Show that DSB signal \( X_{\text{DSB}}(t) = \text{m}(t) \cos(2\pi f_c t) \) may be created by multiplying message signal with periodic signal \( \text{p}(t) \) and through appropriate filtering.

\[
\begin{align*}
\text{m}(t) & \rightarrow X_{\text{m}}(t) \rightarrow \text{filter} \rightarrow X_{\text{DSB}}(t) \\
\text{p}(t) & \rightarrow X_{\text{p}}(t)
\end{align*}
\]

\[
\begin{align*}
X_{\text{DSB}}(t) & = \text{m}(t) \cdot \text{p}(t) = \text{m}(t) \left[ (c_0 + \sum_{n=1}^{\infty} c_n \cos(2\pi n f_0 t)) \right] = \\
& = c_0 \text{m}(t) + \sum_{n=1}^{\infty} c_n \text{m}(t) \cos(2\pi n f_0 t)
\end{align*}
\]

Multiplying \( \text{m}(t) \cdot \text{p}(t) \) generates many DSB bands. The output filter just picks the one centered around desired carrier frequency.
(2) Show that DSB-SC can be demodulated using square law detector given in the figure.

\[ X_{\text{DSB-SC (t)}} \rightarrow (\cdot)^2 \rightarrow \text{LPF} \rightarrow Y_{\text{Hf}} \]

MSE \[ |W_{\text{Hf}}| << 1 \]

\[ X_{\text{DSB-SC (t)}} = A_c (1 + \omega_0 t) \cos (2\pi f_c t) \]

\[ 2f_c = [X_{\text{DSB-SC (t)}}]^2 = A_c^2 (1 + \omega_0 t)^2 \cos^2 (2\pi f_c t) \]

\[ = \frac{A_c^2}{2} (1 + 2\omega_0 t + \omega_0^2 t^2) (1 + \cos 2\pi 2f_c t) \]

\[ = \frac{A_c^2}{2} + \frac{A_c^2 \omega_0 t}{2} + \frac{A_c^2 \omega_0^2 t^2}{2} + \frac{A_c^2}{2} (1 + 2\omega_0 t + \omega_0^2 t^2) \cos (2\pi 2f_c t) \]

Eliminated by LPF.

\[ Y_{\text{Hf}} = \alpha \frac{A_c^2}{2} \left[ \omega_0 t + \frac{\omega_0^2 t^2}{2} \right] \]

Overall gain. Desired signal + distortion.

If \( \omega_0 t \) is small \( \Rightarrow \omega_0 t^2 \) is even smaller. Therefore \( \frac{\omega_0^2 t^2}{2} \ll \omega_0 t \), and

\[ Y_{\text{Hf}} \approx \alpha \frac{A_c^2 \omega_0 t}{2} + \text{some distortion} \]

(3) Using orthogonality of sine & cosine it is possible to transmit two
different signals simultaneously on the same carrier frequency. The scheme
is shown in figure and it is known as QAM (Quadrature Amplitude Modulation.) QAM is used extensively in digital communication systems. Demonstrate that the scheme works by evaluating signals:

\[ X_{\text{Hf}}, Y_{\text{Hf}}, 2 X_{\text{Hf}}, X_{\text{Hf}}, Y_{\text{Hf}}, 2 Y_{\text{Hf}} \]
At the transmitter:

\[ X_1(t) = M_i(t) \cdot \cos(2\pi f_c t + \phi_1) \]
\[ X_2(t) = M_i(t) \cdot \sin(2\pi f_c t) \]
\[ X_{\text{output}}(t) = M_i(t) \cos(2\pi f_c t + \phi_1) + M_o(t) \sin(2\pi f_c t) \]

