Angle modulation (PM & FM)

Amplitude modulation - Signal used to change properties of the amplitude of the carrier.

Angle modulation - either frequency or the phase of the signal is modified.

Properties of the message signal:

1° $w(t)$ is bandlimited, i.e. $N(t) = 0$, $\forall t ; |t| > W$

2° $w(t)$ is a power-type signal

$$P_w = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} w(t)^2 \, dt$$

In general, the angle modulation can be written in a form

$$w(t) = A_c \cos \left[ 2\pi f_c t + \phi(t) \right]$$

- $A_c$ - Magnitude (always constant)
- $f_c$ - Carrier frequency
- $\phi(t)$ - part of the carrier angle that is made dependent on message

There are two ways that $\phi(t)$ may depend on $w(t)$:

1° $\phi(t) = \int P \cdot w(t)$ - Phase modulation ($PM$)

2° $\phi(t) = 2\pi f_c \int_{-T}^{T} w(t) dC$ - Frequency modulation ($FM$)
From formulas 1.8.20 it is seen that FH may be implemented using PM hardware and vice versa. Consider

\[ \text{FH modulated signal implemented using FH modulator} \]

\[ \text{FH modulated signal implemented using PM modulator} \]

Additionally from 1.8.20 one sees that

\[ \text{PM modulated signal implemented using FH demodulator} \]

\[ \text{PH modulated signal implemented using PH modulator} \]

Since integration and differentiation are linear operators (cannot change the content of the signal), FH & PH have quite similar characteristics. The modulation may be seen as process performed on a processed signal (filtered).

Consider an angle modulated carrier.

\[ \text{\( MH(t) = A e^{j \omega_c t} \cos(2\pi f_c t + \phi_H) \)} \]

Instantaneous phase of the cosine is defined as

\[ \Theta(t) = 2\pi f_c t + \phi_0 \] (basically the term under cosine)
Instantaneous frequency is defined as:
\[ f(t) = \frac{1}{2\pi} \frac{d\Theta(t)}{dt} = \frac{f_c}{2\pi} + \frac{1}{\pi} \frac{d\phi(t)}{dt} \]

Maximum phase deviation is defined as:
\[ \Delta \phi_{\text{max}} = k_p \max |\nu(t)| \quad (\text{For PH modulated signal}) \]

Maximum frequency deviation is defined as:
\[ \Delta f_{\text{max}} = k_f \max |\nu(t)| \quad (\text{For FH modulated signal}) \]

**Example 1.** Consider message signal

\[ u(t) = a \cos(2\pi f_c t) \quad - \text{single tone} \]

\[ v(t) = A e^{-j(\theta(t))} \quad - \text{modulated signal} \]

\[ \Theta(t) = 2\pi f_c t + k_p a \cos(2\pi f_c t) \quad - \text{instantaneous phase} \]

\[ \Delta \phi_{\text{max}} = k_p \max |a \cos(2\pi f_c t)| = k_p A \quad - \text{maximum phase deviation} \]

\[ v(t) = A e^{-j(\theta(t))} \quad - \text{modulated signal} \]

\[ f(t) = \frac{1}{2\pi} \frac{d\Theta(t)}{dt} = f_c + k_f a \cos(2\pi f_c t) \quad - \text{instantaneous frequency} \]

\[ \Delta f_{\text{max}} = k_f \max |a \cos(2\pi f_c t)| = k_f A \quad - \text{maximum frequency deviation} \]
In literature, it is common to define modulation indices

\[ \beta_{FM} = \frac{1}{f_0 \cos \omega t} \cdot \Delta f \cdot \cos \omega t \]  

- phase modulation index

\[ \beta_{PM} = \frac{1}{f_0 \cos \omega t} \cdot \Delta f \cdot \sin \omega t \]  

- frequency modulation index

Using modulation indices for the Example 1.

\[ U_{FM} = A_0 \cos(2\pi f_1 t + \beta_{FM} \cos(2\pi f_2 t + \phi_0)) \]

\[ U_{PM} = A_0 \cos(2\pi f_1 t + \beta_{PM} \sin(2\pi f_2 t + \phi_0)) \]

Spatial Coherence of Angle Modulated Signals

- Angle modulation is nonlinear (not a simple hardening of spectrum)
- Determining spatial footprint of FM signal will be done in 3 steps

