(EE 5201) Linear Systems

Lecture 20

Fourier transform

1. \( X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \) \( \rightarrow \) continuous function in frequency domain

2. \( X(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \) \( \rightarrow \) continuous function in time domain

To implement (1) and (2) using computers, one needs to know the relationship between samples of \( x(t) \) and samples of \( X(\omega) \). The sampling in both domains need to satisfy specific sampling criteria.

True domain \quad Frequency domain

\[ x(t) \rightarrow X(\omega) \]

\[ \sum_{n=-\infty}^{\infty} X(\omega) e^{j\omega n} \]

\[ x(t) \rightarrow \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega) e^{j\omega n} \]

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Periodically extended time domain signal.
Reconstruction from Frequency domain samples

Method 1
* Use Frequency domain reconstruction formula to recover \( \hat{X}(\omega) \)
* Use filtering in Frequency domain to generate \( \hat{X}(\omega) \)
* Apply \( \frac{1}{T_0} \int_{-\infty}^{\infty} \hat{X}(\omega)e^{j2\pi f t} \) to determine \( \hat{x}(t) \)

Method 2
* Use sampled Frequency domain to generate sampled and periodically extended time domain
* Pick one period of time domain signal
* Use time domain interpolation to regenerate original signal \( \hat{x}(t) \)

Other methods are possible

Number of Samples

Interestingly, observation

* Number of samples in one period of a signal

\[ N = \frac{T_0}{f_s} \]

* Number of samples in one period in Frequency domain

\[ N_f = \frac{f_s}{f_0} \]

One observes \( N_1 = \frac{T_0}{f_s} = \frac{\sqrt{f_0}}{f_s} = \frac{f_s}{f_0} \equiv N_0 \)

Number of samples in both domains is the same
Discrete Fourier transform (DFT)

Let \( x(nT) \) - \( n \)th sample of the signal in time domain

\( X(e^{j\omega}) \) - \( \omega \)th sample of the signal spectrum in frequency domain

Define \( X_n = T_x X(nT) = \frac{T_0}{N} X(nT) \)

\( X_r = X(r\omega_0) \), where \( \omega_0 = \frac{2\pi}{T_0} \) - frequency domain resolution

DFT Theorem: Samples \( x_n \) and \( X_r \) are related through the formulas

\[ X_r = \sum_{n=0}^{N-1} x_n e^{-j2\pi rn/N} \]  

\[ x_n = \frac{1}{N} \sum_{r=0}^{N-1} X_r e^{j2\pi rn/N} \]

where \( \Omega_0 = \omega_0 T = \frac{2\pi}{N} \)

Explanations (1) and (2) define DFT pair, \( X_n \leftrightarrow X_r \)

Note 1: Other references may have different scaling factors

Note 2: \( x_n, X_r \) are samples of the time domain and frequency domain periodic extensions of the signal. To properly interpret the values, one needs to know \( T_0 \) - sampling time (i.e., \( f_0 \) - sampling rate)

\( \Omega_0 \) - duration of the signal (i.e., \( f_0 \) - resolution in \( f \) domain)

Proof: The sampled signal \( x_{H}(t) \)

\[
\begin{align*}
\mathcal{X}(H) &= \sum_{n=0}^{N-1} x(nT) e^{-j2\pi H nT} \\
\mathcal{F}[x_{H}(t)] &= \sum_{n=0}^{N-1} x(nT) e^{-j2\pi H nT} - \text{Fourier transform of the sampled signal}
\end{align*}
\]
Consider interval \( |w| \leq \omega_s \). If the aliasing is negligible,

\[ X(w) = T_s \overline{X}(\omega) = T_s \sum_{n=0}^{N_o-1} X(nT_s) e^{-j\omega nT_s} \quad |w| \leq \omega_s \]

Also,

\[ X_r = X(r\omega_s) = T_s \sum_{n=0}^{N_o-1} X(nT_s) e^{-j\omega r nT_s} \]

If we let \( \omega_0 T_s = \omega_0 \), then one obtains

\[ \omega_0 = \omega_0 T_0 = 2\pi \frac{f_0}{T_0} = 2\pi \frac{1}{T_0/T_s} = \frac{2\pi}{N_o} \]

Therefore,

\[ X_r = \sum_{n=0}^{N_o-1} T_s X(nT_s) e^{-j\omega nT_s} = \sum_{n=0}^{N_o-1} X(nT_s) e^{-j\omega_0 (m-n) T_s} \]

which is direct form of the DFT.

