Unit Impulse Response

Consider an {\( n \)}th-order difference equation

\[
(F_n + a_1 F_{n-1} + \ldots + a_m F_{n-m}) y[n] = (b_0 F_n + b_1 F_{n-1} + \ldots + b_m) x[n] \quad (\star)
\]

The response of the system to input \( x[n] \)

\[
y[n] = y_1[n] + y_2[n]
\]

\( y_1[n] \) - obtained from the analysis of the original order (characteristic polynomials)

\( y_2[n] \) - obtained as a convolution between input \( x[n] \) and impulse response.

Impulse response may be obtained in two ways.

1) Direct evaluation of difference equation (\( \star \))

2) Closed form solution of difference equation (\( \star \)).

1) Direct evaluation:

\[
\omega(E) \cdot h[n] = P(E) \cdot \delta[n] \quad (\star)
\]

subject to:

\[
h[-1] = h[-2] = \ldots = h[-n] = 0
\]

Equation (\( \star \)) can be easily evaluated for every \( n \geq 0 \) and will provide.

Impulse response of the system.

Example 3.11. Find the impulse response of the system given by

\[
y[n] = 0.5 y[n-1] - 0.16 y[n-2] = 5 \times x[n]
\]
To determine impulse response we substitute $X[n]=\delta[n]$:

$$h[n]=0.6h[n-1]-0.16h[n-2]=5\delta[n],$$
or

$$h[n]=0.6h[n-1]+0.16h[n-2]+5\delta[n]$$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\delta[n]$</th>
<th>$h[n-1]$</th>
<th>$h[n-2]$</th>
<th>$h[n]$</th>
</tr>
</thead>
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<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>5</td>
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<tr>
<td>1</td>
<td>0</td>
<td>5</td>
<td>0</td>
<td>3</td>
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<tr>
<td>2</td>
<td>0</td>
<td>3</td>
<td>5</td>
<td>2.6</td>
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<tr>
<td>3</td>
<td>0</td>
<td>2.6</td>
<td>3</td>
<td>2.04</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>2.04</td>
<td>2.6</td>
<td>1.64</td>
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<tr>
<td>5</td>
<td>0</td>
<td>1.64</td>
<td>2.04</td>
<td>1.3104</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>...</td>
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</tbody>
</table>

**Pro:** Impulse response is easily evaluated. With a simple script one can easily determine value of impulse response for any $n$.

**Con:** The method does not provide closed form of the impulse response. No additional insight that one obtains from the closed form.

2) Closed form of the impulse response

$$h[n]=A_0\delta[n]+y_c[n].u[n]$$

- Proof of impulse response ($\dagger$)
  - Direct coupling of response in the form of
    - Input - output
    - Natural modes

Substitution of ($\dagger$) in ($\star$) one obtains

$$\phi(E).\left(A_0\delta[n]+y_c[n].u[n]\right)=p(E).u[n]$$

$$\phi(E)A_0\delta[n]+\phi(E)y_c[n].u[n]=p(E).u[n]$$
Therefore

\[ A_0 Q(E) \delta[n] = R(E) \delta[n] \], or

\[ A_0 \left( \delta(n+N) + a_1 \delta(n+N-1) + \ldots + a_N \delta[n] \right) = b_0 \delta[n+N] + \ldots + b_N \delta[n] \]

This equation is true for every \( N \). Setting \( N = 0 \), one obtains

\[ A_0(\delta(N) + a_1 \delta(N-1) + \ldots + a_N \delta[0]) = b_0 \delta[0] + \ldots + b_N \delta[0] \], or

\[ A_0 a_N \delta[0] = b_N \implies A_0 = b_N/a_N \]

Therefore

\[ h[n] = b_N/a_N \delta[n] + y_c[n] \cdot u[n] \]

subject to initial conditions \( h[0], h[1], \ldots, h[N-1] \) and may be determined through recursive evaluation of the difference equation.

Example 3.12 Consider the system given by difference equation

\[ y[n] - 0.6 y[n-1] - 0.16 y[n-2] = s[n] \], or

\[ (E^2 - 0.6E - 0.16).y[n] = s \cdot x[n] \]

Step 1

\[ Q(x) = x^2 - 0.6x - 0.16 \]

\[ \lambda_{1,2} = \frac{0.6 \pm \sqrt{0.36 + 0.64}}{2}, \quad \lambda_1 = 0.8, \quad \lambda_2 = -0.2 \]

Step 2

\[ h[n] = b_N/a_N \delta[n] + y_c[n] \cdot u[n] \]

Since \( b_N = 0 \)
\[ h[n] = [c_1(-0.2)^n + 2 \cdot 0.8^n] \times M[n] \]

\[ n = 0 \quad c_1 + c_2 = 5 \]
\[ n = 1 \quad -0.2c_1 + 0.8c_2 = 3 \]

Solving the system yields \( c_1 = 1 \), \( c_2 = 4 \), and therefore:

\[ h[n] = [(-0.2)^n + 4 \cdot 0.8^n] \times M[n] \]

Exercise 3.14: Find impulse response of following LTI systems.

a) \( y[n+1] - y[n] = x[n] \)

\[ y[n] - y[n-1] = x[n-1] \quad b_0 = 1, \quad a_0 = -1 \]

Step 1: \( q(z) = z - 1 \)
\[ q(a) = 2 - 1 = 0 \quad \Rightarrow \quad a = 1 \]
\[ y[n] = c_1 M[n] = c M[n] \]

