(Lecture 20) Probability of Error in M-ary PAM

For M-ary PAM

\[ S_{\text{in}}(t) = \sum_{m=1}^{M} A_m \cdot Y(t), \quad m=1, 2, \ldots, M \]  
\[ \text{PAM signal is one-dimensional} \]

\[ Y(t) = \frac{1}{\sqrt{E_g}} g(t), \quad 0 \leq t \leq T \]

\[ g(t) - \text{pulse shape} \]
\[ E_g - \text{energy of the pulse} \]
\[ E_g = \int_0^T g^2(t) dt \]

\[ A_m \cdot Y(t) \rightarrow HF \rightarrow r \]

\[ r = A_m + n \Rightarrow \text{in this case} \quad r = A_m + n \]

For symmetric PAM, the amplitude values can be expressed as

\[ A_m = \left( 2m - 1 - M \right) \cdot A, \quad m=1, 2, \ldots, M \]

Some examples

\( N = 2 \)
\[ A_1 = 2(1) - 1 - 2 = -1 \text{ A} \]
\[ A_2 = 2(2) - 1 - 2 = 1 \text{ A} \]

\( N = 4 \)
\[ A_1 = -3 \text{ A} \]
\[ A_2 = -1 \text{ A} \]
\[ A_3 = +1 \text{ A} \]
\[ A_4 = +3 \text{ A} \]

e.t.c.
Distance between two adjacent PAM symbols is $2 \cdot A$.

Poucube assumption:

1) Symbols are equiprobable
2) The decision boundaries are set at mid point between symbols.

There are 2 cases to consider:

- $P(e) \text{ Rx mid point}$
- $P(e) \text{ Rx end point}$

From Figure it should be obvious that $P(e) \text{ Rx end point} = \frac{1}{2}$ $P(e) \text{ Rx mid point}$

Probability of error for the end point:

$$A_{w} = (2w - 1 - M) \cdot A \bigg|_{w = M} = (2M - 1 - M)A = (M - 1)A$$

$$A_{w} + n$$

$$n_{i} \sim \mathcal{N}(0, \sigma) , \quad \sigma n = \sqrt{\frac{\omega}{2}}$$
The probability density function is given by:

$$P_d^A(r) = \frac{1}{N_{TH}} \exp \left( - \frac{(r - A_H)^2}{N_0} \right)$$

The probability is:

$$P(e) = \int_{-\infty}^{\infty} \frac{1}{N_{TH} N_0} \exp \left( - \frac{(r - A_H)^2}{N_0} \right) dr$$

Substitution:

$$\frac{(r - (H-1)A)}{N_{TH} N_0} = x$$

$$dr = dx \sqrt{\frac{N_0}{2}}$$

$$P(e) = \int_{-\infty}^{\infty} \frac{1}{N_{TH} N_0} \exp \left( - \frac{x^2}{2} \right) dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left( - \frac{x^2}{2} \right) dx = \frac{A}{N_{TH} N_0}$$

Thus, the probability of even H_n middle point is

$$P_{\text{even}}(e) = 0 \left( \frac{A}{N_{TH} N_0} \right)$$

Therefore, the average probability of any PAN signal is
\[ P(e) = \frac{1}{M} \left[ (N-2) \cdot \text{P}\text{u}\text{d}(e) + 2 \cdot \text{P}\text{e}\text{n}\text{t}(e) \right] \]

\[ = \frac{1}{M} \left[ (N-2) \cdot 2 \cdot \text{P}\text{e}\text{n}\text{t}(e) + 2 \cdot \text{P}\text{e}\text{n}\text{t}(e) \right] = \]

\[ = \frac{2(N-1)}{M} \cdot \text{P}\text{e}\text{n}\text{t}(e) = \frac{N-1}{M} \cdot \text{P}\text{u}\text{d}(e) \]

\[ = \frac{2(N-1)}{M} \cdot Q \left( \frac{A}{N^{1/2}_{\text{No}}} \right) \]