At the receiver:

\[ 2_i(t) = X_{\text{output}}(t) \cdot \cos(2\pi f_c t + \phi_1) = \left[ M_i(t) \cos(2\pi f_c t + \phi_1) + M_o(t) \sin(2\pi f_c t) \right] \cdot \cos(2\pi f_c t + \phi_1) \]
\[ = M_i(t) \cos(2\pi f_c t) \cos(2\pi f_c t + \phi_1) + M_o(t) \sin(2\pi f_c t) \cos(2\pi f_c t + \phi_1) \]
\[ = \frac{1}{2} M_i(t) \cos(\phi_1) - \frac{1}{2} M_o(t) x_c(t) + \frac{1}{2} M_o(t) \sin(2\pi f_c t) \sin(-\phi_1) \]

After the LPF:

\[ y_1(t) = \frac{1}{2} M_i(t) \cos(\phi_1) - \frac{1}{2} M_o(t) \sin(\phi_1) \]

\[ y_2(t) = X_{\text{output}}(t) \sin(2\pi f_c t + \phi_1) = \left[ M_i(t) \cos(2\pi f_c t + \phi_1) + M_o(t) \sin(2\pi f_c t) \right] \sin(2\pi f_c t + \phi_1) \]
\[ = M_i(t) \cos(2\pi f_c t) \sin(2\pi f_c t + \phi_1) + M_o(t) \sin(2\pi f_c t) \cdot \sin(2\pi f_c t + \phi_1) \]
\[ = \frac{1}{2} M_i(t) \sin(2\pi f_c t) + \frac{1}{2} M_o(t) \sin(\phi_1) \]

After the lowpass filter
\[ y_{1t} = \frac{\omega_{1t}}{2} \sin \theta_1 + \frac{\omega_0}{2} \cos \theta_1 \]

If the oscillation at the RX is synchronized with the incoming carrier \( \theta_1 = 0 \) in such scenarios:

\[ y_{1t} = \frac{1}{2} \omega_{1t} \]
\[ y_{2t} = \frac{1}{2} \omega_{0t} \]

In the case when synchronization does not exist, there is a cross-talk between the branches.

4. The heavily multiplier is a nonlinear device followed by an appropriate bandpass filter. Consider the block diagram given below. Find \( y_{1t} \) if

\[ x_{1t} = A \cos [2\pi f_c t + \beta \sin (2\pi f_{in} t)] \]

\[ x_{1t} \rightarrow (\cdot)^2 \rightarrow z_{1t} \rightarrow \text{BPF} \rightarrow y_{1t} \]

\[ z_{1t} = x^2_{1t} = [A \cos [2\pi f_c t + \beta \sin (2\pi f_{in} t)]]^2 = \]

\[ = A^2 \cos^2 [2\pi f_c t + \beta \sin (2\pi f_{in} t)] = \]

\[ = \frac{A^2}{2} \left( 1 + \cos [2\pi f_c \cdot 2\pi f_{in} t + 2\beta \sin (2\pi f_{in} t) \right) = \]

\[ = A^2 \frac{1}{2} + \frac{A^2}{2} \cos (2\pi f_c t + 2\beta \sin (2\pi f_{in} t) \]

After BPF:

\[ y_{1t} = A^2 \frac{1}{2} \cos [2\pi f_c t + 2\beta \sin (2\pi f_{in} t) \]

A doubled carrier frequency and modulation index.
Consider a system given in the figure. Demonstrate that the system can be used for demodulation of FM signal provided that $T$ is small.

\[ 2y(1) = \frac{1}{T} (X_{FM}(1) - X_{FM}(t-2)) \]

For $T$ small, $2y(1) = \frac{dX_{FM}(1)}{dt}$

\[ 2y(1) = \frac{d}{dt} \left( Ac \cos \left( 2\pi f_c t + 2\pi ft \int_0^t w(t) dt \right) \right) = \]

\[ = -Ac \left[ 2\pi f_c + 2\pi ft \text{mod } 1 \right] \sin (2\pi ft + 2\pi ft \int_0^t w(t) dt) \]

\[ = -2\pi Ac \left[ \frac{d_c + kf \text{mod } 1}{\text{mod } 1} \right] \cdot \sin (2\pi ft + 2\pi ft \int_0^t w(t) dt) \]

\[ 2y(1) = -2\pi Ac \left[ \frac{d_c + kf \text{mod } 1}{\text{mod } 1} \right] \]

and \[ y(1) = -2\pi Ac (f_c + kf \text{mod } 1 \cdot w(t)) \]

3) The carrier $C(t) = 100 \cos (2\pi f_c t)$ is frequency modulated using the signal $W(t) = 50 \cos (2\pi ft \cdot \text{mod } 1)$, where $f_c = 100 kHz$. The peak frequency deviation is $20 kHz$.