Step 2: \( h[n] = \frac{b_0}{a_0} \delta[n] + y[n] M[n] \)

\[ h[n] = -1 \delta[n] + c M[n] \]

Step 2: Initial conditions:

\[ h[0] - h[-1] = \delta[-1] \]
\[ h[0] - 0 = 0 \quad \Rightarrow \quad h[0] = 0 \]

\[ h[0] = -1 \delta[0] + c M[0] = -1 + c = 0 \quad \Rightarrow \quad c = 1 \]

d) \[ y[n] = 2x[n] - 2x[n-1], \text{ or} \]
\[ y[n+1] = 2x[n+1] - 2x[n] \]
\[ Q(E) = E + 0 \]
\[ P(E) = 2E - 2 - 1 \ldots \]

In this case, \( a_0 = 0 \), and the impulse response cannot be determined using the formula:
\[ h[n] = \frac{dy}{dn} + y[n-1]u[n] \]

Using direct evaluation:
\[ h[n] = 2 \delta[n] - 2 \delta[n-1] \]
\[ h[0] = 2 \delta[0] - 2 \delta[-1] = 2 \]
\[ h[1] = 2 \delta[1] - 2 \delta[0] = -2 \]
\[ h[2] = 2 \delta[2] - 2 \delta[1] = 0 \ldots \]
\[ h[n] = 2 \delta[n] - 2 \delta[1] \]

System response to external inputs

\[ X[n] = \sum_{m=-\infty}^{\infty} X[m] \delta[n-m] = \]
\[ = x[1] \delta[n-1] + x[0] \delta[n] + \ldots + x[n-1] \delta[n-n] + \ldots \]
Using superposition,

\[ x[n] = \sum_{m=-\infty}^{\infty} x[m] \delta[n-m] \quad \Rightarrow \quad y[n] = \sum_{m=-\infty}^{\infty} x[m] h[n-m] \]

Define convolution in DT domain,

\[ x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m] h[n-m] \quad (*) \]

Some properties of convolutional system,

1) Commutative property \( x_1[n] * x_2[n] = x_2[n] * x_1[n] \)
2) Distributive property \( x_1[n] * (x_2[n] + x_3[n]) = x_1[n] * x_2[n] + x_1[n] * x_3[n] \)
3) Associative property \( x_1[n] * (x_2[n] * x_3[n]) = (x_1[n] * x_2[n]) * x_3[n] \)
4) The shift property

\[ x_1[n] * x_2[n] = c[n] \]

\[ \text{Then } x[n-m] * x[n-p] = c[n-m-p] \]

5) Convolution with an impulse

\[ x[n] * \delta[n] = x[n] \]

Evaluating \((*)\) is valid in general case, if the system is causal and the input is causal, then the output is calculated as,

\[ y[n] = \sum_{m=0}^{n} x[m] h[n-m] = \sum_{m=0}^{\infty} x[n-m] \cdot h[m] \]
Example 3.13. Consider

\[ x[n] = 0.8^n u[n] \]
\[ y[n] = 0.3^n u[n] \]

Determine \( x[n] \ast y[n] \)

\[ c[n] = x[n] \ast y[n] = \sum_{m=0}^{\infty} x[m] \cdot y[n-m] = \]

\[ = \sum_{m=0}^{\infty} 0.8^m \cdot 0.3^{n-m} \cdot u[n-m] = \]

\[ = \sum_{m=0}^{\infty} 0.8^m \cdot 0.3^{n-m} = 0.3^n \sum_{m=0}^{\infty} \left( \frac{0.8}{0.3} \right)^m \]

\[ c[n] = 0.3^n \cdot \left[ 1 + \left( \frac{0.8}{0.3} \right) + \left( \frac{0.8}{0.3} \right)^2 + \ldots + \left( \frac{0.8}{0.3} \right)^n \right] \]

\[ c[n] \cdot \left( \frac{0.8}{0.3} \right) = 0.3^n \cdot \left[ \left( \frac{0.8}{0.3} \right) + \left( \frac{0.8}{0.3} \right)^2 + \ldots + \left( \frac{0.8}{0.3} \right)^{n+1} \right] \]

\[ c[n] \cdot \left( 1 - \frac{0.8}{0.3} \right) = 0.3^n \cdot \left( 1 - \left( \frac{0.8}{0.3} \right)^{n+1} \right) \]

\[ c[n] = 0.3^n \cdot \frac{1 - \left( \frac{0.8}{0.3} \right)^{n+1}}{1 - \frac{0.8}{0.3}} = \frac{0.8^{n+1} - 0.3^{n+1}}{0.5} \]

General results:

\[ S_n = 1 + a + a^2 + \ldots + a^n = \frac{1 - a^{n+1}}{1 - a} \]
\[ S_{oo} = 1 + a + a^2 + \ldots = \frac{1}{1-a}, \quad |a| < 1 \]

Note: Text book provides a table with some commonly encountered convolutional pairs. Excellent exercise is to validate the entries in the table.
Response to complex inputs

In general case  $X[n] = X_r[n] + jX_i[n]$  
$\angle[n] = h_r[n] + j\angle[n]$

$Y[n] = X[n] \times H[n] = (X_r[n] + jX_i[n]) \times (h_r[n] + j\angle[n]) = $

$= (X_r[n] \times h_r[n] - X_i[n] \times \angle[n]) +$

$\angle (X_i[n] \times h_r[n] + X_r[n] \times \angle[n])$

Problems:

3.7-1  3.8-1
3.7-2  3.8-2
3.7-4  3.8-3
2.8-4  2.8-5
2.8-7