ECE 5284 Communication Theory - Lecture 21.

* AWGN - Additive White Gaussian Noise - Channel (under most circumstances this is considered the "best case")

![Diagram of communication process]

- Subject of study

Geometrical representation of signal functions:

Consider a set of continuous functions defined on interval [a, b]. This set along with addition and multiplication by scalar which are defined in usual way forms a vector space - Functional vector space.

Definition of dot product (inner product) between two functions on interval I:

\[ \langle s(t), s'(t) \rangle = \int_{a}^{b} s(t) s'(t) \, dt \]

for real \( s(t), s'(t) \)

\[ \langle s(t), s'(t) \rangle = \int_{a}^{b} s(t) s'(t) \, dt \]
It is easily shown that following properties hold.

1) \( \langle S_i(t), S_j(t) \rangle = \langle S_j(t), S_i(t) \rangle \) - associativity with scalar multiplication.

2) \( \langle S_i(t), S_j(t) + S_k(t) \rangle = \langle S_i(t), S_j(t) \rangle + \langle S_i(t), S_k(t) \rangle \)

Proof: \( \langle S_i(t), S_j(t) + S_k(t) \rangle = \int_{\Delta} S_i(t) \cdot (S_j(t) + S_k(t)) \, dt = \int_{\Delta} S_i(t) \cdot S_j(t) \, dt + \int_{\Delta} S_i(t) \cdot S_k(t) \, dt = \langle S_i(t), S_j(t) \rangle + \langle S_i(t), S_k(t) \rangle \).

**Definition of the signal norm**

\[ \| S_i(t) \| = \left( \langle S_i(t), S_i(t) \rangle \right)^{1/2} = \left( \int_{\Delta} S_i(t) \cdot S_i(t) \, dt \right)^{1/2} = \sqrt{E_i} \]

where \( E_i \) is the energy of the signal \( S_i(t) \) on the interval \( \Delta \).

**Definition of orthogonality**

\( S_i(t) \) is orthogonal to \( S_j(t) \) if their dot product is equal to zero. That is:

\[ S_i(t) \text{ is orthogonal to } S_j(t) \iff \langle S_i(t), S_j(t) \rangle = \int_{\Delta} S_i(t) \cdot S_j(t) \, dt = 0 \]

\[ \| S_i(t) \| = \left( \int_{\Delta} S_i(t) \cdot S_i(t) \, dt \right)^{1/2} \]

The number of waveforms that may be defined on interval \( \Delta \) is infinite.

* For communication one selects \( N \) forms, \( S_i(t) \), \( 1 \leq i \leq N \)

* \( N \) selected waveforms are called independent if

\[ \langle S_1(t), S_2(t), S_3(t), \ldots, S_N(t) \rangle \equiv 0 \iff \forall i, a_i = 0, \forall i \]
In general the selected waveforms are dependent but there is a subset $N \in M$ that consists the basis of the space. One way to determine the basis of the space is through Gram-Schmidt orthogonalization process.

**Gram-Schmidt Orthogonalization**

Consider a set of $M$ waveforms $\{S(m(t), m=1, 2, ..., M\}$ defined on interval $I$. The goal is to find $N \leq M$ mutually independent and orthogonal waveforms that can be used to express $\{S(t)\}$. In other words, one seeks to determine the basis of the signal space. The basis may be determined through the following procedure:

$\Psi_1(t) = \frac{S_1(t)}{\sqrt{E_1}}$, where $E_1 = \int |S_1(t)|^2 dt$

$\delta_1(t) = S_2(t) - \langle S_1(t), \Psi_1(t) \rangle \Psi_1(t)$

$\Psi_2(t) = \frac{\delta_2(t)}{\sqrt{E_2}}$, where $E_2 = \int |\delta_2(t)|^2 dt$

... General ...

$\Psi_k(t) = \frac{\delta_k(t)}{\sqrt{E_k}}$, where

$E_k = \int |\delta_k(t)|^2 dt$

... Orthogonalization process continues until all $M$ waveforms are exhausted.

**Example 1.** Apply GS orthogonalization to the set of signals given in the figure. Determine $\Psi_k(t)$ and the size of the basis of the signal space.
I = [0, 2] signaling interval.

\( n = 4 \) (number of waveforms in the signaling set) - usually \( n = 2^n \), n - number

1° \( E_1 = \int_0^2 |S_1(t)|^2 \, dt = 2 \), \( \psi_1 = \frac{1}{\sqrt{2}} S_1(t) \)

2° \( d_2(t) = S_2(t) - \langle S_2(t), \psi_1(t) \rangle \psi_1(t) \)

\( \langle S_2(t), \psi_1(t) \rangle = \int_0^2 S_2(t) \psi_1(t) \, dt = \int_0^1 \frac{1}{\sqrt{2}} - 1 \, dt + \int_1^2 \frac{1}{\sqrt{2}} \, (-1) \, dt = 0 \)

\( d_2(t) = S_2(t) \), \( E_2 = \int_0^2 |d_2(t)|^2 \, dt = \int_0^2 1 \, dt = 2 \)

\( \psi_2(t) = \frac{1}{\sqrt{2}} S_2(t) \)

3° \( d_3(t) = S_3(t) - \langle S_3(t), \psi_1(t) \rangle \psi_1(t) - \langle S_3(t), \psi_2(t) \rangle \psi_2(t) \)

\( \langle S_3(t), \psi_1(t) \rangle = 0 \)

\( \langle S_3(t), \psi_2(t) \rangle = -\sqrt{2} \)

\( d_3(t) = S_3(t) + \sqrt{2} \psi_2(t) \)

