Unmodulated carrier with a sum of parasitic sinusoids:
\[ r(t) = \frac{A_c \cos(2\pi f_c t)}{1 + \sum_{k=1}^{\infty} \text{unmodulated} \cdot \cos(2\pi (f_c + f_k) t + \theta_k)} \]

One may rewrite \((\mathbf{r})\) in a form given by:
\[ r(t) = \left[ A_c + \sum_{k=1}^{\infty} \text{parasitic} \cdot \cos(2\pi (f_c + f_k) t + \theta_k) \right] \cdot \cos(2\pi f_c t) - \left[ \sum_{k=1}^{\infty} \text{parasitic} \cdot \sin(2\pi (f_c + f_k) t + \theta_k) \right] \cdot \sin(2\pi f_c t) = H(t) \cdot \cos[2\pi f_c t + \psi(t)] \]

Signal \( r(t) \) is passed through the limiter that eliminates the parasitic in the signal magnitude. In other words, after the limiter \( H(t) \rightarrow A_c \).

\[ 2_1 h(t) = A_c \cos[2\pi f_c t + \psi(t)] \]

\[ \psi(t) = \text{atan} \left[ \frac{\sum_{k=1}^{\infty} \text{parasitic} \cdot \sin(2\pi (f_c + f_k) t + \theta_k)}{A_c + \sum_{k=1}^{\infty} \text{parasitic} \cdot \cos(2\pi (f_c + f_k) t + \theta_k)} \right] \]

Under assumption of high SIR, \( A_c \gg \sum_{k=1}^{\infty} \text{parasitic} \cdot \cos(2\pi (f_c + f_k) t + \theta_k) \)
\[ \psi(t) \approx \omega t \tan \left( \sum_{k=1}^{m} \frac{U_{m_k}}{A_c} \sin(2\pi f_{m_k} t + \theta_e) \right) \]

\[ \approx \sum_{k=1}^{m} \frac{U_{m_k}}{A_c} \sin(2\pi f_{m_k} t + \theta_e) \]

Therefore,

\[ x(t) \approx A \cos \left( 2\pi f_c t + \sum_{k=1}^{m} \frac{U_{m_k}}{A_c} \sin(2\pi f_{m_k} t + \theta_e) \right) \]

The output:

\[ e_0(t) = \frac{k}{2\pi} \frac{d}{dt} \sin \left[ \sqrt{P_0} \left( \sum_{k=1}^{m} \frac{U_{m_k}}{A_c} + \sum_{k=1}^{m} \frac{U_{m_k}}{A_c} \cos(2\pi f_{m_k} t + \theta_e) \right) \right] \]

and

\[ y(t) = k \sum_{k=1}^{m} \frac{U_{m_k}}{A_c} \sin(2\pi f_{m_k} t + \theta_e) \]

This result demonstrates that under assumption of high signal to interference ratio a bound approximation holds. The system becomes linearized.

Note: Each component gets scaled by its frequency offset from the carrier.

\[ \text{at the carrier frequency} \]

\[ \text{at the baseband} \]
FM signal & white noise

Consider a single small sinusoidal component. At the output of the FM receiver this component produces an output given by

\[ \Delta V_{\text{out}} = k \cdot \frac{\Delta V_{\text{in}} \cdot f_{\text{in}}}{\Delta f} \rightarrow \text{magnitude} \]

\[ \frac{1}{2} \Delta V_{\text{out}}^2 = k^2 \cdot \frac{\Delta V_{\text{in}}^2 \cdot f_{\text{in}}^2}{2 \Delta f^2} \rightarrow \text{power} \]

\[ P_{\text{out}} = \frac{1}{2} \frac{k^2 f_{\text{in}}^2}{\Delta f^2} \]

where \( P_{\text{out}} \) = power of the small sinusoidal component & \( S \) is the power of the carrier.

Now let \( \Delta f \to 0 \) (continuous version of the noise spectrum)

\[ \frac{dP_{\text{out}}}{df} = \frac{1}{2} k^2 f^2 \cdot \frac{dC_{\text{in}}}{s} \]

\[ \frac{dP_{\text{out}}}{df} = \left( \frac{1}{2} k^2 f^2 \right) \frac{dC_{\text{in}}}{s} \]

\( \Rightarrow \) PSD at the output

\[ \frac{dP_{\text{in}}}{df} = \frac{1}{2} k^2 f^2 \]

\( \Rightarrow \) PSD at the input
\[ S_{\text{No}}(\phi) = \frac{1}{2} \frac{k^2 T^2}{S} \cdot S_{\text{in}}(\phi) \]

It is assumed as white, so
\[ S_{\text{No}}(\phi) = \frac{1}{2} \frac{k^2 T^2}{S} \]

No \(-\) it is not \(N_0/2\) since we only considered single-sided PSD in the derivation so far.

