(Lecture 15) Pulse Amplitude Modulation (PAM)

PAM - Information is conveyed by the amplitude of transmitted signal.

Remember - signaling is through AWGN channel with infinite BW.

Baseband signaling

* The most simple PAM is binary PAM
* In binary PAM the mapping is performed as:

\[ (0) \rightarrow A \Psi(t) = S_1(t) \]
\[ (1) \rightarrow -A \Psi(t) = S_2(t) \]

* This kind of signaling is sometimes referred to as the nonipodal signaling.
* Binary PAM = one signal in the vector space per bit
* Generalization of PAM = M-ary PAM
  M-ary PAM combines several bits and assigns them one of M possible amplitude levels.

Example: Consider the mapping given in the following table
In general, the PAM signals $s_{mt} = A_m g_{it(t)}$, $m=1,2,...,M$, $0 \leq t \leq T$

where

$A_m$ - one of possible $M$ amplitude levels

$g_{it(t)}$ - pulse shape

Vector representation of PAM signal

* Dimension of PAM space is equal to $M$

$A_1 A_2 ... A_2 ... A_M$

* Basis vector of PAM space $\psi_{1t} = g_{1t(t)}/\sqrt{E_g}$

* In general, to limit the bandwidth necessary for transmission, $g_{it(t)}$ has rounded edges.

Typical shape of basis vector for PAM signals.
Energy of the baseband PAM signal

\[ E = \int_{-\infty}^{\infty} s(t) \, dt = \sum_{n=-\infty}^{\infty} a_n \int_{-\infty}^{\infty} g(t+nT) \, dt = A v^2 \sum_{n=-\infty}^{\infty} g^2(nT) \, dt = A v^2 E_g \]

where \( E_g \) is the energy of the pulse signal \( g(t) \).

**Bandpass PAM Signal**

*To obtain bandpass PAM signal, we perform AM modulator at a sinusoidal carrier. Message signal is baseband PAM.*

**Baseband Signal**

\[ s(t) \Rightarrow \cos(2\pi ft) \Rightarrow S(t) = A v g(t) \cos(2\pi ft) \]

\[ \cos(2\pi ft) \]

Therefore

\[ E_{\text{up}} = \int_{-\infty}^{\infty} E_{\text{up}}^2 \, dt = \int_{-\infty}^{\infty} \left( A v \cos(2\pi ft) \right)^2 \, dt = A v^2 \int_{-\infty}^{\infty} g^2(t) \cos^2(2\pi ft) \, dt = A v^2 \int_{-\infty}^{\infty} g^2(t) \left[ \frac{1}{2} + \cos(2 \cdot 2\pi ft) \right] \, dt \]

\[ E_{\text{up}} = \frac{A v^2}{2} g^2(t) = \frac{1}{2} \left( A v g^2(t) \right) = \text{one half of the baseband energy.} \]
**Geometric representation of PAH signal**

**Basic properties for PAH signal**

\[ S(t) = A_m \psi(t), \quad m = 0, 1, \ldots, M = 2^k \]

- **\( k \)**: number of bits per symbol
- **\( \psi(t) \)**: pulse shape used for signaling
- **\( E_g \)**: energy of \( \psi(t) \)

**Corresponding vectors**

\[ S(t) \rightarrow S_w = N \sqrt{E_g} A_m, \quad m = 0, 1, \ldots, M \]

*Important parameter is the Euclidean distance between two signal points*

\[ d_{wn} = ||S_w - S_n|| = \sqrt{\sum (S_w - S_n)^2} = \sqrt{E_g (A_m - A_n)^2} \]

*PAH signals have different energies*

\[ E_w = S_w^2 = (N \sqrt{E_g} A_m)^2 = E_g A_m^2, \quad m = 0, 1, \ldots, M \]

For equally probable PAH signals, the average energy is given by

\[ E_{\text{avg}} = \frac{1}{M} \sum_{m} E_w = \frac{E_g}{M} \sum_{m=1}^{M} A_m^2 \]