1) Find the spectrum when \( U_{FM} \) is a tone
2) Find the spectrum when \( U_{FM} \) is a periodic signal
3) Find the spectrum for any band limited \( U_{FM} \)

Modulation with single tone

\[ U_{PM} = A_0 \cos(2\pi f_1 t + \beta \sin(2\pi f_2 t + \phi_0)) \]

\( \beta \) - modulation index (can be either FM or PM)

One way write

\[ U_{PM} = \Re \{ A_0 e^{j 2\pi f_1 t} \sin(2\pi f_2 t + \phi_0) \} \]
Since \( \sin(2\pi ft) \) is periodic, it follows that \( 2\pi f \in \mathbb{Z} \Rightarrow 2\pi ft \in \{2\pi n|n \in \mathbb{Z}\} \) is periodic as well. The period is \( T = \frac{1}{f_n} \).

2.1(f) \quad \text{Periodic \& Fourier Series}

\[
2\pi f = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n ft}, \quad \text{where } (x)(x)
\]

\[
C_n = \frac{1}{T} \int_{0}^{T} 2\pi \cos(2\pi n ft) \, dt = \frac{1}{f_n} \int_{0}^{\frac{1}{f_n}} e^{j2\pi n ft} \cdot \sin(2\pi ft) \, dt
\]

\[
= \frac{1}{f_n} \int_{0}^{\frac{1}{f_n}} e^{j \left( 2\pi n ft - 2\pi n \frac{t}{f_n} \right)} \, dt
\]

Substituting \( x = 2\pi ft \), \( dt = dx / (2\pi f_n) \)

Therefore,

\[
C_n = \frac{1}{2\pi f_n} \int_{0}^{\pi} e^{j \left( 2\pi n x - nx \right)} \, dx
\]

\[
= \frac{1}{2\pi} \int_{0}^{\pi} e^{j \left( 2\pi n x - nx \right)} \, dx = i^{n}(n) (x)
\]

Where \( i^n(n) \) is the Bessel function of the first kind and order \( n \), that is evaluated at the point \( n \).

Substituting \( \psi \rightarrow (\pi x) \), one obtains

\[
2.1(f) = \sum_{n=-\infty}^{\infty} i^n(n) e^{j2\pi n ft}
\]

Therefore

\[
2.1(f) = Re \left\{ \sum_{n=-\infty}^{\infty} i^n(n) e^{j2\pi n ft} \right\} \in \mathbb{R}
\]
\[
\cos(2\pi ft + 2\pi f_0 t) = \sum_{n=-\infty}^{\infty} a_n e^{j2\pi f_0 t/n}
\]

or
\[
\cos(2\pi ft + 2\pi f_0 t) = \sum_{n=-\infty}^{\infty} a_n e^{j2\pi f_0 t/n}
\]

Notes:
1° Spectrum is infinite even when the message signal is a simple tone.
2° Spectrum components occur at integer increments of modulation frequency.
3° We need to know to evaluate Bessel function of the first kind.

Evaluation of \( J_n(\beta) \):

1° Every Bessel is defined by its Taylor Series:
\[
J_n(\beta) = \sum_{k=0}^{\infty} \frac{(-1)^k (\beta/2)^{n+2k}}{k! (2k)!}
\]

For example:
\[
J_0(0.5) = \sum_{k=0}^{\infty} \frac{(-1)^k (0.5/2)^{n+2k}}{k! (2k)!} =
\]
\[
= 1 - \frac{(-1)(0.5/2)^2}{1!1!} + \frac{(-1)^2(0.5/2)^4}{2!2!} + \frac{(-1)^3(0.5/2)^6}{3!3!}
\]
\[
= 0.9385
\]
Two properties to be noted

1) For small $\beta$

$$J_n(\beta) \approx \frac{\beta^n}{n!} \quad \text{(keeping only first term of the Taylor series)}$$

2) For negative $n$

$$J_{-n}(\beta) = \begin{cases} J_n(\beta), & n \text{ even} \\ -J_n(\beta), & n \text{ odd} \end{cases}$$

Handouts:
- Bessel function graph
- Bessel function table

Homework: Exercise 4.1, 4.2, 4.3 & 4.4
Assignment: none.