\( E_3 = \int_0^2 |d_3(t)|^2 \, dt = 1 \)

\( \psi_3(t) = S_3(t) + \sqrt{2} \psi_2(t) \)
Signal $\Psi_1(t)$ may be expressed as a linear combination of previously existing signals $(\Psi_1(t), \Psi_2(t), \ldots, \Psi_k(t))$. The dimension of the basis is $k = 3$, and orthonormal basis of the signal space is the basis set known. Every signal in the set may be expressed as a sum of combinations of the signals from basis set, that is

$$ S_{\Psi_1} = \frac{1}{2\pi} \sum_{k=1}^{2} \left< S_{\Psi_1}, \Psi_k(t) \right> \Psi_k(t), \quad W = 1, 2, \ldots , 4 $$

or:

$$ S_{\Psi_1} = \left< S_{\Psi_1}, \Psi_1(t) \right> \Psi_1(t) = \int S_{\Psi_1}(t) \Psi_1(t) \, dt $$

where $\left< S_{\Psi_1}, \Psi_k(t) \right>$ is the inner product of $S_{\Psi_1}(t)$ and $\Psi_k(t)$. The first three basis vectors are:

$$ \Psi_1(t) = -\frac{1}{N_2} \Psi_1(t), \quad \Psi_2(t) = \frac{1}{N_2} \Psi_1(t), \quad \Psi_3(t) = \frac{1}{N_2} \Psi_1(t) $$

The orthogonality of the basis vectors ensures that the signal can be reconstructed accurately.

\[ d\Psi_1(t) = \frac{3}{2} \left< S\Psi_1, \Psi_2 \right> \Psi_2(t) - \frac{3}{2} \left< S\Psi_1, \Psi_3 \right> \Psi_3(t) \]

\[ <S\Psi_1, \Psi_1(t)> = \int_0^T S\Psi_1(t) \Psi_1(t) \, dt = \int_0^T \frac{3}{N_2} \Psi_1(t) \, dt = \frac{3}{N_2} \]
The energy of the signal \( s(t) \)

\[
E_s = \langle s(t), s(t) \rangle = \frac{1}{T} \int_{-T/2}^{T/2} s(t)s^*(t) dt = \sum_{k=1}^{N} \left( \frac{2}{T} \text{Re} \{ \sum_{p=1}^{M} S_{k,p} \} \right)^2
\]

\[
= \sum_{k=1}^{N} \frac{2}{T} \text{Re} \{ \sum_{p=1}^{M} S_{k,p} \} \left( \frac{2}{T} \text{Re} \{ \sum_{p=1}^{M} S_{k,p} \} \right) dt =
\]

\[
= \sum_{k=1}^{N} \frac{2}{T} \text{Re} \{ \sum_{p=1}^{M} S_{k,p} \} \left( \frac{2}{T} \text{Re} \{ \sum_{p=1}^{M} S_{k,p} \} \right)
\]

Based on the basic decomposition, every signal \( s(t) \) may be mapped into a vector

\[
\text{vec}(s(t)) \rightarrow \text{vec}(s(t)) = [s_1, s_2, \ldots, s_M]^T \text{ point in an } N \text{-dimensional vector space.}
\]

Energy of the signal is \( \| s(t) \|^2 \), where \( \| s(t) \| \) is Euclidean norm of the vector.

One also notes

\[
\langle s(t), s(t) \rangle = \frac{1}{T} \int_{-T/2}^{T/2} s(t)s^*(t) dt =
\]

\[
= \int \left( \sum_{k=1}^{N} S_{k,p} \right)(\sum_{p=1}^{M} S_{k,p}) dt =
\]

\[
= \sum_{k=1}^{N} \sum_{p=1}^{M} S_{k,p}^2 \left( \sum_{p=1}^{M} S_{k,p} \right) dt =
\]

\[
= \sum_{k=1}^{N} \sum_{p=1}^{M} S_{k,p}^2 \left( \sum_{p=1}^{M} S_{k,p} \right) =
\]

\[
= \sum_{k=1}^{N} \sum_{p=1}^{M} S_{k,p}^2
\]

\[
= s(t)^T s(t)
\]

Inner product between two signals give the same result as inner product between the corresponding vectors.
Summation

\[
\begin{align*}
\mathbf{S}_w &= [\mathbf{S}_{w1}, \mathbf{S}_{w2}, \ldots, \mathbf{S}_{wT}]^T \quad \text{corresponding vector} \\
\mathbf{S}_w &= \langle \mathbf{S}_{w1}, \mathbf{S}_{w2} \rangle = \int \mathbf{S}_{w1} \cdot \mathbf{S}_{w2} \, dt \quad \text{corresponding coordinate} \\
\langle \mathbf{S}_{w1}, \mathbf{S}_{w2} \rangle &= \mathbf{S}_{w1}^T \mathbf{S}_{w2} \quad \text{dot product} \\
\langle \mathbf{S}_{w1}, \mathbf{S}_{w2} \rangle &= \| \mathbf{S}_{w1} \|^2 \iff \| \mathbf{S}_{w1} \| = \| \mathbf{S}_{w2} \| = \| \mathbf{S}_{w} \| = \mathbf{E}_x \quad \text{energy of the signal} \\
\langle \mathbf{S}_{w1}, \mathbf{S}_{w2} \rangle &= 0 \iff \mathbf{S}_{w1}^T \mathbf{S}_{w2} = 0 \quad \text{orthogonality}
\end{align*}
\]

Exercise (15): none
Assignment: 7.3 & 7.4