Lecture 22) Digital Transmission Through Bandlimited Channels

- Transmission channel is modeled as a linear filter with bandlimited behavior.
- In practice, all channels are bandlimited.
- The reasons can be:
  1. Channel physical limitations
     - Example: twisted pair has characteristics of a low-pass filter, with 8 dB passband rolloff.
  2. Hull-pair requirements
     - Limits imposed to allow sharing of the spectrum.
     - Example: FM radio station is assigned 200 kHz of spectrum allowing speech of multiple FM radio stations in the same geographical region.

Model of communication through bandlimited channels:

\[ \text{Sw(t)} \rightarrow C(t) \rightarrow \text{RX} \rightarrow \text{NTI} \]

- \( \text{Sw(t)} \) - digital symbol (one of the two from the transmission alphabet)
- \( C(t) \) - impulse response of the channel (assumed time invariant)
- \( \text{NTI} \) - AWGN with PSD = \( N_0 \) (single sided)
- \( \text{RX} \) - receive subsection.

Illustration of the bandlimited transmission:

\[ \text{Sw(t)} = A \cdot \text{I}(t+\frac{T}{2}) \]

The channel is modeled as an ideal low-pass filter with passband frequency of \( \frac{1}{T} \).
The problem is easily analyzed in frequency domain.

\[ S_{\text{out}}(f) = A \prod \frac{\omega_i}{\omega_c} \Rightarrow S_{\text{out}}(f) = AT \text{sinc}(Tf) \]

Therefore,

\[ S_{\text{out}}(f) = S_{\text{out}}(f) \cdot C(f) = AT \text{sinc}(Tf) \prod \frac{\omega_i}{\omega_c} e^{-j2\pi f T_0} \]

And

\[ S_{\text{out}}(f) = AT \int_{-\infty}^{\infty} \text{sinc}(Tf) \prod \frac{\omega_i}{\omega_c} e^{j2\pi f (t-\delta)} df \]

\[ S_{\text{out}}(f) = AT \int_{-\infty}^{\infty} \text{sinc}(Tf) e^{j2\pi f (t-\delta)} df \]

\[ = AT \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin[2\pi f (t-\delta)]}{\omega_c T} \cos[2\pi f (t-\delta)] df \]

\[ = AT \left\{ \int_{0}^{\infty} \frac{\sin[2\pi f (t-\delta + T/2)]}{\omega_c T} df + \int_{0}^{\infty} \frac{\sin[2\pi f (t-\delta - T/2)]}{\omega_c T} df \right\} \]

(used trig identity: \( \sin A \cos B = \frac{1}{2} \sin(A+B) + \frac{1}{2} \sin(A-B) \))

After substitution for first integral \( 2\pi f (t-\delta + T/2) = x \) \( df = \frac{dx}{2\pi (t-\delta + T/2)} \)

Second integral \( 2\pi f (t-\delta - T/2) = x \)
One obtains

\[ 2\pi \text{Wc}(t-\lambda+\frac{1}{2}) \]

Solve:

\[ A \cdot T \int_0^{2\pi (t-\lambda+\frac{1}{2})} \frac{\sin(x)}{x} \frac{dx}{2\pi (t-\lambda+\frac{1}{2})} \]

\[ = A \int_0^{2\pi \text{Wc}(t-\lambda+\frac{1}{2})} \frac{\sin(x)}{x} \frac{dx}{2\pi (t-\lambda+\frac{1}{2})} \]

\[ = \frac{A}{\pi} \left\{ \int_0^{2\pi \text{Wc}(t-\lambda+\frac{1}{2})} \frac{\sin(x)}{x} \, dx - \int_0^{2\pi \text{Wc}(t-\lambda+\frac{1}{2})} \frac{\sin(x)}{x} \, dx \right\} \]

Integrals cannot be evaluated in closed form.
They appear frequently and are defined as separate functions:

\[ \text{Si}(t) = \int_0^t \frac{\sin(x)}{x} \, dx \quad \text{integral sine} \]

Therefore:

\[ \text{Sol(t)} = \frac{A}{\pi} \left\{ \text{Si}[2\pi \text{Wc}(t-\lambda+\frac{1}{2})] - \text{Si}[2\pi \text{Wc}(t-\lambda+\frac{1}{2})] \right\}(\lambda) \]

Simplifying under assumption \( \lambda \to 0 \) (just translation in time):

\[ \text{Sol(t)} = \frac{A}{\pi} \left\{ \text{Si}[2\pi \text{WcT} \frac{1}{4} - \frac{1}{2}] - \text{Si}[2\pi \text{WcT} \frac{1}{4} - \frac{1}{2}] \right\} \]

1) Graph of \( \text{Si}(t) \) vs. \( \text{Si}(t) \)

![Graph of Si(t)](image-url)
Consider new output of the system for various $W_cT$ products.

$$S_n(t_n) = \frac{A}{T} \left( S_n \left[ 2\pi W_cT \left( t_n + \frac{1}{2} \right) \right] - S_n \left[ 2\pi W_cT \left( t_n - \frac{1}{2} \right) \right] \right)$$

Notes:

1. The signal of the output is distorted.
2. The level of distortion is inversely proportional to the channel BW ($\propto W_c$).
3. Distortion introduces "noise" into the output signal.

- Output signal exists in time that is outside of signaling period.
- Signal in interval $(\frac{1}{2}, \frac{3}{2})$ will "caus" signals coming after. - Inter symbol interference.
4) Since channel is always bandlimited, the signal needs to be adjusted so that it fits within the allocated spectrum.