Average energy of \( H \)-any path:

\[ E_{av} = \frac{1}{M} \sum_{m=1}^{N} E_{m} = \frac{1}{M} \sum_{m=1}^{N} \int_{0}^{T} \left[ (2m-1-H)A \psi_{m}(t) \right]^{2} dt = \]

\[ = \frac{1}{M} \sum_{m=1}^{N} \left( 2m-1-H \right) A^{2} \int_{0}^{T} \psi_{m}^{*}(t) \psi_{m}(t) dt = \]

\[ = \frac{H^{2}-1}{M} \sum_{m=1}^{N} A^{2} \cdot \frac{1}{M} \cdot \frac{H(H^{2}-1)}{3} \]

\[ = A^{2} \cdot \frac{H^{2}-1}{3} \Rightarrow A = \left[ \frac{3 \cdot E_{av}}{H^{2}-1} \right]^{1/2} \]

Therefore, average probability of error:

\[ P(e) = \frac{2(H-1)}{M} \cdot Q \left[ \left( \frac{3 \cdot E_{av}}{(H^{2}-1) N^{1/2}_{\text{No}}} \right)^{1/2} \right] \]

\[ = \frac{2(H-1)}{M} \cdot Q \left[ \frac{6 \cdot E_{av}}{(H^{2}-1) N^{1/2}_{\text{No}}} \right] \]
Frequently we define $P(e)$ as a function of average power

$$P_{av} = E_{av} / T \Rightarrow E_{av} = T \cdot P_{av}$$

$$P_s(e) = \frac{2(H-1)}{M} \cdot Q \left[ \left( \frac{6 \cdot \log_2(H) \cdot E_{bat}}{M^2 - 1} \cdot N_o \right)^{1/2} \right]$$

- Different PAMs have different symbol rates for same bit rate
- To make comparison between different PAMs, probability of symbol error is expressed as a function of energy per bit

$$E_{bat} = \frac{E_{av}}{\log_2(H)} \Rightarrow E_{av} = \log_2(H) \cdot E_{bat}.$$

Therefore

$$P_s(e) = \frac{2(H-1)}{M} \cdot Q \left[ \left( \frac{6 \cdot \log_2(H) \cdot E_{bat}}{M^2 - 1} \cdot N_o \right)^{1/2} \right] \quad \text{or}$$

$$P_s(e) = \frac{2(H-1)}{M} \cdot Q \left[ \left( \frac{6 \cdot \log_2(H) \cdot SNR_b}{M^2 - 1} \right)^{1/2} - 1 \right]$$

$\Delta P_s(e)$

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$SNR_b = \log_{10} \left( \frac{E_{bat}}{N_o} \right)$

$SNR_b$ - Signal to noise ratio per bit

$SNR_b \text{ [dB]}$

Note: To maintain same symbol error $SNR_b$ needs to be increased.
Probability of error in PSK modulation

\[ y_m = m \frac{2\pi}{M}, \quad m = 0, 1, \ldots, M-1 \]

\[ s_m = \sqrt{E_s} \left[ \cos \left( m \frac{2\pi}{M} \right) \sin \left( m \frac{2\pi}{M} \right) \right] \]

PSK - two dimensional

Constellation of \( M \)-PSK signals

*For analysis we assume perfect synchronisation of the \( M \) oscillators

Probability of symbol error

\[ \Psi_{i-1} = (i-1) \frac{2\pi}{M} \]

\[ \Psi_i = i \cdot \frac{2\pi}{M} \]

\[ \Psi_{i+1} = (i+1) \frac{2\pi}{M} \]

Decision boundaries

\[ T_1 = (i-1) \frac{2\pi}{M} + \frac{\pi}{M} = \frac{(2i-1)\pi}{M} \]

\[ T_2 = i \cdot \frac{2\pi}{M} + \frac{\pi}{M} = \frac{(2i+1)\pi}{M} \]

\[ 1 - P_e(e) = \int_{-\infty}^{\infty} \int_{0}^{T_2} pdf \left( e | s_i \right) \, dA \]

- Probability of correctly receiving \( e \)th symbol.