1) Determine the amplitude and frequency of all signal components that carry more than 10% of signal power.

2) Determine the BW using Carson's rule.
1) \[ \Delta f_{\text{FM}} = 20k \text{Hz}, \quad \beta = \frac{\Delta f_{\text{FM}}}{WN} = \frac{20k \text{Hz}}{10^4 \text{Hz}} = 2 \]

\[ X_{\text{FM}}(t) = 100 \cos \left[ 2\pi \cdot 10^8 t + 2 \cdot \sin \left[ \frac{2\pi}{10^4 \text{Hz}} \cdot t \right] \right] = \]

\[ = 100 \sum_{n=-\infty}^{\infty} \sin \left[ 2\pi \cdot 10^8 \left( 10^8 + n \cdot 10^4 \text{Hz} \right) t \right] \]

Component Power / 10^4 \% of total power (total power = 1/2 \times 10^4)

- n = 0 \[ \frac{1}{2} \sin^2 \left( \frac{2\pi}{10^4 \text{Hz}} t \right) = 0.0051 \]
- n = 1 \[ \frac{1}{2} \sin^2 \left( \frac{2\pi}{10^4 \text{Hz}} t \right) = 0.0016 \]
- n = 2 \[ \frac{1}{2} \sin^2 \left( \frac{2\pi}{10^4 \text{Hz}} t \right) = 0.0002 \]
- n = 3 \[ \frac{1}{2} \sin^2 \left( \frac{2\pi}{10^4 \text{Hz}} t \right) = 0.0001 \]
- n = 4 \[ \frac{1}{2} \sin^2 \left( \frac{2\pi}{10^4 \text{Hz}} t \right) = 0.0000 \]

Power spectrum

2) Using Carson's rule: \[ BW = 2 \cdot (\Delta f_{\text{FM}} + WN) = 2 \cdot (20k \text{Hz} + 10^4 \text{Hz}) = 60 \text{kHz} \]

3. An FM signal is given as:

\[ M(t) = 100 \cos \left[ 2\pi f_c t + 100 \int_{-\infty}^{t} w(t) \, dt \right] \]

where \[ w(t) \] is shown in Figure below.

4) Sketch the instantaneous frequency as a function of time.
5) Determine peak frequency deviation.
1) \[ \Theta(t) = \int_0^t 9\pi^2 \varphi(t) + 100 \int_0^t \psi(r) \, dr \]

\[ \varphi(t) = \frac{1}{2\pi} \cdot \frac{d\Theta(t)}{dt} = \frac{1}{2\pi} \frac{d}{dt} \left[ 9\pi^2 \varphi(t) + 100 \int_0^t \psi(r) \, dr \right] = \]

\[ = \frac{\varphi(t) + 100}{2\pi} \psi(t) \]

In this case:

\[ \varphi(t) = \frac{\varphi(t) + 100}{2\pi} \sum_{k=0}^{\infty} \left(-1\right)^{k+1} \pi^2 \left| \frac{t-\frac{1}{2}}{\pi} \right|^{2k} \]

\[ = \frac{\varphi(t) + 250}{2\pi} \sum_{k=0}^{\infty} \left(-1\right)^{k+1} \pi^2 \left| \frac{t-\frac{1}{2}}{\pi} \right|^{2k} \]

2) Maximum Frequency Deviation

\[ \Delta f_{\text{max}} = \frac{d\Theta(t)}{dt} = 79.58 \text{ Hz} \]