = \begin{cases} u(x,\alpha_l) - g(x,\alpha_l), & x \in \overline{\Omega(\alpha_l)} \\ 0, & x \notin \overline{\Omega(\alpha_l)} \end{cases}$$

Obviously, U_l is uniformly bounded with respect to l. Hence, from (3.27) we derive that

$$\int\limits_{\Omega(t)} (u-g)\omega(x)dx \le \int\limits_{B\Omega} |U_l(x)|dx + C_2 \cdot meas(\Omega(\alpha_l)\backslash B\Omega), \tag{3.28}$$

where the constant C_2 does not depend on l. From Lebesgue's theorem it follows that

$$\lim_{l\to+\infty} \int\limits_{B\Omega} |U_l(x)|dx = 0.$$

Hence, passing to the limit $l \to +\infty$, from (3.28), (3.2) follows. As it is explained earlier, from (3.2), (3.1) with $j = 1$ follows. Similarly, we prove (3.1) (step by step) for each $j = 2, \dots, k$. Theorem 2.2 is proved. The proofs of the Theorems 2.3 and 2.4 are similar to the given proof.

References

1. U.G. Abdulla, On the Dirichlet problem for the nonlinear diffusion equation in non-smooth domains, *Journal of Mathematical Analysis and Applications*, 260, 2 (2001), 384-403.

2. U.G. Abdulla, On the Dirichlet problem for reaction-diffusion equations in non-smooth domains, *Nonlinear Analysis*, 47, 2 (2001), 765-776.

3. O.A. Ladyzhenskaya, V.A. Solonnikov and N.N. Uralceva, Linear and Quasilinear Equations of Parabolic Type, *American Mathematical Society*, Providence RI, 1968.

4. G.M.Lieberman, Second Order Parabolic Differential Equations, *World Scientific*, 1996.