**Baseband filter**
\[ S_{\text{in}}(f) = \frac{1}{2} \frac{k^2 T^2}{S} \]

No

**Calculation of the S/N relationship**

\[ \text{S/N}_\text{in} = \text{Power of utility} / \text{Power of noise} = \frac{\sqrt{2} k T^2}{N_0 \cdot B} \]

\[ = \frac{\sqrt{2} k T^2}{g(1+\beta) \cdot W \cdot N_0} = \frac{k T^2}{A(1+\beta) \cdot W \cdot N_0} \] (\%)

**At the output of the RX**

\[ V(t) = A_1 e^{-\alpha t} \cos[2\pi f_0 t + 2\pi kf \int_0^t \text{wn}(c) dc] \]

\[ S(f) = K \cdot k f \text{ wn}(f) = K \cdot \frac{B \cdot W}{\text{max} \{ |\text{wn}(f)| \}} \]

\[ S/N \text{ out} = \text{Power of} S(f) / \text{Power of} \text{ noise} \]

\[ \text{Power of} S(f) = k^2 \beta^2 \omega^2 \text{ wn}^2(1) = k^2 \beta^2 \omega^2 \text{ Pwn} \]
\[ P_{\text{SNR in}} = 2 \int_0^W S(t) dF = 2 \int_0^W \frac{f^2}{R} \frac{V_0}{2} \]

\[ = \frac{1}{2} \frac{\frac{L}{R} V_0}{S} \frac{W^2}{2} \int_0^W = \frac{\frac{L}{R} V_0}{S} \frac{W^2}{2} \quad \text{"noise quenching"} \]

\[ \frac{S}{N}_{\text{out}} = \frac{k^2 \beta^2 W^2 R_{\text{in}}}{k^2 R_0 W^2} \left( \frac{A_0^2}{2} \right) = 3 \frac{L^2}{R_0} \frac{R_{\text{in}}}{W} \]

From (x) \( \Rightarrow \frac{1}{2} A_0^2 = \frac{S}{N}_{\text{in}} \quad \text{No BW} = 2 (\frac{S}{N}_{\text{in}}) \left( 1 + \beta^2 \right) \quad W \cdot R_0 \]

After substitution

\[ \frac{S}{N}_{\text{out}} = \frac{3 L^2}{R_0} \frac{R_{\text{in}}}{W} \]

\[ = 3 \beta^2 \left( 1 + \beta^2 \right) \left( 2 R_{\text{in}} \right) \left( \frac{S}{N}_{\text{in}} \right) \text{in} \quad (\times x) \]

Processing gain

Another way to write (\times x)

\[ \frac{S}{N}_{\text{out}} = 6 \beta^2 \left( 1 + \beta^2 \right) 2 R_{\text{in}} \frac{A_0^2/2}{2 \left( 1 + \beta^2 \right) \cdot W R_0} = 3 \beta^2 \left( 1 + \beta^2 \right) R_{\text{in}} \frac{A_0^2/2}{W R_0} \]

Which is the same as the reaction (S.3.24 in the textbook)

**Example.** Consider an FM signal modulated with parameters

\[ M(t) = \cos \left[ 2 \pi f_{\text{in}} t \right], \quad \text{where } f_{\text{in}} = 1 \text{kHz} \]

\[ f_0 = 5 \]

\[ N_0 = 4 \times 10^{-7} \frac{W W}{Hz} \quad \text{(single sided)} \]

Required SNR at the output is 50 dB
1) Estimate required $S/N_{in}$

2) If the only TX power is 0 dBm, determine maximum sustainable path loss.

\[ (S/N)_{out} = 3(1+\beta) \beta^2 (2P_{in}) \cdot (S/N)_{m} \Rightarrow \]

\[ (S/N)_{in} = \frac{(S/N)_{out}}{3(1+\beta) \beta^2 (2P_{in})}, \quad P_{in} = 1/2 \]

Theorem: $(S/N)_{in} = (S/N)_{out} - 10 \log (3(1+\beta) \beta^2 \cdot 2 \cdot 1/2)$

\[ = 50 \text{dB} - 10 \log (3(1+3) \beta^2) = \]

\[ = 50 \text{dB} - 26.58 \text{dB} \Rightarrow 23.42 \text{dB} \]

The receiver processing gain is 26.58 dB (close to 500 hours)

3) Power of the noise at the input

\[ N = 2(1+\beta) \cdot W \cdot N_s = 2(1+5) \cdot 1000 \times 4 \times 10^{-17} \text{W/mW} = 4.8 \times 10^{-13} \text{W} \]

\[ N [\text{dBm}] = 10 \log \left( 4.8 \times 10^{-15} \text{W/mW} \right) = -123.19 \text{dBm} \]

\[ S_{in} = N + (S/N)_{in} = -123.19 \text{dBm} + 23.42 \text{dB} = -99.77 \text{dBm} \]

Maximum path loss

\[ P_{loss} = S_{in} - S_{in} = 0 \text{dBm} - (-99.72 \text{dBm}) = 99.72 \text{dB} \approx 100 \text{dB}. \]
Conclusions

1) The output S/N is proportional to $\beta^2$. The increase of $\beta$ increases output S/N.

2) The increase of S/N is obtained at the expense of $\beta$. Frequency modulation is a "Spread Spectrum" communication system.

3) Increase in $\beta$ (and output S/N) is limited by the requirement $S >> 1$.
   
   - When $S >> N$, $S/N$ benefits from processing gain
   - When $S < S_{threshold}$, $S/N$ decreases rapidly

4) In FH PSK, the output noise is not constant. It increases as $\beta^2$. This means that not all parts of the message signal spectrum are affected in the same manner. Higher frequencies in the message signal spectrum are affected more.

Homework: (13) Exercises: 8.10, 8.11, 8.12, 8.13
Assignment: 5.8 & 5.9 (a)