**Example:** Consider the case of \( M = 4 \), \( A_1 = -3, A_2 = -1, A_3 = 1, A_4 = 3 \)

\[ E_{\text{avg}} = \frac{E_g}{4} \left( (-3)^2 + (-1)^2 + 1^2 + 3^2 \right) = 5 E_g \]
In general, for symmetric PAM we have:

\[ A_M = (2M-1-1) \], \quad M=1,2,...,H

\[ E_{CM} = E_C (H^2-1)/2 \]

\[ \rightarrow \quad \text{PAM signal is one dimensional} \]

\[ \sim \sqrt{E_C} \]

Bandpass waveform of PAM signal:

\[ u(t) = S_M \psi(t) \], \quad M=1,2,...,H

\[ \psi(t) = \frac{g(t)}{\sqrt{E_C}} \cdot \sqrt{2} \cos(2\pi f_c t) \quad \text{basis function} \]

\[ S_M = \sqrt{E_C/2} \cdot A_M \]

\[ \sqrt{E_C/2} A_1 \quad \sqrt{E_C A_2} \quad \sqrt{E_C A_3} \quad \sqrt{E_C A_4} \ldots \]

\[ \downarrow \quad \text{waveform} \]

\[ \| S_M - S_n \|^2 = \sqrt{E_C} - S_{n^2} = \sqrt{E_C} (H^2 - A_n^2) \]

The only change between passband and baseband representation of PAM signals is in the scaling factor \( \sqrt{2} \)
Two dimensional signal vectors

Two signals are orthogonal over interval \((0,T)\) if

\[
\int_0^T S_1(t) S_2(t) \, dt = 0
\]

from which \( \langle S_1(t), S_2(t) \rangle = \int_0^T S_1(t) S_2(t) \, dt \).

Two examples of orthogonal signals are given as:

\[ S_1(t) = \begin{cases} A & \text{for } 0 \leq t < T \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \]

\[ S_2(t) = \begin{cases} A & \text{for } T \frac{1}{2} \leq t < T \\ 0 & \text{otherwise} \end{cases} \]

Energy of all signals are equal

\[
E = \int_0^T S_1^2(t) \, dt = \int_0^T S_2^2(t) \, dt = \int_0^{T\frac{1}{2}} S_1^2(t) \, dt + \int_{T\frac{1}{2}}^T S_2^2(t) \, dt = A^2 T
\]

Each of the two sets can be used as a basis for a 2-dimensional vector space. Let us select the second one for the basis

\[ \Psi_1(t) = \sqrt{2/T} , \quad 0 \leq t < T/2 \]

\[ \Psi_2(t) = \sqrt{2/T} , \quad T/2 \leq t \leq T \]

\[ \int_0^T \Psi_1^2(t) \, dt = \int_0^T \Psi_2^2(t) \, dt = 1 \]
Then, the signals $S_{1(t)}$ and $S_{2(t)}$ can be represented as a linear combination of $\Psi_{1(t)}$ and $\Psi_{2(t)}$.

$$S_{1(t)} = S_{11} \Psi_{1(t)} + S_{21} \Psi_{2(t)} \quad \&$$
$$S_{2(t)} = S_{21} \Psi_{1(t)} + S_{22} \Psi_{2(t)}$$

**Corresponding vectors**:

$$S_{1} = \begin{bmatrix} A \overline{N} T_{1/2} & A \overline{N} T_{1/2} \end{bmatrix}^T$$

$$\varepsilon_{1} = \| S_{1} \|^{2} = A^{2} T_{1/2} + A^{2} T_{1/2} = A^{2} T$$

$$S_{2} = \begin{bmatrix} A \overline{N} T_{1/2} & -A \overline{N} T_{1/2} \end{bmatrix}^T$$

$$\varepsilon_{2} = \| S_{2} \|^{2} = A^{2} T_{1/2} + A^{2} T_{1/2} = A^{2} T$$