The power spectrum of Digitally Modulated Signals

We will study these results for band-limited PAM modulated signal.

Note: Both PSK & QAM can be seen as superposition of two PAM signals.

**Baseband PAM's signal**

\[ v(t) = \sum_{n=-\infty}^{\infty} A_n \psi(t-nT) \]  

- \( A_n \) = sequence of amplitudes
- \( \psi(t) \) = basis vector
- \( E_q \) = energy of the pulse.

**PSD evaluation**

\[ R_{xx}(\tau) = E[v(t)v(t+\tau)] \]

\[ = E = \sum_{n=-\infty}^{\infty} A_n \psi(t-nT) \sum_{m=-\infty}^{\infty} A_m \psi(t-mT+\tau) \]

\[ = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E[AnA_m] \psi(t-nT) \psi(t-mT+\tau) \]

*Ef AnA_m* - property of input random source. Usually assumed that input random sources are WSS with given autocorrelation function.
\[ R_a(w) = E_z \sum_n A_n a_n w^j \]

Therefore

\[ R_a(t+\tau, t) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} R_a(n-m) \psi(t-n\tau) \psi(t-m\tau+\tau) \]

Substituting \( w-n=k \Rightarrow w=k+n \)

\[ R_a(t+\tau, t) = \sum_{k=-\infty}^{\infty} R_a(k) \sum_{n=-\infty}^{\infty} \psi(t-n\tau) \psi(t-n\tau-k\tau+\tau) \quad (x) \]

Additionally,

\[ E_z \psi(t) = E_z \sum_{n=-\infty}^{\infty} A_n \psi(t-n\tau) = \mu \sum_{n=-\infty}^{\infty} \psi(t-n\tau) \quad (xy) \]

From (xy) - the mean of \( \psi(t) \) is periodic, with period \( T \).

From (x) - the autocorrelation of \( \psi(t) \) is periodic with period \( T \).

(1H) - not shown here is specifically speaking, (WK suggested as presented before) do not apply.

However, both mean and correlation are periodic \( \Rightarrow \) PSD can be obtained as a Fourier transform of time averaged correlation function.

\[ R_a(t, \tau) \]

\[ \bar{R}_a(\tau) \]

- time domain average of the autocorrelation function.
\[
\overline{R}_\nu(t) = \frac{1}{T} \int_{-T/2}^{T/2} R_\nu(t, t+\tau) \, d\tau
\]

\[
= \frac{1}{T} \int_{-T/2}^{T/2} \sum_{k=0}^{\infty} R_\alpha(k) \sum_{n=-\infty}^{\infty} \psi(t-nt) \psi(t-nt-kT+\tau) \, d\tau
\]

\[
= \sum_{k=0}^{\infty} R_\alpha(k) \sum_{n=-\infty}^{\infty} \int_{-T/2}^{T/2} \psi(t-nt) \psi(t-nt-kT+\tau) \, d\tau
\]

\[
= \sum_{k=0}^{\infty} R_\alpha(k) \sum_{n=-\infty}^{\infty} \int_{-T/2}^{T/2} \psi(x) \psi(x-kT+\tau) \, dx
\]

\[
= \sum_{k=0}^{\infty} R_\alpha(k) \int_{-\infty}^{\infty} \psi(x) \psi(x+\tau-kT) \, dx
\]

\[
= \sum_{k=0}^{\infty} R_\alpha(k) R_\psi(t-kT)
\]

(1.8) is obtained without assistance.

Where \( R_\psi(t) = \int_{-\infty}^{t} \psi(x) \psi(x+\tau) \, dx \) - autocorrelation function of \( \psi(t) \)

\[
S_\delta(t) = \int_{-\infty}^{\infty} R_\psi(t) e^{-j2\pi f \tau} \, d\tau = \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} R_\alpha(k) R_\psi(t-kT) e^{-j2\pi f \tau} \, d\tau
\]

\[
= \sum_{k=0}^{\infty} R_\alpha(k) \int_{-\infty}^{\infty} R_\psi(t-kT) e^{-j2\pi f \tau} \, d\tau
\]

\[\tau-kT=\tau \implies \tau = \tau + kT\]

\[
S_\psi(t) = \int_{-\infty}^{\infty} R_\psi(t) e^{-j2\pi f \tau} \, d\tau \int_{-\infty}^{\infty} R_\psi(\tau) e^{j2\pi f \tau} \, d\tau
\]

\[
S(f) = \sum_{k=0}^{\infty} R_\alpha(k) e^{-j2\pi k f} \int_{-\infty}^{\infty} R_\psi(t) e^{-j2\pi f \tau} \, d\tau
\]

\[\overline{S(f)} = \sum_{k=0}^{\infty} R_\alpha(k) e^{-j2\pi k f} \int_{-\infty}^{\infty} \overline{R_\psi(t)} e^{-j2\pi f \tau} \, d\tau
\]

\[\min_{k=1} \overline{|S(f)|^2} = 1 \overline{|R_\psi(t)|^2} \]
Finally we obtain

\[ S_a(f) = \frac{1}{T} \cdot S_x(f) \cdot |\psi(f)|^2 \]

where

- \( T \) - period of the symbol
- \( S_x(f) \) - PSD of the discrete process generating information sequence
- \( \psi(f) \) - Fourier transform of the basis vector

Since \( \psi(t) = \frac{1}{\sqrt{T}} \cdot g(t) \)

\[ \psi(f) = \frac{1}{\sqrt{T}} \cdot |G(f)| \]

\( G(f) \) - Fourier transform of the transmission pulse

Homework problem: (8.1)