In the reverse direction, consider

\[ \sum_{r=0}^{N_o-1} X_r e^{j\omega_0 r} = \sum_{r=0}^{N_o-1} \left( \sum_{n=0}^{N_o-1} X_n e^{-j\omega_0 n} \right) e^{j\omega_0 r} \]

\[ = \sum_{n=0}^{N_o-1} X_n \sum_{r=0}^{N_o-1} e^{j\omega_0 (m-n) T_s} \]

Consider sum

\[ \sum_{r=0}^{N_o-1} e^{j\omega_0 (m-n) T_s} = \begin{cases} \frac{1 - e^{-j\omega_0 (m-n) T_s}}{1 - e^{-j\omega_0 T_s}} & \text{for} \quad m \neq n \\ N_o & \text{for} \quad m = n \end{cases} \]

Therefore,

\[ \sum_{r=0}^{N_o-1} X_r e^{j\omega_0 r} = \sum_{n=0}^{N_o-1} X_n \sum_{r=0}^{N_o-1} e^{j\omega_0 (m-n) T_s} = X_m \]
Therefore \( X_m = \sum_{r=0}^{N_0-1} X_r e^{-i \frac{2\pi r}{N_0}} \).

Since \( X_r \) is periodic, one can use only \( N_0 \) samples. It is however customary to use samples from 0 to \( N_0 - 1 \).

Signal is truncated in time domain to duration of \( T_0 \).

\[ \text{DFT} \quad N_0 = \frac{T_0}{T_s} \]

Signal is sampled in time domain with the sampling interval \( T_s \).

\( T_s \leq \frac{1}{2\pi} \) B - bandwidth of the signal.

Sampled version of the spectrum of truncated signal.

\( f_0 = \frac{1}{T_0} \) - resolution in frequency domain.

\[ X(r) = X[r(\omega)] = T_s \sum_{h=0}^{N_0-1} X[n] e^{-j \frac{2\pi r}{N_0} T_s} \]

\[ = T_s \sum_{h=0}^{N_0-1} X[n(\omega)] e^{-j \frac{2\pi r}{T_s} T_0} \]

\[ = T_s \sum_{n=0}^{N_0-1} X[n(\omega)] e^{-j \frac{2\pi r}{N_0} T_0}. \]

Note: DFT transforms two numbers into other \( N_0 \) numbers. It is only the knowledge of \( T_0, T_s \) that allows interpretation of the data.
Choice of $T_s$ and $T_o$ (and $N_o$)

DFT can only be used if the suitable choice of $T_s$ and $T_o$.

Step 1: Decide on $B$ - bandwidth of the signal

One needs to be aware that due to truncation in time domain, the signal becomes unlimited in frequency domain.

$$X_{T_o}(f) = \mathcal{F} \left\{ \left[ U(t) - U(t - T_o) \right] \cdot X(t) \right\}$$

$$X_{T_o}(f) = \mathcal{F} \left\{ \left[ U(t) - U(t - T_o) \right] \right\} \cdot X(f) = \frac{1}{T_o} \left[ \Pi \left( \frac{t - T_o}{T_o} \right) \right] \cdot X(f)$$

Note: $B$ needs to be larger than the bandwidth of the signal to avoid aliasing that is due to truncation in time domain.

Step 2: Once $B$ is known, the sampling frequency in time domain is set as

$$f_s \geq 2B \quad \text{or} \quad T_s \leq \frac{1}{2B}$$

Step 3: The resolution in frequency domain is determined by $T_o$. If one knows what $f_o$ is (i.e., separation of samples in frequency domain), then
The bandwidth of the signal is \( B = 10 \text{ kHz} \) and required resolution in frequency domain is 100 Hz. Determine \( N_0 \).

\[ N_0 = \frac{T_0}{T_s} \]

Increasing \( N_0 \) from 200 to 256 may be used two ways:

1) Reduce aliasing error
2) Improve frequency resolution
a) Reducing aliasing error

\[ f_s = \frac{f_0}{N_0} = \frac{f_0}{N_0} = \frac{1}{T_{\text{max}} \times 256} = 2.56 \text{ kHz} \]

b) Improving resolution

\[ T_0 = N_0 \times T_s = 256 \times (50 \mu s) = 12.8 \mu s \]

\[ f_0 = \frac{1}{T_0} = 78.125 \text{ Hz} \quad \text{(improved resolution in frequency domain)} \]

c) Combination of the two:

\[ T_s = 45 \mu s, \quad T_0 = 256 \times T_s = 11.5 \mu s, \quad f_0 = \frac{1}{T_0} = 86.96 \text{ Hz} \]

Problems

8.5-1
8.5-2
8.5-3
8.5-4