Single Side Band (SSB) Amplitude Modulation

1. For a message signal with bandwidth \( W \), both DSB-SC and SSB require bandwidth \( 2W \).
2. The information in the sideband is redundant.
3. Spectrum is frequently the most valuable resource.
4. A method is needed to transmit just a single sideband and to recover the signal from a single sideband transmission.

**Idea:**

\[
\begin{align*}
\text{Message Signal} & \xrightarrow{\text{Modulation}} \text{Modulated Signal} \\
& \xrightarrow{\text{Ideal Filter}} \text{Sideband of Interest}
\end{align*}
\]

**Spectral Characteristics of SSB-AM**

- \( M(t) \) $\leftrightarrow$ \( M(f) \) — message signal
- \( U_{ssb}(t) \) $\leftrightarrow$ \( \frac{A_c}{2} \text{ } M(f+c) + \frac{A_c}{2} \text{ } M(f-c) \) — modulated signal

**Transfer Function of the Ideal Filter**

\[
|H(f)| = \begin{cases} 
1, & |f| \geq f_c \\
0, & \text{otherwise}
\end{cases}
\]

Using step function, defined as \( \mu(x) = \begin{cases} 1, & x > 0 \\
0, & x < 0 \end{cases} \)
The transfer function of the Aller may be written as:

\[ H(f) = \mu (1 - \frac{f}{f_c}) + \mu (1 + \frac{f}{f_c}) \]

When the DSB-SC signal is passed through the high pass Aller, one obtains

\[ U_{ssb}(f) = \frac{A_c}{2} H(f - \frac{f_c}{2}) \mu (1 - \frac{f}{f_c}) + \frac{A_c}{2} H(f + \frac{f_c}{2}) \mu (1 - \frac{f}{f_c}) \]

or equivalently

\[ U_{ssb}(f) = \frac{A_c}{2} H(f - \frac{f_c}{2}) \mu (1 - \frac{f}{f_c}) \text{ upper sideband} + \frac{A_c}{2} H(f + \frac{f_c}{2}) \mu (1 - \frac{f}{f_c}) \text{ lower sideband} \]

Taking inverse Fourier transform.

\[ U_{ssb}(t) = \frac{A_c}{2} \text{ FT} \{ M(t) \} \text{ FT}^{-1} \{ H(f) \} \in \mathbb{F} \text{ + } \]

\[ \frac{A_c}{2} \text{ FT} \{ M(t) \} \text{ FT}^{-1} \{ H(f) \} \in \mathbb{F} \text{ e}^{-j\omega_c f t} \quad (xx) \]

From Table 2.1, it may be easily shown

\[ \text{ FT}^{-1} \{ U(\frac{1}{2}) \} = \frac{1}{2} \delta (t) + \frac{j}{2\pi t} \]

\[ \text{ FT}^{-1} \{ U(-\frac{1}{2}) \} = \frac{1}{2} \delta (t) - \frac{j}{2\pi t} \]

On substituting into (xx), one obtains

\[ U_{ssb}(t) = \frac{A_c}{2} \text{ WH} \{ \frac{1}{2} \delta (t) + \frac{j}{2\pi t} \} e^{j\omega_c f t} + \]

\[ \frac{A_c}{2} \text{ WH} \{ \frac{1}{2} \delta (t) - \frac{j}{2\pi t} \} e^{-j\omega_c f t} \]
\[ U_{SSB}(\omega_0 t) = \frac{A_c}{2} \left[ \hat{W}(t) + j \hat{W}(t) \right] e^{j2\pi f_{offset}} + \frac{A_c}{2} \left[ \hat{W}(t) - j \hat{W}(t) \right] e^{-j2\pi f_{offset}} \]

where:

\[ \hat{W}(t) = W(t) \ast \tilde{S}(t) \rightarrow \text{ convolution with } \tilde{S}(t) \text{ gives the same signal} \]

\[ \hat{W}(t) = \frac{1}{j \omega_0} \text{ } \rightarrow \text{ some form of filtering (yet to be explained)} \]

\[ U_{SSB}(\omega_0 t) = \frac{A_c}{2} W(t) \left[ e^{j\omega_0 \omega_0 t} + e^{-j\omega_0 \omega_0 t} \right] + \]

\[ - \frac{A_c}{2} \left( \hat{W}(t) \left[ e^{j2\pi f_{offset}} - e^{-j2\pi f_{offset}} \right] \right) \]

\[ = \frac{A_c}{2} W(t) \cos(2\pi f_{offset}) = \frac{A_c}{2} \hat{W}(t) \sin(2\pi f_{offset}) \] - time domain representation for USB

Summary:

\[ U_{SSB-\text{up}}(t) = W(t) \cos(2\pi f_{offset}) + \hat{W}(t) \sin(2\pi f_{offset}) \]

\[ U_{SSB-\text{down}}(t) = W(t) \cos(2\pi f_{offset}) - \hat{W}(t) \sin(2\pi f_{offset}) \]

What is signal \( \hat{W}(t) \)?

\( \hat{W}(t) = W(t) \ast \frac{1}{j \omega_0} = \int_{-\infty}^{t} W(t') \frac{1}{j \omega_0 (t'-t)} dt' \)

\( \hat{W}(t) \) - Hilbert transform of \( W(t) \) (another integral transform).

Hilbert transform is equivalent to filtering with a filter with unit impulse response \( h(t) = \frac{1}{j \omega_0} \)

Frequency response of the Hilbert transform filter:

\( H(f) = F \left\{ \text{sign}(f) \right\} = F \left\{ \frac{1}{j \omega_0} \right\} \cos(f) = -j \text{sign}(f) \)
\[ H(f) = -j \text{sign}(f) = \begin{cases} -j, & f > 0 \\ 0, & f = 0 \\ +j, & f < 0 \end{cases} \]

\[ \frac{1}{\pi} |H(f)| \]

**Notes:**

1. **Hilbert transform preserves magnitude of the signal spectrum.**
2. **Hilbert shifts the phase of every signal component by \(-\pi/2\) (90°).**

**Proof:**

\[
\cos(\alpha - \pi/2) = \cos(\alpha) \cos(\pi/2) + \sin(\alpha) \sin(\pi/2) = \sin \alpha
\]

**Nature:** 
Hilbert transform turns every cosine component in the signal spectrum into a sinusoidal component.

**Proof:**

\[
\hat{u}_{ssb}(t) = \hat{u}_H \hat{u}(t) = \hat{u}_H \hat{u}(t) \sin(\hat{\omega}_s t + \phi) = \\
\left[ \hat{u}_H(t) \hat{\omega}_s(t) \right]^{1/2} \cos(2\pi f_H + \phi) \sin(\hat{\omega}_s t + \phi)
\]

*Amplitude modulation*  
*Phase modulation*

*Amplitude of \(\hat{u}_{ssb}(t)\) is not proportional to \(\hat{u}_H(t)\). It is dependent on \(\hat{u}_H(t)\) in an complicated nonlinear fashion.*

*SSB modulates not only magnitude but also the phase of the carrier.*
Demodulation of SSB - AM

Demodulator block diagram

$$\begin{align*}
X(t) & = \text{Ussb}(t) \cdot 2\cos(2\pi ft) \\
\text{Ussb}(t) & = \hat{W}t + \hat{W} t \cos(2\pi ft) + \hat{W} t \sin(2\pi ft) \\
X(t) & = [\hat{W}t + \hat{W} t \cos(2\pi ft)] \cdot 2\cos(2\pi ft) \\
& = 2\hat{W} t \cos^2(2\pi ft) + 2\hat{W} t \sin(2\pi ft) \cos(2\pi ft) \\
& = \frac{\hat{W}t + \hat{W} t \cos(2\pi ft)}{2} + \frac{\hat{W}t \sin(2\pi ft)}{2} \\
\text{band} & \text{band} \quad \text{high frequency}
\end{align*}$$

After the low-pass filter the high frequency component is eliminated and therefore one obtains:

$$y(t) = \hat{W} t$$

* The SSB-AM is demodulated correctly. That is, the phase of locally generated carrier signal needs to be the same as the phase of the incoming carrier. If the synchronization is not perfect there is a level of cross-talk. (i.e. mixing of \(\hat{W} t\) & \(\hat{W} t\))
Signal Multiplexing in Frequency Domain

Modulation translate the spectrum of the signal from the baseband to the carrier frequency.

Occupied bandwidth $\rightarrow 2W$ for 8SB-SC-1C or $W$ for SSB

Using different $f_c$ allows multiplexing (i.e. broadcasting simultaneously) more than one signal through the same line.

Example:

\[
U_1(f) = \frac{1}{2} H_1(f) + \frac{1}{2} H_1(f - f_c)
\]

\[
U_2(f) = \frac{1}{2} H_2(f + f_c) + \frac{1}{2} H_2(f - f_c)
\]

End up the signals through the same channel:

\[
U_1(f) + U_2(f)
\]

Two signals co-exist in time, but are separable in frequency domain. This method of multiplexing is called Frequency Division Multiplexing (FDM).
To make multiplexing even more efficient, systems with AM modulation frequently use SSB-AM.

Examples: Analog telephony SSB-AM + FDM
AM radio: DSB-SC + FDM

Homework (5)

Exercise (Problem book): 3.9, 3.10, 3.13, 3.14 & 3.15
Homework (Textbook): 3.9, 3.13 & 3.16